Semigroup C*-algebras

Xin Li

Queen Mary University of London (QMUL)

Plan for the talks

Plan for my talks:

- Constructions and examples
- Nuclearity and amenability
- K-theory
- Classification, and some questions

Plan for the talks

Plan for my talks:

- Constructions and examples
- Nuclearity and amenability
- K-theory
- Classification, and some questions

Reference: Survey paper “Semigroup C*-algebras” (arXiv).

Semigroup C*-algebras. The construction

Let P be a left cancellative semigroup.
Left cancellation means that for all $p, x, y \in P$, $px = py$ implies $x = y$.

Semigroup C*-algebras. The construction

Let P be a left cancellative semigroup.
Left cancellation means that for all $p, x, y \in P$, $px = py$ implies $x = y$.

For $p \in P$, define $V_p : \ell^2 P \to \ell^2 P$, $\delta_x \mapsto \delta_{px}$.

Semigroup C*-algebras. The construction

Let P be a left cancellative semigroup.
Left cancellation means that for all $p, x, y \in P$, $px = py$ implies $x = y$.

For $p \in P$, define $V_p : \ell^2 P \to \ell^2 P$, $\delta_x \mapsto \delta_{px}$.

A C*-algebra is an algebra of bounded linear operators on a Hilbert space, invariant under adjoints, closed in the norm topology.

Semigroup C*-algebras. The construction

Let P be a left cancellative semigroup.
Left cancellation means that for all $p, x, y \in P$, $px = py$ implies $x = y$.

For $p \in P$, define $V_p : \ell^2 P \to \ell^2 P$, $\delta_x \mapsto \delta_{px}$.

A C*-algebra is an algebra of bounded linear operators on a Hilbert space, invariant under adjoints, closed in the norm topology.

Definition

$C^*_\lambda(P) := C^*(\{V_p \colon p \in P\}) \subseteq \mathcal{L}(\ell^2 P)$

Semigroup C*-algebras. Examples

- Positive cones:
  $P = \{x \in G\colon\ x \geq e\}$ in a left-ordered group $(G, \geq)$, where $\geq$ is a total order on G with $y \geq x \Rightarrow gy \geq gx$ for all $y, x, g \in G$.

Semigroup C*-algebras. Examples

- Positive cones:
  $P = \{x \in G\colon x \geq e\}$ in a left-ordered group $(G, \geq)$, where $\geq$ is a total order on G with $y \geq x \Rightarrow gy \geq gx$ for all $y, x, g \in G$.

  First example $\mathbb{N} \subseteq \mathbb{Z}$: $C^*_\lambda(\mathbb{N})$ is the Toeplitz algebra [Coburn].

Semigroup C*-algebras. Examples

- Positive cones:
  $P = \{x \in G \colon x \geq e\}$ in a left-ordered group $(G, \geq)$, where $\geq$ is a total order on G with $y \geq x \Rightarrow gy \geq gx$ for all $y, x, g \in G$.

  First example $\mathbb{N} \subseteq \mathbb{Z}$: $C^*_\lambda(\mathbb{N})$ is the Toeplitz algebra [Coburn].

  For $G \subseteq (\mathbb{R}, +)$, $P = G \cap [0, \infty)$: $C^*_\lambda(P)$ determines P [Douglas].

Semigroup C*-algebras. Examples

- Positive cones:
  $P = \{x \in G\colon\ x \geq e\}$ in a left-ordered group $(G, \geq)$, where $\geq$ is a total order on G with $y \geq x \Rightarrow gy \geq gx$ for all $y, x, g \in G$.

  First example $\mathbb{N} \subseteq \mathbb{Z}$: $C^*_\lambda(\mathbb{N})$ is the Toeplitz algebra [Coburn].

  For $G \subseteq (\mathbb{R}, +)$, $P = G \cap [0, \infty)$: $C^*_\lambda(P)$ determines P [Douglas].

  What about positive cones in general left-ordered groups?

