C^*-algebras arising from integral and rational dynamics

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Motivation

Cuntz (2008) introduces $\mathcal{Q}_{\mathbb{N}}$ as the universal C^*-algebra generated by isometries $\{s_n\}_{n\in\mathbb{N}}$ and a unitary u satisfying

$$s_m s_n = s_{mn}, \quad s_n u = u^n s_n, \quad \text{and} \quad \sum_{k=0}^{n-1} u^k s_n s_n^* u^{-k} = 1.$$

Larsen and Li (2012) define $\mathcal{Q}_2$ as the universal C^*-algebra generated by an isometry s_2 and a unitary u satisfying

$$s_2 u = u^2 s_2 \quad \text{and} \quad s_2 s_2^* + u s_2 s_2^* u^* = 1.$$

We think of $\mathcal{Q}_{\mathbb{N}}$ as coming from the set $S = \{\text{all primes}\}$, and $\mathcal{Q}_2$ as coming from $S = \{2\}$. It is computed that

$$K_0(\mathcal{Q}_{\mathbb{N}}) \cong \mathbb{Z}^\infty \cong K_1(\mathcal{Q}_{\mathbb{N}}) \quad \text{and} \quad K_0(\mathcal{Q}_2) \cong \mathbb{Z} \cong K_1(\mathcal{Q}_2).$$

The more general versions can have torsion in their K-groups.

The algebras $\mathcal{Q}_S$

Definition

Let S be a set of mutually relatively prime numbers ≥ 2.
Define the algebra $\mathcal{Q}_S$ as the universal C^*-algebra generated by a unitary u and isometries $\{s_p\}_{p \in S}$ satisfying

(i) $s_p^* s_q = s_q s_p^*$,

(ii) $s_p u = u^p s_p$, and

(iii) $\sum_{k=0}^{p-1} e_{k+p\mathbb{Z}} = 1$

for all $p, q \in S$, where $e_{k+p\mathbb{Z}} := u^k s_p s_p^* u^{-k}$.

Let $(\xi_n)_{n \in \mathbb{Z}}$ denote the standard orthonormal basis for $\ell^2(\mathbb{Z})$. If we define

$$U\xi_n = \xi_{n+1} \quad \text{and} \quad S_p \xi_n = \xi_{pn},$$

then U and $\{S_p\}_{p \in S}$ satisfy (i)–(iii). This representation on $\ell^2(\mathbb{Z})$ is faithful, so we can also think of $\mathcal{Q}_S$ as a subalgebra of $B(\ell^2(\mathbb{Z}))$.

Crossed product description

Let H^+ be the submonoid of $\mathbb{N}^\times$ generated by S. Then

$$\mathcal{Q}_S \cong (\mathcal{D}_S \rtimes \mathbb{Z}) \rtimes^e H^+ \cong \mathcal{D}_S \rtimes^e (\mathbb{Z} \rtimes H^+),$$

where

$$\mathcal{D}_S = C^*\{e_{k+q\mathbb{Z}} : q \in S, k \in \mathbb{Z}\}.$$

Set $N = \mathbb{Z}[\{\frac{1}{p} : p \in S\}] \subseteq \mathbb{Q}$ and let H be the subgroup of $\mathbb{Q}_+^\times$ generated by S. Then it follows from the dilation theory of Laca that

$$\mathcal{Q}_S \sim_M C_0(\Omega) \rtimes N \rtimes H,$$

where Ω is the completion of N w.r.t. to the subgroup topology generated by $\{h\mathbb{Z} : h \in H\}$, and the action is the natural $ax + b$-action.
Let Δ be the closure of $\mathbb{Z}$ in Ω, then $\mathcal{D}_S \cong C(\Delta)$.
Moreover, $\mathcal{Q}_S$ is isomorphic to the full corner of $C_0(\Omega) \rtimes N \rtimes H$ cut down by the projection $\chi_\Delta \in C_0(\Omega)$.
If $P = \{p \in \mathbb{N}^\times : p \text{ prime and } p|q \text{ for some } q \in S\}$,
then $\Delta \simeq \prod_{p\in P} \mathbb{Z}_p$ and $\Omega \simeq \prod'_{p \in P} \mathbb{Q}_p$.

