Metric Diophantine approximation: solubility of inhomogeneous wave equation

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August 23, 2015

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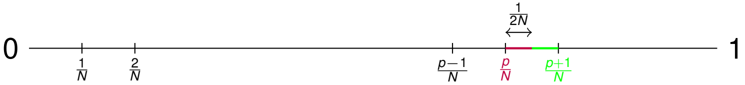
CARMA

How well can a real number be approximated by rationals?

- Qualitatively: The set of rational numbers is dense, therefore for any real number x we can construct a sequence of rational numbers r_n such that $r_n \to x$ as $n \to \infty$.
- Quantitatively: What happens if the denominators of rational numbers are equal to some integer value N?



$$\left| x - \frac{p}{q} \right| \leq \frac{1}{2N}$$

- What happens if the denominators are bounded by some value N?

If the denominators are bounded by some value N?

Theorem (Dirichlet 1842)

For any real number α and any positive integer N, there exists a rational p/q with positive denominator $q \leq N$, such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN}.$$

Since $q \leq N$, it immediately follows that

Corollary

For any irrational $\alpha \in \mathbb{R}$ there exists infinitely many rationals p/q such that

$$\left|\alpha - \frac{p}{q}\right| < q^{-2}. \tag{1}$$

Is it possible to do better?

Theorem (Hurwitz 1891)

For any irrational real number α there exist infinitely many integers p and $q > 0$ such that

$$|\alpha - p/q| \leq 1/\sqrt{5}q^2 .$$

For $\alpha = \frac{\sqrt{5}+1}{2}$, $\sqrt{5}$ in the above inequality is best possible.

A real number α is said to be badly approximable if there exists a constant $c = c(\alpha) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}, \quad \text{for all integers } p \text{ and } q > 0$$

All quadratic irrationals are badly approximable

$$W(\tau) := \{\alpha \in \mathbb{R} : \left|\alpha - \frac{p}{q}\right| \leq q^{-\tau} \text{ for i.m.} (p, q) \in \mathbb{Z} \times \mathbb{N}\}.$$

How big are the sets $W(\tau)$ for $\tau > 2$, Bad and $\mathcal{L}$?

Khintchine's Theorem

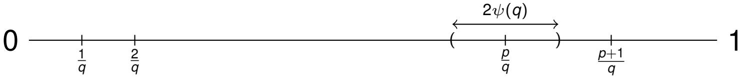
An *approximating function* is a function $\psi : \mathbb{N} \to \mathbb{R}^+$ such that $\psi(r) \to 0$ as $r \to \infty$.

- $W(\psi) := \{\alpha \in \mathbb{I} : |\alpha - p/q| < \psi(q) \text{ for i.m. } (p, q) \in \mathbb{Z} \times \mathbb{N}\}$.

Theorem (Khintchine(24, 25))

Let ψ be an approximating function. Then

$$|W(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q\psi(q) < \infty \\ 1 & \text{if } \sum_{q=1}^{\infty} q\psi(q) = \infty \quad \psi \textit{ is decreasing}. \end{cases}$$



Open Problems

- *Duffin-Schaeffer* Conjecture (1941). *For any function* $\psi : \mathbb{N} \mapsto \mathbb{R}^+$

$$|W(\psi) \cap \mathbb{I}| = 1 \quad \textit{if} \quad \sum_{q=1}^{\infty} \varphi(q)\psi(q) \; = \; \infty.$$

- *Littlewood's* Conjecture (1930). For any real α and β and any $\epsilon > 0$ there are infinitely many positive integers q such that

$$q\|q\alpha\|\|q\beta\| < \epsilon.$$

Theorem (Einsiedler–Katok–Lindenstrauss 2006)

The set of exceptions to the Littlewood's conjecture has zero Hausdorff dimension.

Open Problems

Baker–Schmidt Problem (1970)

For any non-degenerate submanifold $\mathcal{M}$ of $\mathbb{R}^n$,

$$\dim(W(\tau) \cap \mathcal{M}) = \frac{n+1}{\tau+1} + \dim \mathcal{M} - 1$$

- Klienbock–Margulis (1996). Almost all points on $\mathcal{M}$ are not very well approximable ($\mathcal{M}$ is extremal).