Semigroup C*-algebras. Examples

- Right-angled Artin monoids [Crisp-Laca]:
  Let $\Gamma = (V, E)$ be a graph with $E \subseteq V \times V$.
  $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \, \forall \, (v, w) \in E \rangle^+$.

Semigroup C*-algebras. Examples

- Right-angled Artin monoids [Crisp-Laca]:
  Let $\Gamma = (V, E)$ be a graph with $E \subseteq V \times V$.
  $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \, \forall (v, w) \in E \rangle^+$.

  e.g. $\mathbb{N} \times \mathbb{N}$: $C_\lambda^*(\mathbb{N} \times \mathbb{N}) \cong C_\lambda^*(\mathbb{N}) \otimes C_\lambda^*(\mathbb{N})$.

Semigroup C*-algebras. Examples

- Right-angled Artin monoids [Crisp-Laca]:
  Let $\Gamma = (V, E)$ be a graph with $E \subseteq V \times V$.
  $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \, \forall \, (v, w) \in E \rangle^+$.

  e.g. $\mathbb{N} \times \mathbb{N}$: $C_\lambda^*(\mathbb{N} \times \mathbb{N}) \cong C_\lambda^*(\mathbb{N}) \otimes C_\lambda^*(\mathbb{N})$.
  e.g. $\mathbb{N} * \mathbb{N}$: $C_\lambda^*(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$, where $\mathcal{T}_2$ is the Toeplitz extension of the Cuntz algebra $\mathcal{O}_2$, $0 \to \mathcal{K} \to \mathcal{T}_2 \to \mathcal{O}_2 \to 0$.

Semigroup C*-algebras. Examples

- Right-angled Artin monoids [Crisp-Laca]:
  Let $\Gamma = (V, E)$ be a graph with $E \subseteq V \times V$.
  $A_\Gamma^+ := \langle \{\sigma_v\colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \, \forall \, (v, w) \in E \rangle^+$.

  e.g. $\mathbb{N} \times \mathbb{N}$: $C_\lambda^*(\mathbb{N} \times \mathbb{N}) \cong C_\lambda^*(\mathbb{N}) \otimes C_\lambda^*(\mathbb{N})$.
  e.g. $\mathbb{N} * \mathbb{N}$: $C_\lambda^*(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$, where $\mathcal{T}_2$ is the Toeplitz extension of the Cuntz algebra $\mathcal{O}_2$, $0 \to \mathcal{K} \to \mathcal{T}_2 \to \mathcal{O}_2 \to 0$.

  What about general graph products?

Semigroup C*-algebras. Examples

- Right-angled Artin monoids [Crisp-Laca]:
  Let $\Gamma = (V, E)$ be a graph with $E \subseteq V \times V$.
  $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \, \forall \, (v, w) \in E \rangle^+$.

  e.g. $\mathbb{N} \times \mathbb{N}$: $C^*_\lambda(\mathbb{N} \times \mathbb{N}) \cong C^*_\lambda(\mathbb{N}) \otimes C^*_\lambda(\mathbb{N})$.
  e.g. $\mathbb{N} * \mathbb{N}$: $C^*_\lambda(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$, where $\mathcal{T}_2$ is the Toeplitz extension of the Cuntz algebra $\mathcal{O}_2$, $0 \to \mathcal{K} \to \mathcal{T}_2 \to \mathcal{O}_2 \to 0$.

  What about general graph products?
  What about general Artin monoids?
  e.g. Braid monoids, e.g. $B_3^+ = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle^+$

Semigroup C*-algebras. Examples

- Right-angled Artin monoids [Crisp-Laca]:
  Let $\Gamma = (V, E)$ be a graph with $E \subseteq V \times V$.
  $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \, \forall \, (v, w) \in E \rangle^+$.

  e.g. $\mathbb{N} \times \mathbb{N}$: $C_\lambda^*(\mathbb{N} \times \mathbb{N}) \cong C_\lambda^*(\mathbb{N}) \otimes C_\lambda^*(\mathbb{N})$.
  e.g. $\mathbb{N} * \mathbb{N}$: $C_\lambda^*(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$, where $\mathcal{T}_2$ is the Toeplitz extension of the Cuntz algebra $\mathcal{O}_2$, $0 \to \mathcal{K} \to \mathcal{T}_2 \to \mathcal{O}_2 \to 0$.