Boundary quotients of semigroup C^*-algebras

$\mathcal{Q}_S$ can also be constructed from either $\mathbb{N} \rtimes H^+$ or $\mathbb{Z} \rtimes H^+$ using the theory of boundary quotients of semigroup C^*-algebras.

Relatively primeness of S gives that $\mathbb{N} \rtimes H^+$ and $\mathbb{Z} \rtimes H^+$ are right LCM. Both are also left Ore semigroups with enveloping group $N \rtimes H \subseteq \mathbb{Q} \rtimes \mathbb{Q}_+^\times$, where still $N = \mathbb{Z}[\{\frac{1}{p} : p \in S\}]$.

First, note that $(\mathbb{N} \rtimes H^+, N \rtimes H)$ forms a quasi-lattice ordered group. Hence we can form the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes H^+, N \rtimes H)$ using the work of Nica, which coincides with $C^*(\mathbb{N} \rtimes H^+)$.

To define $C^*(\mathbb{Z} \rtimes H^+)$, we use Xin Li's theory of semigroup C^*-algebras, which generalizes Nica's approach.

Boundary quotients were introduced by Crisp and Laca for quasi-lattice ordered groups, and later generalized to right LCM semigroups by several people. In this setting

$$\mathrm{BQ}(\mathbb{N} \rtimes H^+) = \mathrm{BQ}(\mathbb{Z} \rtimes H^+) = \mathcal{Q}_S.$$

Kirchberg algebra

Theorem

For every set S of relatively prime numbers, the algebra $\mathcal{Q}_S$ is a unital Kirchberg algebra in the UCT class.
Consequently, the K-theory is a complete isomorphism invariant for $\mathcal{Q}_S$ (Kirchberg-Phillips).

We can use that $\mathcal{Q}_S$ is a full corner of $C_0(\Omega) \rtimes N \rtimes H$ to see this:
The $ax + b$-action of $N \rtimes H$ on Ω is minimal, locally contractive, and topologically free, implying that $C_0(\Omega) \rtimes N \rtimes H$ is purely infinite and simple (Archbold-Spielberg, Laca-Spielberg).

Separability, nuclearity, UCT hold because:
$N \rtimes H$ is discrete countable amenable, $C_0(\Omega)$ is commutative separable, and the transformation groupoid of $(\Omega, N \rtimes H)$ is amenable (Tu).

Main theorem

Let S be a set of mutually relatively prime numbers. Define

$$g = \gcd\{p-1 : p \in S\} = \max\{q \in \mathbb{N}^\times : q|p-1 \text{ for all } p \in S\}.$$

If $2 \in S$, then $g = 1$.

Theorem

$$K_i(\mathcal{Q}_S) \cong \mathbb{Z}^{2^{|S|-1}} \oplus T_i, \quad i = 0, 1,$$

where T_0 and T_1 are torsion groups, which are finite if S is finite. Moreover, if $g = 1$, then T_0 and T_1 are both trivial.

Case $|S| = 1$, i.e., $S = \{r\}$ is previously studied (Hirshberg, Katsura). These are graph C^*-algebras and their K-theory is given by

$$K_0(\mathcal{Q}_S) \cong \mathbb{Z} \oplus \mathbb{Z}/(r-1), \quad [1]_0 = (0,1), \quad K_1(\mathcal{Q}_S) = \mathbb{Z}.$$

Hence, $\mathcal{Q}_{\{r\}} \cong \mathcal{Q}_{\{q\}}$ if and only if $r = q$.

Two generator case and conjecture

Theorem

Assume $|S| = 2$, i.e., $S = \{q, r\}$ and $g = \gcd\{q-1, r-1\}$. Then

$$K_0(\mathcal{Q}_S) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/g\mathbb{Z}, \quad [1]_0 = (0,1), \quad K_1(\mathcal{Q}_S) = \mathbb{Z}^2 \oplus \mathbb{Z}/g\mathbb{Z}.$$

E.g. $\mathcal{Q}_{\{4,13\}} \cong \mathcal{Q}_{\{7,10\}}$.

