"RECENT DEVELOPMENTS IN FIXED POINT AND BANACH SPACE THEORY."

WORKSHOP IN HONOUR OF BRAILEY SIMS, UNIVERSITY OF NEWCASTLE

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Abstract

I will discuss three different recent results in *fixed point and Banach space theory.* Aspects of each were strongly influenced by Brailey's life and career as an extremely fine person, mathematician, teacher and mentor.

Parts (1), (2) and (3) were all partially inspired by Brailey's wonderful seminar series at Kent State University in 1986, entitled: *"The Existence Question for Fixed Points of Nonexpansive Maps."* As a new Ph.D. student at KSU, meeting and interacting with Brailey during my first year was one of the highlights of my Ph.D. studies - and my life, both generally and mathematically.

Further, my interest in characterizing the fixed point property in c_0, which led in part to Item (3), was initiated by a paper of Enrique Llorens-Fuster and Brailey: *"The fixed point property in c_0"*, Canadian Mathematical Bulletin 41(4) (1998), 413-422.

1. PART (1): INTRODUCTION

In 1965, Browder [B1] proved: [♠] [For every closed, bounded, convex (non-empty) subset C of a Hilbert space $(X, \|\cdot\|)$, for all nonexpansive mappings $T \colon C \to C$

[i.e., $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$], T has a fixed fixed point in C.] Soon after, also in 1965, Browder [B2] and Göhde [G] (independently) generalized the result [♠] to uniformly convex Banach spaces $(X, \|\cdot\|)$; e.g., $X = L^p$, $1 < p < \infty$, with its usual norm $\|\cdot\|_p$.
 Later in 1965, Kirk [K] further generalized [♠] to all reflexive Banach spaces X with normal structure: those spaces such that all non-trivial closed, bounded, convex sets C have a smaller radius than diameter.

Spaces $(X, \|\cdot\|)$ with the property of Browder [♠] became known as spaces with the "fixed point property for nonexpansive mappings" (FPP (n.e.)).

Concerning Kirk's theorem, we may ask what further generalizations are possible? After 50 years, it remains an open question as to whether or not every reflexive Banach space $(X, \|\cdot\|)$ has the fixed point property for nonexpansive maps.

Recently (in 2009), Domínguez Benavides [5] proved the following intriguing result: [Given a reflexive Banach space $(X, \|\cdot\|)$, there exists an equivalent norm $\|\cdot\|_\sim$ on X such that $(X, \|\cdot\|_\sim)$ has the fixed point property for nonexpansive mappings]. This improves a theorem of van Dulst [8] for separable reflexive Banach spaces.

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In contrast to this result, the non-reflexive Banach space $(\ell^1, \|\cdot\|_1)$, the space of all absolutely summable sequences, with the absolute sum norm $\|\cdot\|_1$, fails the fixed point property for nonexpansive mappings. E.g., let $C := \{(t_n)_{n \in \mathbb{N}} : \text{ each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1\}$. This is a closed, bounded, convex subset of ℓ^1. Let $T \colon C \to C$ be the right shift map on C; i.e.,

$$T(t_1, t_2, t_3, \dots) := (0, t_1, t_2, t_3, \dots) .$$

T is clearly $\|\cdot\|_1$-nonexpansive (being an isometry) and fixed point free on C.

Recently (in 2008), in another significant development, Lin [13] provided the first example of a non-reflexive Banach space $(X, \| \cdot \|)$ with the fixed point property for nonexpansive mappings. Lin verified this fact for $(\ell^1, \| \cdot \|_1)$ with the equivalent norm $\|| \cdot \||$ given by

$$\||x\|| = \sup_{k \in \mathbb{N}} \frac{8^k}{1 + 8^k} \sum_{n=k}^{\infty} |x_n|, \text{ for all } x = (x_n)_{n \in \mathbb{N}} \in \ell^1 .$$

What about $(c_0, \|\cdot\|_\infty)$, the Banach space of real-valued sequences that converge to zero, with the absolute supremum norm $\|\cdot\|_\infty$? This is another non-reflexive Banach space of great importance in Banach space theory. It also fails the fixed point property for nonexpansive mappings. E.g., let

$$C := \{(t_n)_{n \in \mathbb{N}} : \text{ each } t_n \geq 0, \ 1 = t_1 \geq t_2 \geq \cdots \geq t_n \geq t_{n+1}$$
$$\longrightarrow 0, \text{ as } n \to \infty\} \ .$$

Let $U \colon C \to C$ be the natural right shift map.

$$U(t_1, t_2, t_3, \dots) := (1, t_1, t_2, t_3, \dots) \ .$$

Then U is a $\|\cdot\|_\infty$-nonexpansive (isometric, actually) map with no fixed points in the closed bounded convex set C.