Generalised Baker–Schmidt Problem (GBSP)

Determine Hausdorff measure for $W(\psi) \cap \mathcal{M}$, especially the convergent case.

- Beresnevich–Dickinson–Velani (2006). Divergent case of GBSP holds.
- *Hussain* (2015). GBSP holds for a parabola.
- *Huang, J. J.* (To appear, CRELLE). GBSP holds for any non-degenerate planar curve.
- *Badziahin–Harrap–Hussain* (preprint). Inhomogeneous GBSP holds for any non-degenerate planar curve.

A Diophantine Problem

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be periodic in each of its variables x_1, x_2 and t, with periods α, β and γ respectively. Assume also that f is a smooth function of each variable. The inhomogeneous wave equation is given by the PDE

$$\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} - \frac{\partial^2 u(\mathbf{x},t)}{\partial x_1^2} - \frac{\partial^2 u(\mathbf{x},t)}{\partial x_2^2} = f(\mathbf{x},t), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2,\ t \in \mathbb{R}, \qquad (2)$$

where u is a smooth, periodic solution with the same periods as f.
The smoothness conditions on f are equivalent to the property that it has a Fourier series expansion of the form

$$f(\mathbf{x},t) = \sum_{(a,b,c)\in\mathbb{Z}^3} f_{a,b,c} \exp\left(2\pi i \left[\frac{a}{\alpha}x_1 + \frac{b}{\beta}x_2 + \frac{c}{\gamma}t\right]\right),$$

in which the coefficients $f_{a,b,c}$ decay faster than the reciprocal of any polynomial in (a, b, c) as $\max\{|a|, |b|, |c|\}$ tends to infinity. Any smooth solution u to (2) must satisfy a similar Fourier expansion.

That is $u(x,t) = \sum_{(a,b,c)\in\mathbb{Z}^3} u_{a,b,c} \exp\left(2\pi i\left[\frac{a}{\alpha}x_1 + \frac{b}{\beta}x_2 + \frac{c}{\gamma}t\right]\right)$
Substituting it into (2) and comparing coefficients, we have

$$u_{a,b,c} = \frac{\gamma^2}{4\pi^2}\frac{f_{a,b,c}}{a^2\frac{\gamma^2}{\alpha^2} + b^2\frac{\gamma^2}{\beta^2} - c^2}$$

For u to be smooth it suffices to verify that

$$\left|a^2\frac{\gamma^2}{\alpha^2} + b^2\frac{\gamma^2}{\beta^2} - c^2\right| \geq C\max\{|a|,|b|\}^{-\tau},$$

for some $C > 0, \tau > 1$ for all $(a,b,c) \in \mathbb{Z}^3$ with $a \neq 0$. That means, this condition can only fail if for all $\tau > 1$ the inequality

$$\left|a^2\frac{\gamma^2}{\alpha^2} + b^2\frac{\gamma^2}{\beta^2} - c^2\right| < \max\{|a|,|b|\}^{-\tau},$$

holds for infinitely many $(a,b,c) \in \mathbb{Z}^3$ with $a \neq 0$.

A general Diophantine problem

Consider a general differential operator

$$\frac{\partial^p}{\partial t^p} - \frac{\partial^n}{\partial x_1^n} - \frac{\partial^m}{\partial x_2^m}.$$

For any triple $(n, m, p) \in \mathbb{N}^3$ *and any approximating function* ψ *define* $W^p_{n,m}(\psi)$ *to be the set of vectors* $\mathbf{x} = (x_1, x_2) \in [0, 1)^2$ *for which the inequality*

$$|a^n x_1 + b^m x_2 - c^p| < \psi(h_{a,b})$$

holds for infinitely many $(a, b, c) \in \mathbb{N}^2 \times \mathbb{Z}_{\geq 0}$.