When $|S| \geq 3$ and $g > 1$, we can only show that $|T_i|$ divides $g^{2^{|S|-2}}$.

Conjecture

For $|S| \geq 2$ we have

$$T_0 \cong (\mathbb{Z}/g\mathbb{Z})^{2^{|S|-2}} \cong T_1,$$

and consequently, $\mathcal{Q}_S \cong \mathcal{Q}_{S'}$ if and only if $|S| = |S'|$ and $g = g'$.

The algebras $\mathcal{Q}_H$

Definition

Let $H \subseteq \mathbb{Q}^\times$. Define the algebra $\mathcal{Q}_H$ as the universal C^*-algebra generated by a unitary u and partial isometries $\{s_h\}_{h\in H}$ satisfying

(i) $s_h^* = s_{h^{-1}}$ and $s_h^* s_h s_g = s_h^* s_{hg}$ for all $g, h \in H$.

(ii) $s_h u^q = u^p s_h$ when $h = p/q$.

(iii) $\sum_{k=0}^{p-1} e_{k+p\mathbb{Z}} = 1$,

where $e_{k+p\mathbb{Z}} := u^k s_h s_h^* u^{-k}$ and $h = p/q$.

Let $(\xi_n)_{n\in\mathbb{Z}}$ denote the standard orthonormal basis for $\ell^2(\mathbb{Z})$. If we define

$$U\xi_n = \xi_{n+1} \quad \text{and} \quad S_h\xi_n = \xi_{hn} \text{ if } hn \in \mathbb{Z} \text{ and } 0 \text{ else},$$

then U and $\{S_h\}_{h\in H}$ satisfy (i)–(iii). This representation on $\ell^2(\mathbb{Z})$ is faithful, so we can also think of $\mathcal{Q}_H$ as a subalgebra of $B(\ell^2(\mathbb{Z}))$. Moreover, if $h = p/q$ with $\gcd(p, q) = 1$, then $S_h = S_p S_q^*$ so that $S_h^* S_h = E_{q\mathbb{Z}}$ and $S_h S_h^* = E_{p\mathbb{Z}}$.

Kirchberg algebra

Set $N = \mathbb{Z}[\{h : h \in H\}] \subseteq \mathbb{Q}$ and let Ω be the completion of N w.r.t. to the subgroup topology generated by $\{h\mathbb{Z} : h \in H\}$, i.e., N is dense in Ω. Let Δ be the closure of $\mathbb{Z}$ in Ω.

Proposition

The algebra $\mathcal{Q}_H$ embeds into $C_0(\Omega) \rtimes N \rtimes H$ as a full corner, cut down by the projection $\chi_\Delta \in C_0(\Omega)$.
Thus, when H is infinite, $\mathcal{Q}_H$ is a UCT Kirchberg algebra, and its K-theory is a complete isomorphism invariant (Kirchberg-Phillips).

Partial actions

Definition

A partial action of a group G on a set X is a collection $\{D_g\}_{g\in G}$ of subsets of X, and a collection of maps $\{\theta_g\}_{g\in G}$, $\theta_g\colon D_{g^{-1}} \to D_g$ such that

$$D_e = X, \quad \theta_e = \mathrm{id}_X$$

$$\theta_{gh} \text{ is an extension of } \theta_g \circ \theta_h$$

Example

Every $H \subseteq \mathbb{Q}^\times$ acts partially on $\mathbb{Z}$ as follows: For $h = p/q \in H$ with $\gcd(p, q) = 1$, set $D_h = p\mathbb{Z}$, and define $\theta_h\colon q\mathbb{Z} \to p\mathbb{Z}$ by $qn \to pn$.

Generalized partial C^*-dynamical systems

Let $\mathcal{B}_H$ be the C^*-subalgebra of $\mathcal{Q}_H$ generated by u and projections $\{e_{k+q\mathbb{Z}} : k \in \mathbb{Z}, 1/q \in N\}$. Recall that $\mathcal{B}_H \cong \mathcal{D}_H \rtimes \mathbb{Z}$ is a Bunce-Deddens algebra.