  What about general graph products?
  What about general Artin monoids?
  e.g. Braid monoids, e.g. $B_3^+ = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle^+$
- Baumslag-Solitar monoid [Spielberg]:
  $\left\langle a, b \mid ab^c = b^d a \right\rangle^+$ or $\left\langle a, b \mid a = b^d a b^c \right\rangle^+$
  What about general graphs of semigroups?

Semigroup C*-algebras. Examples

- Right-angled Artin monoids [Crisp-Laca]:
  Let $\Gamma = (V, E)$ be a graph with $E \subseteq V \times V$.
  $A_\Gamma^+ := \langle \{\sigma_v \colon v \in V\} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \, \forall \, (v, w) \in E \rangle^+$.

  e.g. $\mathbb{N} \times \mathbb{N}$: $C_\lambda^*(\mathbb{N} \times \mathbb{N}) \cong C_\lambda^*(\mathbb{N}) \otimes C_\lambda^*(\mathbb{N})$.
  e.g. $\mathbb{N} * \mathbb{N}$: $C_\lambda^*(\mathbb{N} * \mathbb{N}) \cong \mathcal{T}_2$, where $\mathcal{T}_2$ is the Toeplitz extension of the Cuntz algebra $\mathcal{O}_2$, $0 \to \mathcal{K} \to \mathcal{T}_2 \to \mathcal{O}_2 \to 0$.

  What about general graph products?
  What about general Artin monoids?
  e.g. Braid monoids, e.g. $B_3^+ = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle^+$
- Baumslag-Solitar monoid [Spielberg]:
  $\left\langle a, b \mid ab^c = b^d a \right\rangle^+$ or $\left\langle a, b \mid a = b^d a b^c \right\rangle^+$
  What about general graphs of semigroups?
- Thompson monoid $\langle x_0, x_1, x_2, \ldots \mid x_n x_k = x_k x_{n+1} \text{ for } k < n \rangle^+$

Semigroup C*-algebras. Examples

- $R^\times$ or $R \rtimes R^\times$ for an integral domain R.
  $R^\times = R \setminus \{0\}$, $R \rtimes R^\times = R \times R^\times$ as sets, multiplication given by $(d, c)(b, a) = (d + cb, ca)$.

Semigroup C*-algebras. Examples

- $R^\times$ or $R \rtimes R^\times$ for an integral domain R.
  $R^\times = R \setminus \{0\}$, $R \rtimes R^\times = R \times R^\times$ as sets, multiplication given by $(d,c)(b,a) = (d + cb, ca)$.
  e.g. R ring of algebraic integers in number field,
  e.g. $\mathbb{Z}[i]$ or $\mathbb{Z}[\zeta]$ (ζ: root of unity)

Semigroup C*-algebras. Examples

- $R^\times$ or $R \rtimes R^\times$ for an integral domain R.
  $R^\times = R \setminus \{0\}$, $R \rtimes R^\times = R \times R^\times$ as sets, multiplication given by $(d, c)(b, a) = (d + cb, ca)$.
  e.g. R ring of algebraic integers in number field,
  e.g. $\mathbb{Z}[i]$ or $\mathbb{Z}[\zeta]$ (ζ: root of unity)
- $P \subseteq \mathbb{Z}^n$ finitely generated.

Semigroup C*-algebras. Examples

- $R^\times$ or $R \rtimes R^\times$ for an integral domain R.
  $R^\times = R \setminus \{0\}$, $R \rtimes R^\times = R \times R^\times$ as sets, multiplication given by $(d, c)(b, a) = (d + cb, ca)$.
  e.g. R ring of algebraic integers in number field,
  e.g. $\mathbb{Z}[i]$ or $\mathbb{Z}[\zeta]$ (ζ: root of unity)
- $P \subseteq \mathbb{Z}^n$ finitely generated.
  e.g. numerical semigroups: $P \subseteq \mathbb{Z}$ finitely generated, e.g.
  $P = \mathbb{N} \setminus \{1\} = \{0, 2, 3, 4, \ldots\}$.