It is natural to ask whether there is a c_0-analogue of Lin's theorem about ℓ^1. It remains an open question as to whether or not there exists an equivalent norm $\|\cdot\|_\sim$ on $(c_0, \|\cdot\|_\infty)$ such that $(c_0, \|\cdot\|_\sim)$ has the fixed point property for nonexpansive mappings. However, if we weaken the nonexpansive condition to "asymptotically nonexpansive", then the answer is "no". In 2000, Dowling, Lennard and Turett [7] showed that for every equivalent renorming $\|\|\cdot\|\|$ of $(c_0, \|\cdot\|_\infty)$, there exists a closed, bounded, convex set C and an asymptotically nonexpansive mapping $T\colon C \to C$ [i.e., there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $[1, \infty)$ such that $k_n \xrightarrow[n]{} 1$, and for all $n \in \mathbb{N}$, for all $x, y \in C$, $\|\|T^n x - T^n y\|\| \leq k_n \|\|x - y\|\|]$ such that T has no fixed point.

In contrast to this, note that in 1972, Goebel and Kirk [9] proved that for all uniformly convex spaces $(X, \|\cdot\|)$, for every closed, bounded, convex set $C \subseteq X$, for all eventually asymptotically nonexpansive maps $T \colon C \to C$, T has a fixed point in C.

In a recent paper of Lennard and Nezir [12] (Nonlinear Analysis 95 (2014), 414-420), using the above-described theorem of Domínguez Benavides and the Strong James' Distortion Theorems, we proved that if a Banach space is a Banach lattice, or has an unconditional basis, or is a symmetrically normed ideal of operators on an infinite-dimensional Hilbert space, then it is reflexive if and only if it has an equivalent norm that has the fixed point property for *cascading nonexpansive mappings* (see Definition 2.1).

This new class of mappings strictly includes nonexpansive mappings. *Cascading nonexpansive mappings* are analogous to *asymptotically nonexpansive mappings*, but examples show that neither of these two classes of mappings contain the other. One of these examples is due to Łukasz Piasecki. He invented an asymptotically nonexpansive map that is not cascading nonexpansive. This example will be described in the paper currently being prepared by Lennard, Nezir and Piasecki [LNP].

Note that via Lin's example (described above) of an equivalent renorming $\|\|\cdot\|\|$ of $(\ell^1, \|\cdot\|_1)$ such that $(\ell^1, \|\|\cdot\|\|)$ has the fixed point property for nonexpansive

mappings, in our theorem one cannot replace "cascading nonexpansive mappings" by "nonexpansive mappings".

2. Reflexivity iff the perturbed cascading nonexpansive FPP in Banach lattices

Let C be a closed bounded convex subset of a Banach space $(X, \|\cdot\|)$. Let $T : C \longrightarrow C$ be a mapping. Let $C_0 := C$ and

$$C_1 := \overline{\mathrm{co}}\big(T(C)\big) \subseteq C \ .$$

Clearly C_1 is a closed bounded convex set in C. Let $x \in C_1$. Then

$$Tx \in T(C_1) \subseteq T(C) \subseteq \overline{\mathrm{co}}\,(T(C)) = C_1 \ .$$

So, T maps C_1 into C_1. Inductively, for all $n \in \mathbb{N}$ we define

$$C_n := \overline{\mathrm{co}}\big(T(C_{n-1})\big) \subseteq C_{n-1} \ .$$

Similarly to above, it follows that T maps C_n into C_n.

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Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space and C be a closed bounded convex subset of X. Let $T : C \longrightarrow C$ be a mapping and $(C_n)_{n \in \mathbb{N}}$ be defined as above. We say that T is *cascading nonexpansive* if there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $[1, \infty)$ such that $\lambda_n \xrightarrow[n]{} 1$, and for all $n \in \mathbb{N}$, for all $x, y \in C_n$,

$$\|Tx - Ty\| \leq \lambda_n \|x - y\|.$$

Note that every cascading nonexpansive mapping is norm-to-norm continuous; and every nonexpansive map is cascading nonexpansive.

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Cascading nonexpansive mappings arise naturally in Banach spaces $(X, \|\cdot\|)$ that contain an isomorphic copy of ℓ^1 or c_0. Examples of such spaces are Banach spaces isomorphic to a nonreflexive Banach lattice, and nonreflexive Banach spaces with an unconditional basis. (See, for example, Lindenstrauss and Tzafriri [15] 1.c.5 and [14] 1.c.12.) Another class of such spaces are Banach spaces isomorphic to a nonreflexive symmetrically normed ideal of operators on an infinite-dimensional Hilbert space. (See Peter Dodds and Lennard [4].)

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space that contains an isomorphic copy of ℓ^1 or c_0. Then there exists a closed bounded convex set $C \subseteq X$ and an affine cascading nonexpansive mapping $T : C \longrightarrow C$ such that T is fixed point free.*

The proof uses the Strong James' Distortion Theorem for ℓ^1 and c_0 ([6], [7]).