The group H acts partially by α on $\mathcal{B}_H$, where each α_h is a *-isomorphism from its domain $D_{h^{-1}} = e_{q\mathbb{Z}}\mathcal{B}_H e_{q\mathbb{Z}}$ to its range $D_h = e_{p\mathbb{Z}}\mathcal{B}_H e_{p\mathbb{Z}}$. In terms of the generators u and e_X, the map α_h for $h = p/q$ with $\gcd(p,q) = 1$ is defined by

(i) for $X \subseteq q\mathbb{Z}$ by $\alpha_h(e_X) := e_{hX}$, and

(ii) for $n \in q\mathbb{Z}$ by $\alpha_h(u^n) = u^{hn}$.

Remark

In Exel's definition of a partial C^*-dynamical systems, the domains are required to be ideals, which is not the case here. We would still like to think about our $\mathcal{Q}_H$ as a partial crossed product $\mathcal{B}_H \rtimes_\alpha^{\text{part}} H$.

Main theorem

Let $H \subseteq \mathbb{Q}_+^\times$ be nontrivial and choose a minimal generating set $\{p_i/q_i\}_{i\in I}$ such that $gcd(p_i, q_i) = 1$ and $p_i > q_i$ for all $i \in I$. Define

$$g = \mathrm{gcd}\{p_i - q_i : i \in I\} = \max\{r \in \mathbb{N}^\times : r|(p_i - q_i) \text{ for all } i \in I\}.$$

Theorem

Let $H \subseteq \mathbb{Q}_+^\times$ be nontrivial of rank $m \geq 1$, i.e., $H \cong \mathbb{Z}^m$. Then

$$K_i(\mathcal{Q}_H) \cong \mathbb{Z}^{2^{m-1}} \oplus T_i, \quad i = 0, 1,$$

where T_0 and T_1 are torsion groups, which are finite if H is finitely-generated. If $g = 1$, then $T_0 = T_1 = 0$.
Moreover, there is a C^*-subalgebra $\mathcal{A}_H$ of $\mathcal{Q}_H$ such that

$$K_i(\mathcal{A}_H) = T_i \text{ for } i = 1, 2.$$

Two-generator case and conjecture

Remark (one-generator case)

Let $H = \langle p/q \rangle$, with $\gcd(p, q) = 1$ and $p > q$.
Then $T_0 = \mathbb{Z}/(p-q)\mathbb{Z}$ and $T_1 = 0$.

Theorem (two-generator case)

Assume $H = \langle p_1/q_1, p_2/q_2 \rangle \cong \mathbb{Z}^2$, and set $g = \gcd\{p_1 - q_1, p_2 - q_2\}$.
Then

$$K_0(\mathcal{Q}_H) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/g\mathbb{Z}, \quad [1]_0 = (0, 1), \quad K_1(\mathcal{Q}_H) = \mathbb{Z}^2 \oplus \mathbb{Z}/g\mathbb{Z}.$$

Conjecture

When the rank m of H is at least 2, we have

$$K_i(\mathcal{A}_H) = T_i \cong (\mathbb{Z}/g\mathbb{Z})^{2^{m-2}} \text{ for } i = 1, 2,$$

and consequently, $\mathcal{Q}_H \cong \mathcal{Q}_{H'}$ if and only if $m = m'$ and $g = g'$.

Techniques involved in the proof

Step 1 (comparing with real dynamics)

Again, let H be the subgroup of $\mathbb{Q}^\times$.
Recall that $N = \mathbb{Z}[\{h : h \in H\}]$ and Ω is a completion of N, and that $\mathcal{Q}_H \sim_M C_0(\Omega) \rtimes N \rtimes H$.
Then we apply a "duality theorem" (Kaliszewski-O.-Quigg, 14):

$$C_0(\Omega) \rtimes_{ax+b} (N \rtimes H) \sim_M C_0(\mathbb{R}) \rtimes_{ax+b} (N \rtimes H)$$

Hence, the problem is to compute the K-theory of $C_0(\mathbb{R}) \rtimes N \rtimes H$.