Semigroup C*-algebras. Examples

- $R^\times$ or $R \rtimes R^\times$ for an integral domain R.
  $R^\times = R \setminus \{0\}$, $R \rtimes R^\times = R \times R^\times$ as sets, multiplication given by $(d, c)(b, a) = (d + cb, ca)$.
  e.g. R ring of algebraic integers in number field,
  e.g. $\mathbb{Z}[i]$ or $\mathbb{Z}[\zeta]$ (ζ: root of unity)

- $P \subseteq \mathbb{Z}^n$ finitely generated.
  e.g. numerical semigroups: $P \subseteq \mathbb{Z}$ finitely generated, e.g.
  $P = \mathbb{N} \setminus \{1\} = \{0, 2, 3, 4, \ldots\}$.

- $P \nsubseteq G$, e.g. Zappa-Szép products, e.g. from self-similar groups [Nekrashevych, Brownlowe-Ramagge-Robertson-Whittaker]

Inverse semigroups

To construct full semigroup C*-algebras, we first need to isolate a class of isometric representations. We need the notion of inverse semigroups.

Inverse semigroups

To construct full semigroup C*-algebras, we first need to isolate a class of isometric representations. We need the notion of inverse semigroups.

Definition

An inverse semigroup is a semigroup S with the property that for every $x \in S$, there is a unique $y \in S$ with $x = xyx$ and $y = yxy$. We write $y = x^{-1}$ (or $y = x^*$).

Inverse semigroups

To construct full semigroup C*-algebras, we first need to isolate a class of isometric representations. We need the notion of inverse semigroups.

Definition

An inverse semigroup is a semigroup S with the property that for every $x \in S$, there is a unique $y \in S$ with $x = xyx$ and $y = yxy$. We write $y = x^{-1}$ (or $y = x^*$).

Every inverse semigroup can be realized ...

- ... as partial bijections on a fixed set, where multiplication is given by composition (wherever it makes sense);

Inverse semigroups

To construct full semigroup C*-algebras, we first need to isolate a class of isometric representations. We need the notion of inverse semigroups.

Definition

An inverse semigroup is a semigroup S with the property that for every $x \in S$, there is a unique $y \in S$ with $x = xyx$ and $y = yxy$. We write $y = x^{-1}$ (or $y = x^*$).

Every inverse semigroup can be realized ...

- ... as partial bijections on a fixed set, where multiplication is given by composition (wherever it makes sense);
- ... as partial isometries on a Hilbert space whose source and range projections commute, where multiplication is composition.

The left inverse hull

Let P be a left cancellative semigroup with identity e.

The left inverse hull

Let P be a left cancellative semigroup with identity e.

Definition

The left inverse hull $I_l(P)$ of P is the inverse semigroup generated by the partial bijections $P \to pP$, $x \mapsto px$.

The left inverse hull

Let P be a left cancellative semigroup with identity e.

Definition

The left inverse hull $I_l(P)$ of P is the inverse semigroup generated by the partial bijections $P \to pP$, $x \mapsto px$.

So $I_l(P)$ is the smallest semigroup of partial bijections on P which is closed under inverses and contains $\{P \to pP, x \mapsto px \colon p \in P\}$.

The left inverse hull

Let P be a left cancellative semigroup with identity e.

Definition

The left inverse hull $I_l(P)$ of P is the inverse semigroup generated by the partial bijections $P \to pP$, $x \mapsto px$.

So $I_l(P)$ is the smallest semigroup of partial bijections on P which is closed under inverses and contains $\{P \to pP, x \mapsto px \colon p \in P\}$.

Alternatively, $I_l(P)$ is the smallest semigroup of partial isometries on $\ell^2 P$ which is closed under adjoints and contains $\{V_p \colon p \in P\}$.

Idempotents in inverse semigroups

Let S be an inverse semigroup.