Here, we have assigned a natural height $h_{a,b} := \max(a^n, b^m)$ to each pair (a, b) of positive integers.

How big is $W^p_{n,m}(\psi)$?

How big is $Bad^p_{n,m}(\psi)$?

How big is $Bad^p_{n,m}(\psi)$?

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A Khintchine Type Theorem

Theorem (Harrap–Hussain–Kristensen, 2015)

Assume that either $n = m = 1$ *or* $\gcd(n, m) \geq 2$*. Then, for every approximating function* ψ *we have that*

$$\left|W^{p}_{n,m}(\psi)\right| = \begin{cases} 0, & \displaystyle\sum_{(a,b)\in\mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} < \infty. \\ 1, & \displaystyle\sum_{(a,b)\in\mathbb{N}^2} \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} = \infty. \end{cases}$$

A Jarnik Type Theorem

Theorem (Harrap–Hussain–Kristensen, 2015)

Let ψ be an approximating function and assume that either $n = m = p = 1$ or $\gcd(n, m) \geq 2$. Let f be a dimension function such that $r^{-2}f(r)$ is monotonic and for notational convenience let $g : r \to r^{-1}f(r)$ be another dimension function. Then,

$$\mathcal{H}^f\left(W^p_{n,m}(\psi)\right) = \begin{cases} 0, & \displaystyle\sum_{(a,b)\in\mathbb{N}^2} g\left(\frac{\psi(h_{a,b})}{h_{a,b}}\right) h_{a,b}^{1/p} < \infty. \\ \mathcal{H}^f\left([0,1)^2\right), & \displaystyle\sum_{(a,b)\in\mathbb{N}^2} g\left(\frac{\psi(h_{a,b})}{h_{a,b}}\right) h_{a,b}^{1/p} = \infty. \end{cases}$$

Corollary

Assume that either $n = m = p = 1$ *or* $\gcd(n, m) \geq 2$. *Let* $\tau > 1$, *then*

$$\dim_H\left(W^p_{n,m}\left(\psi : r \to r^{-\tau}\right)\right) = 1 + \min\left\{1, \frac{\frac{1}{n} + \frac{1}{m} + \frac{1}{p}}{\tau + 1}\right\}.$$

It also implies for example that $\mathcal{H}^s\left(W^p_{n,m}(\psi : r \to r^{-\tau})\right) = \infty$ at the critical exponent $s = \dim_H\left(W^p_{n,m}(\psi : r \to r^{-\tau})\right)$.

Corollary

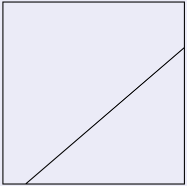
If f *is smooth and periodic in* x_1, x_2, t *with periods* α, β *and* γ *respectively then the given PDE is solvable with u smooth and periodic with the same periods whenever* $(\gamma^2/\alpha^2, \gamma^2/\beta^2)$ *does not belong to*

$$\bigcap_{\tau > 1} W^p_{n,m}\left(\psi : r \to r^{-\tau}\right),$$

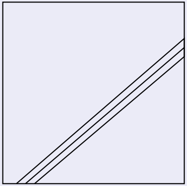
a null set of Hausdorff dimension 1.

Sketch of the proof: the convergence case

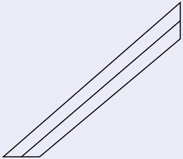
(Borel–Cantelli Lemma)



$y = (-a^n/b^m)x + c^p/b^m$

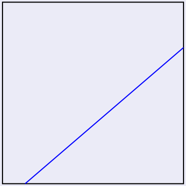


$\ell_{a,b}(c)$



$width \asymp \psi(h_{a,b})/\sqrt{a^{2n} + b^{2m}}$

(Hausdorff–Cantelli lemma)



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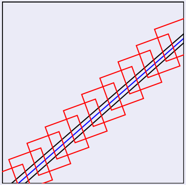
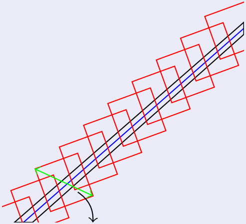




$width \asymp f\left(\frac{\psi(h_{a,b})}{h_{a,b}}\right)$

Divergence case: sketch

Lemma

Let Ω be an open subset of $\mathbb{R}^t$ and let E be a Borel subset of $\mathbb{R}^t$. If there exist strictly positive constant r_0 such that for any ball B in Ω of radius $r(B) < r_0$ we have

$$|E \cap B| \gg |B|, \tag{3}$$

where the implied constant is independent of B, then E has full measure in Ω.