Techniques involved in the proof

Step 2 (decomposition)

The embedding $C_0(\mathbb{R}) \rtimes H \hookrightarrow C_0(\mathbb{R}) \rtimes N \rtimes H$ induces an injection in K-theory onto the free abelian part of $K_*(C_0(\mathbb{R}) \rtimes N \rtimes H) = K_*(\mathcal{Q}_H)$. The action of H is homotopic to the trivial action, so

$$K_*(C_0(\mathbb{R}) \rtimes H) = K_*(C_0(\mathbb{R}) \otimes C^*(H)) = K_*(C_0(\mathbb{R}) \otimes C^*(\mathbb{Z}^m)) = \mathbb{Z}^{2^{m-1}}$$

There is a certain H-invariant subalgebra $A \subset C_0(\mathbb{R}) \rtimes N$ such that

$$K_*(C_0(\mathbb{R}) \rtimes N \rtimes H) \cong K_*(C_0(\mathbb{R}) \rtimes H) \oplus K_*(A \rtimes H),$$

where $K_*(A \rtimes H)$ is a torsion group.

(for this, one first shows that $K_*(C_0(\mathbb{R}) \rtimes N) \cong K_*(C_0(\mathbb{R})) \oplus K_*(A)$)

Techniques involved in the proof

Step 3 (the torsion part)
The algebra $\mathcal{A}_H$ can be described as follows:
Consider $M_H \subset \mathcal{B}_H \cong C(\Delta) \rtimes \mathbb{Z} = \chi_\Delta(C_0(\Omega) \rtimes N)\chi_\Delta$.

There is an H-invariant C^*-subalgebra B of $C_0(\Omega) \rtimes N$, such that M_H is a full corner of B cut down by χ_Δ.

Since $\mathcal{Q}_H$ embeds as a full corner of $C_0(\Omega) \rtimes N \rtimes H$ cut down by χ_Δ, we can find a C^*-subalgebra $\mathcal{A}_H$ of $\mathcal{Q}_H$ that embeds as a full corner of $B \rtimes H$ cut down by χ_Δ.

There exists an H-equivariant isomorphism $C_0(\Omega) \rtimes N \cong C_0(\mathbb{R}) \rtimes N$, and then the A above is defined as the image of B under this map.

The partial action of H on $\mathcal{B}_H$ restricts to a partial action on $M_H \subset \mathcal{B}_H$, where the domains become $D_h = e_{q\mathbb{Z}} M_H e_{q\mathbb{Z}}$. One might then think of $\mathcal{A}_H$ as $M_H \rtimes_\alpha^{\text{part}} H$.

Other descriptions of $\mathcal{A}_S$

In general, it remains to find good descriptions of $\mathcal{A}_H$, but in the original case where H is generated by a set S of mutually relatively prime numbers, we have that

$$\text{Define } \mathcal{A}_S = C^*\{u^m s_p \mid p \in S, 0 \leq m \leq p-1\} \subset \mathcal{Q}_S.$$

Moreover, recall that

$$\mathcal{Q}_S \cong (\mathcal{D}_S \rtimes \mathbb{Z}) \rtimes^e H^+,$$

and the UHF-algebra M_{d^∞}, for $d = \prod_{p \in S} p$, is a subalgebra of $\mathcal{D}_S \rtimes \mathbb{Z}$ invariant under H^+, so

$$\mathcal{A}_S \cong M_{d^\infty} \rtimes^e H^+,$$

and we can show that $\mathcal{A}_S$ is a UCT Kirchberg algebra.

For $|S| \geq 2$, both K-groups of $\bigotimes_{p \in S} \mathcal{O}_p$ are $(\mathbb{Z}/g\mathbb{Z})^{2^{|S|-2}}$.
Hence, our conjecture about $K_*(\mathcal{Q}_S)$ is equivalent with the following:

Conjecture (restated for $\mathcal{Q}_S$)

The algebra $\mathcal{A}_S$ is isomorphic to $\bigotimes_{p \in S} \mathcal{O}_p$.