Definition

$E := \{x^{-1}x \colon x \in S\} = \{xx^{-1} \colon x \in S\} = \{e \in S \colon e = e^2\}$ is the semilattice of idempotents of S. For $e, f \in E$, set $e \leq f$ if $e = ef$.

Idempotents in inverse semigroups

Let S be an inverse semigroup.

Definition

$E := \{x^{-1}x\colon x \in S\} = \{xx^{-1}\colon x \in S\} = \{e \in S\colon e = e^2\}$ is the semilattice of idempotents of S. For $e, f \in E$, set $e \leq f$ if $e = ef$.

For $I_l(P)$, $E = \{q_n^{-1} p_n \cdots q_1^{-1} p_1 P\colon q_i, p_i \in P\} =: \mathcal{J}_P$.
$\mathcal{J}_P$ always contains all principal right ideals pP, $p \in P$, and if we have $\mathcal{J}_P \setminus \{\emptyset\} = \{pP\colon p \in P\}$, then we say that P is right LCM.

Idempotents in inverse semigroups

Let S be an inverse semigroup.

Definition

$E := \left\{x^{-1}x\colon x \in S\right\} = \left\{xx^{-1}\colon x \in S\right\} = \left\{e \in S\colon e = e^2\right\}$ is the semilattice of idempotents of S. For $e, f \in E$, set $e \leq f$ if $e = ef$.

For $I_l(P)$, $E = \left\{q_n^{-1}p_n \cdots q_1^{-1}p_1 P\colon q_i, p_i \in P\right\} =: \mathcal{J}_P$.
$\mathcal{J}_P$ always contains all principal right ideals pP, $p \in P$, and if we have $\mathcal{J}_P \setminus \{\emptyset\} = \{pP\colon p \in P\}$, then we say that P is right LCM.

$\widehat{E} = \{\chi : E \to \{0, 1\}$ non-zero semigroup homomorphism$\}$, with the topology of pointwise convergence, is the spectrum of E.
$\widehat{E}$ can be identified with the space of filters via $\chi \leftrightarrow \chi^{-1}(1)$.

Idempotents in inverse semigroups

Let S be an inverse semigroup.

Definition

$E := \{x^{-1}x\colon x \in S\} = \{xx^{-1}\colon x \in S\} = \{e \in S\colon e = e^2\}$ is the semilattice of idempotents of S. For $e, f \in E$, set $e \leq f$ if $e = ef$.

For $I_l(P)$, $E = \{q_n^{-1}p_n \cdots q_1^{-1}p_1 P\colon q_i, p_i \in P\} =: \mathcal{J}_P$.
$\mathcal{J}_P$ always contains all principal right ideals pP, $p \in P$, and if we have $\mathcal{J}_P \setminus \{\emptyset\} = \{pP\colon p \in P\}$, then we say that P is right LCM.

$\widehat{E} = \{\chi : E \to \{0,1\}$ non-zero semigroup homomorphism$\}$, with the topology of pointwise convergence, is the spectrum of E.
$\widehat{E}$ can be identified with the space of filters via $\chi \leftrightarrow \chi^{-1}(1)$.

Exercise: Work out $\mathcal{J}_P$ and $\widehat{\mathcal{J}_P}$ for $P = \mathbb{N} * \mathbb{N}$.

C*-algebras of inverse semigroups

C*-algebras of inverse semigroups

Let S be an inverse semigroup. For $s \in S$, define $\lambda_s : \ell^2 S \to \ell^2 S$ by $\delta_x \mapsto \delta_{sx}$ if $s^{-1}s \geq xx^{-1}$ and $\delta_x \mapsto 0$ otherwise.

C*-algebras of inverse semigroups

Let S be an inverse semigroup. For $s \in S$, define $\lambda_s : \ell^2 S \to \ell^2 S$ by $\delta_x \mapsto \delta_{sx}$ if $s^{-1}s \geq xx^{-1}$ and $\delta_x \mapsto 0$ otherwise.