Divergence case: sketch

Lemma ((Second) Borel–Cantelli)

*Let E_t be a sequence of measurable sets which are **quasi-independent on average**; that is, the sequence E_t satisfies*

$$\sum_{t=1}^{\infty} |E_t| = \infty \tag{4}$$

and there exists some strictly positive constant α for which

$$\sum_{s,t=1}^{Q} |E_s \cap E_t| \leq \frac{1}{\alpha} \left(\sum_{t=1}^{Q} |E_t| \right)^2$$

for all $Q \geq 1$. Then,

$$\left| \limsup_{t \to \infty} E_t \right| \geq \alpha.$$

Estimating the measure of $\ell_{a,b} \cap B$

$$\left|\ell_{a,b} \cap B\right| \quad \asymp \quad |B| \, \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}}. \tag{5}$$

Thus

$$\sum_{(a,b)\in\mathbb{N}^2} \left|\ell_{a,b} \cap B\right| \quad \asymp \quad \sum_{(a,b)\in\mathbb{N}^2} |B| \, \frac{\psi(h_{a,b})}{h_{a,b}^{1-1/p}} = \infty.$$

Estimating the measure of $\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B$

$$\left|\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B\right| \asymp |B| \frac{\psi(h_1)\psi(h_2)}{h_1^{1-1/p} h_2^{1-1/p}} \cdot \left(1 + \frac{1}{r h_1^{1/p} \sin\alpha}\right). \quad (6)$$

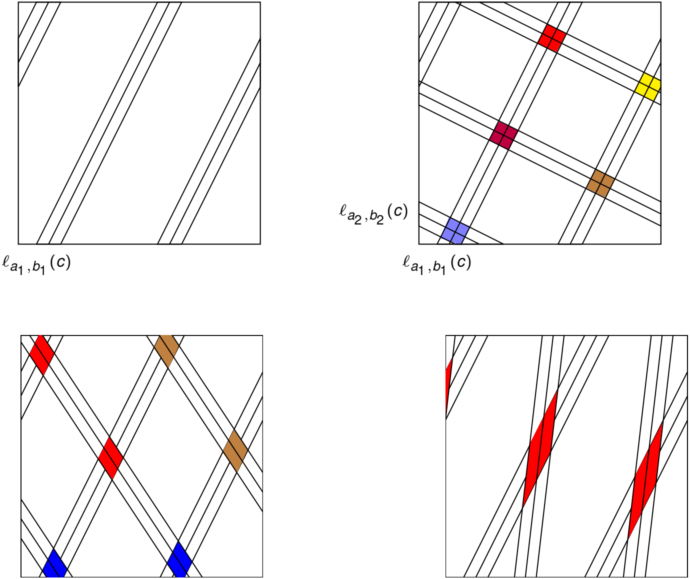
We now split our calculation into three exhaustive subcases depending on the size of the angle α.

- $$\sin\alpha \geq \frac{1}{r h_1^{1/p}}. \quad (7)$$

- $$\frac{1}{r h_1^{1/p}} \geq \sin\alpha \geq \frac{1}{r^{1+\frac{k}{M}} h_1^{\frac{k}{pN}} h_2^{\frac{1}{p}}}, \quad (8)$$

- $$\sin\alpha < r^{-(1+\frac{k}{M})} h_1^{-\frac{k}{pN}} h_2^{-\frac{1}{p}}. \quad (9)$$

Estimating the measure of $\ell_{a_1,b_1} \cap \ell_{a_2,b_2} \cap B$



Number Theory Down Under

18-19 September, 2015
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