THE UNIVERSITY OF
NEWCASTLE
AUSTRALIA

Varieties of Smoothness and Rotundity

J.R.Giles
Faculty of Science and Information Technology
School of Mathematical and Physical Sciences
August 2015

Banach space X, dual X^*

$$\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$$

Natural embedding of X into X^{**}, $x \mapsto \hat{x}$

$$\hat{x}(f) = f(x) \text{ for all } f \in X^*, \|\hat{x}\| = \|x\|$$

onto is reflexive.

$$\text{Ball } B(X) \equiv \{x \in X : \|x\| \leq 1\}$$

$$\text{Sphere } S(X) \equiv \{x \in X : \|x\| = 1\}$$

$$x \in S(X), \;\; \lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

exists all $y \in S(X)$, $Gateaux$ dble at x,

uniformly all $y \in S(X)$, $Fréchet$ dble at x.

Theme: To determine those Banach spaces which enjoy particular Euclidean space properties.

Euclidean space - continuous convex functions are dble on a dense G_δ.

1933 Mazur: separable Banach space has G dble on a dense G_δ.

1968 Asplund: Banach space separable dual has Fr. dble on a dense G_δ.

Activity of 1970s:

characterisation *Asplund spaces* - where cts convex Fr. dble on a dense G_δ (David Gregory 1979)

X with Fr. dble norm on $S(X)$ is Asplund.

Weak Asplund spaces - where cts convex G dble on a dense G_δ.

X with G dble norm on $S(X)$ is weak Asplund (P-P-N 1990).

Rotundity conditions

• $X(UR)$ for all $x, y \in S(X)$, given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\|x - y\| < \epsilon \text{ when } \|x + y\| > 2 - \delta$$

(X reflexive, Ringrose 1958).

• $X(WUR)$ for all $x, y \in S(X)$, given $\epsilon > 0$ there exists $\delta(\epsilon, f) > 0$ such that

$$|f(x - y)| < \epsilon \text{ when } \|x + y\| > 2 - \delta \text{ for each } f \in S(X^*)$$

(X Asplund, Hajek 1996).

• $X^*(W^*UR)$ for all $f, g \in S(X^*)$, given $\epsilon > 0$ there exists $\delta(\epsilon, z) > 0$ such that

$$|(f - g)(z))| < \epsilon \text{ when } \|f + g\| > 2 - \delta \text{ for each } z \in S(X)$$

Related Smoothness Conditions

- $X(UF)$

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} = f_x(y) \text{ uniformly all } x, y \in S(X).$$

$$X(UF) \Leftrightarrow X^*(UR)$$

- $X(UG)$

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} = f_x(y) \text{ uniformly all } x \in S(X).$$

(Zajicek 1983).

$$X(WUR) \Leftrightarrow X^*(UG) \Leftrightarrow X^{**}(W^*UR)$$

- $X(VS)$

$$\lim_{\lambda \to 0} \frac{\|\hat{x} + \lambda F\| - \|\hat{x}\|}{\lambda} = \hat{f}_x(F)$$

(X smooth at $x \in S(X)$ and X^{**} smooth at $\hat{x} \in S(\hat{X})$.)

($X(VS)$ Asplund, $X^*(VS)$ on $S(X^*)$ reflexive.)

Continuity Characterisations of Differentiability

$$x \in S(X), D(x) \equiv \{f \in S(X^*) : f(x) = 1\},$$

$x \mapsto D(x)$ is w^* uscts (i)

support mapping $x \mapsto f_x$ where $f_x \in D(x)$.

subgradient inequality for $x \in S(X)$

$$f_x(y) \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(y) \text{ all } y \in S(X)$$

for $\delta > 0$ and reverse for $\delta < 0$ (ii)

(i) and (ii) $\Rightarrow$ norm is G dble at x iff $D(x)$ is singleton.

(1) $x \mapsto f_x(y)$ for each $y \in S(X)$ cts at $x \in S(X) \Leftrightarrow G$ dble norm at x.

(2) $x \mapsto f_x$ cts at $x \in S(X)$ $\Leftrightarrow$Fr dble norm at x.

(3) $x \mapsto f_x$ uniformly cts on $S(X) \Leftrightarrow UF$ dble norm on $S(X)$.

(4) $x \mapsto f_x(y)$ for each $y \in S(X)$ uniformly cts on $S(X) \Leftrightarrow UG$ dble norm on $S(X)$.

(5) $x \mapsto f_x(F)$ for each $F \in S(X^{**})$ cts at $x \in S(X) \Leftrightarrow VS$ norm at x.

Problem (Andrew Yorke, PhD 1977)

$X^*(WUR)$

X not necessarily reflexive, (Hajek, 1996).

$X^*(WUR)$ iff for each $F \in S(X^{**})$

$$\lim_{\lambda \to 0} \frac{\|\hat{x} + \lambda F\| - \|\hat{x}\|}{\lambda} = \hat{f}_x(F) \text{ uniformly all } x \in S(X).$$

iff for each $F \in S(X^{**})$, $\hat{x} \mapsto \hat{f}_x(F)$ uniformly cts on $S(X)$

X Asplund and X^* Asplund