Definition

$C^*_\lambda(S) := C^*(\{\lambda_s \colon s \in S\}) \subseteq \mathcal{L}(\ell^2 S)$.

C*-algebras of inverse semigroups

Let S be an inverse semigroup. For $s \in S$, define $\lambda_s : \ell^2 S \to \ell^2 S$ by $\delta_x \mapsto \delta_{sx}$ if $s^{-1}s \geq xx^{-1}$ and $\delta_x \mapsto 0$ otherwise.

Definition

$C^*_\lambda(S) := C^*(\{\lambda_s \colon s \in S\}) \subseteq \mathcal{L}(\ell^2 S)$.

(If $0 \in S$, replace $\ell^2 S$ by $\ell^2 S^\times$, where $S^\times = S \setminus \{0\}$.)

C*-algebras of inverse semigroups

Let S be an inverse semigroup. For $s \in S$, define $\lambda_s : \ell^2 S \to \ell^2 S$ by $\delta_x \mapsto \delta_{sx}$ if $s^{-1}s \geq xx^{-1}$ and $\delta_x \mapsto 0$ otherwise.

Definition

$$C^*_\lambda(S) := C^*(\{\lambda_s \colon s \in S\}) \subseteq \mathcal{L}(\ell^2 S).$$

(If $0 \in S$, replace $\ell^2 S$ by $\ell^2 S^\times$, where $S^\times = S \setminus \{0\}$.)

The full inverse semigroup C*-algebra is given by

Definition

$$C^*(S) := C^*\left(\{v_s\}_{s \in S} \,\middle|\, v_s v_t = v_{st},\ v_s^* = v_{s^{-1}}\ (\textit{and } v_0 = 0 \textit{ if } 0 \in S)\right).$$

C*-algebras of inverse semigroups

Let S be an inverse semigroup. For $s \in S$, define $\lambda_s : \ell^2 S \to \ell^2 S$ by $\delta_x \mapsto \delta_{sx}$ if $s^{-1}s \geq xx^{-1}$ and $\delta_x \mapsto 0$ otherwise.

Definition

$$C^*_\lambda(S) := C^*(\{\lambda_s \colon s \in S\}) \subseteq \mathcal{L}(\ell^2 S).$$

(If $0 \in S$, replace $\ell^2 S$ by $\ell^2 S^\times$, where $S^\times = S \setminus \{0\}$.)

The full inverse semigroup C*-algebra is given by

Definition

$$C^*(S) := C^* \left(\{v_s\}_{s \in S} \;\middle|\; v_s v_t = v_{st},\, v_s^* = v_{s^{-1}} \,(\textit{and } v_0 = 0 \textit{ if } 0 \in S) \right).$$

By construction, there is a homomorphism $C^*(S) \to C^*_\lambda(S)$, $v_s \mapsto \lambda_s$.

Full semigroup C*-algebras

Full semigroup C*-algebras

Let P be a left cancellative semigroup. Its full semigroup C*-algebra is

Definition

$C^*(P) := C^*(I_l(P))$.

Full semigroup C*-algebras

Let P be a left cancellative semigroup. Its full semigroup C*-algebra is

Definition

$C^*(P) := C^*(I_l(P))$.

Note: We say that $0 \in I_l(P)$ if $\emptyset \to \emptyset$ lies in $I_l(P)$, i.e., $\emptyset \in \mathcal{J}_P$.

Full semigroup C*-algebras

Let P be a left cancellative semigroup. Its full semigroup C*-algebra is

Definition

$C^*(P) := C^*(I_l(P))$.

Note: We say that $0 \in I_l(P)$ if $\emptyset \to \emptyset$ lies in $I_l(P)$, i.e., $\emptyset \in \mathcal{J}_P$.

By universal property, there is a canonical homomorphism $C^*(P) \to C^*_\lambda(P)$ called the left regular representation.

Full semigroup C*-algebras

Let P be a left cancellative semigroup. Its full semigroup C*-algebra is

Definition

$C^*(P) := C^*(I_l(P))$.

