Semigroup C*-algebras. The Toeplitz condition and K-theory

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Theorem (L)

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- A right-angled Artin monoid embeds into its right-angled Artin group, and this embedding is Toeplitz. More generally, the Toeplitz condition is preserved under graph products.

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In general, if P embeds into a group, then there is a universal embedding $P \hookrightarrow G_{\mathrm{univ}}$. It turns out that if there is a group embedding $P \hookrightarrow G$ which is Toeplitz, then $P \hookrightarrow G_{\mathrm{univ}}$ must be Toeplitz.

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- Let $P = \left\langle a, b \mid a = b^d a b^c \right\rangle^+$. Then $G_{\mathrm{univ}} = \left\langle a, b \mid a = b^d a b^c \right\rangle$.

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  Hence there is no embedding $P \hookrightarrow G$ which is Toeplitz.

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Theorem (Cuntz-Echterhoff-L)

Let $P \subseteq G$. Assume that P satisfies independence, $P \subseteq G$ is Toeplitz, and that G satisfies the Baum-Connes conjecture with coefficients. Then

$$K_*(C_\lambda^*(P)) \cong \bigoplus_{[X] \in G \backslash \mathcal{J}_{P \subseteq G}^{\times}} K_*(C_\lambda^*(G_X)).$$

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$$K_*(C_\lambda^*(R \rtimes R^{\times})) \cong \bigoplus_{[\mathfrak{a}] \in Cl_K} K_*(C_\lambda^*(\mathfrak{a} \rtimes R^*)).$$

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For instance, Bernoulli shifts $\Gamma \curvearrowright \{0, \ldots, N\}^\Gamma$ satisfy independence.