Note: We say that $0 \in I_l(P)$ if $\emptyset \to \emptyset$ lies in $I_l(P)$, i.e., $\emptyset \in \mathcal{J}_P$.

By universal property, there is a canonical homomorphism $C^*(P) \to C^*_\lambda(P)$ called the left regular representation.

Examples:

- $C^*(\mathbb{N}) \cong C^*(v \mid v^* v = 1)$.

Full semigroup C*-algebras

Let P be a left cancellative semigroup. Its full semigroup C*-algebra is

Definition

$C^*(P) := C^*(I_l(P))$.

Note: We say that $0 \in I_l(P)$ if $\emptyset \to \emptyset$ lies in $I_l(P)$, i.e., $\emptyset \in \mathcal{J}_P$.

By universal property, there is a canonical homomorphism $C^*(P) \to C^*_\lambda(P)$ called the left regular representation.

Examples:

- $C^*(\mathbb{N}) \cong C^*(v \mid v^* v = 1)$.
- $C^*(\mathbb{N} \times \mathbb{N}) \cong C^*(v_a, v_b \mid v_a^* v_a = 1 = v_b^* v_b, v_a v_b = v_b v_a, v_a^* v_b = v_b v_a^*)$.

Full semigroup C*-algebras

Let P be a left cancellative semigroup. Its full semigroup C*-algebra is

Definition

$C^*(P) := C^*(I_l(P))$.

Note: We say that $0 \in I_l(P)$ if $\emptyset \to \emptyset$ lies in $I_l(P)$, i.e., $\emptyset \in \mathcal{J}_P$.

By universal property, there is a canonical homomorphism $C^*(P) \to C^*_\lambda(P)$ called the left regular representation.

Examples:

- $C^*(\mathbb{N}) \cong C^*(v \mid v^* v = 1)$.
- $C^*(\mathbb{N} \times \mathbb{N}) \cong C^*(v_a, v_b \mid v_a^* v_a = 1 = v_b^* v_b, v_a v_b = v_b v_a, v_a^* v_b = v_b v_a^*)$.
- $C^*(\mathbb{N} * \mathbb{N}) \cong C^*(v_a, v_b \mid v_a^* v_a = 1 = v_b^* v_b,\ v_a v_a^* v_b v_b^* = 0)$.

Full semigroup C*-algebras

Full semigroup C*-algebras

If P is right reversible, i.e., $Pp \cap Pq \neq \emptyset$ for all $p, q \in P$, or if P is right LCM, then

$$C^*(P) \cong C^* \left(\begin{array}{c|c} \{e_X \colon X \in \mathcal{J}_P\} & e_X^* = e_X = e_X^2;\ v_p^* v_p = 1; \\ \cup \{v_p \colon p \in P\} & e_\emptyset = 0 \text{ if } \emptyset \in \mathcal{J}_P,\ e_P = 1, \\ & e_{X \cap Y} = e_X \cdot e_Y; \\ & v_{pq} = v_p v_q; \\ & v_p e_X v_p^* = e_{pX} \end{array} \right)$$

Full semigroup C*-algebras

If P is right reversible, i.e., $Pp \cap Pq \neq \emptyset$ for all $p, q \in P$,
or if P is right LCM, then

$$C^*(P) \cong C^* \left(\begin{array}{c} \{e_X \colon X \in \mathcal{J}_P\} \\ \cup \{v_p \colon p \in P\} \end{array} \middle| \begin{array}{c} e_X^* = e_X = e_X^2;\ v_p^* v_p = 1; \\ e_\emptyset = 0 \text{ if } \emptyset \in \mathcal{J}_P,\ e_P = 1, \\ e_{X \cap Y} = e_X \cdot e_Y; \\ v_{pq} = v_p v_q; \\ v_p e_X v_p^* = e_{pX} \end{array} \right)$$

So $C^*(P) \cong D(P) \rtimes_\alpha P$, where $D(P) = C^*(\{e_X \colon X \in \mathcal{J}_P\}) \subseteq C^*(P)$,
and $\alpha_p : D(P) \to D(P),\ e_X \mapsto e_{pX}$.