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then there exists an equivalent norm $\|\cdot\|_\sim$ on X such that for every closed bounded convex subset C of X, for all $\|\cdot\|_\sim$-cascading nonexpansive mappings $T : C \longrightarrow C$, T has a fixed point in C.*

Proof. By Domínguez Benavides [5], there exists an equivalent norm $\|\cdot\|_\sim$ on X such that for every closed bounded convex subset E of X, for all $\|\cdot\|_\sim$-nonexpansive mappings $U : E \longrightarrow E$, U has a fixed point in E. Fix an arbitrary closed bounded convex subset C of X. Let $T : C \longrightarrow C$ be a $\|\cdot\|_\sim$-cascading nonexpansive mapping. As above, let $C_0 := C$ and $C_n := \overline{\mathrm{co}}\big(T(C_{n-1})\big)$, for all $n \in \mathbb{N}$. By hypothesis there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $[1, \infty)$ such that $\lambda_n \xrightarrow[n]{} 1$, and for all $n \in \mathbb{N}$, for all $x, y \in C_n$,
$$\|Tx - Ty\|_\sim \leq \lambda_n \|x - y\|_\sim.$$

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Now X is reflexive, and so C is weakly compact. By Zorn's Lemma there exists a closed bounded convex (non-empty) set $D \subseteq C$ such that D is a *minimal invariant set for T*. I.e., $T(D) \subseteq D$, and if F is a closed bounded convex (non-empty) subset of D with $T(F) \subseteq F$, then $F = D$. It follows that $\overline{\mathrm{co}}\big(T(D)\big) = D$. Let $D_0 := D$ and $D_n := \overline{\mathrm{co}}\big(T(D_{n-1})\big)$, for all $n \in \mathbb{N}$. Inductively, we see that for all $n \in \mathbb{N}$, $D = D_n \subseteq C_n$. Therefore, by our hypotheses on T, for all $x, y \in D$, for all $n \in \mathbb{N}$,

$$\|Tx - Ty\|_\sim \leq \lambda_n \|x - y\|_\sim \,.$$

But $\lambda_n \xrightarrow[n]{} 1$. Consequently, for all $x, y \in D$,

$$\|Tx - Ty\|_\sim \leq \|x - y\|_\sim \,;$$

i.e., T is $\|\cdot\|_\sim$-nonexpansive on D. By Domínguez Benavides [5], T has a fixed point in $D \subseteq C$. $\qquad\square$

Combining Theorem **2.2**, the remarks preceding that theorem, and Theorem **2.3**, we get the following "fixed point property" characterization of reflexivity in Banach lattices, Banach spaces with an unconditional basis, and symmetrically normed ideals of operators on an infinite-dimensional Hilbert space.

Theorem 2.4. *[Lennard and Nezir [12]] Let $(X, \|\cdot\|)$ be a Banach lattice, or a Banach space with an unconditional basis, or a symmetrically normed ideal of operators on an infinite-dimensional Hilbert space. Then the following are equivalent.*
(1) X is reflexive.
(2) There exists an equivalent norm $\|\cdot\|_\sim$ on X such that for all closed bounded convex sets $C \subseteq X$ and for all $\|\cdot\|_\sim$-cascading nonexpansive mappings $T : C \longrightarrow C$, T has a fixed point in C.

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3. The FPP in general B-spaces and mappings of cascading nonexpansive type

Definition 3.1. Let $(X, \|\cdot\|)$ be a Banach space and C be a closed bounded convex subset of X. Let $T : C \longrightarrow C$ be a mapping and $(C_n)_{n \in \mathbb{N}_0}$ be defined as above. We say that T is of *cascading nonexpansive type* if T is norm-to-norm continuous and

$$\limsup_{m \longrightarrow \infty} \sup_{x \in C_m} \limsup_{\mu \longrightarrow \infty} \sup_{z \in C_\mu} \big[\|Tx - Tz\| - \|x - z\| \big] \leq 0 \, .$$

Note that every *nonexpansive map* **is** *cascading nonexpansive,* **every** *cascading nonexpansive mapping* **is** *of cascading nonexpansive type,* **and every** *mapping of cascading nonexpansive type* **is** *norm-to-norm continuous.*

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Remark 3.2. There exists a mapping of cascading nonexpansive type that is not cascading nonexpansive.

4. REFLEXIVITY IFF THE PERTURBED FPP FOR MAPPINGS OF CASCADING NONEXPANSIVE TYPE

Theorem 4.1 (Lennard, Nezir, Piasecki, in preparation). *Let $(X, \|\cdot\|)$ be a Banach space. Then the following are equivalent.*

(1) X is reflexive.

(2) There exists an equivalent norm $\|\cdot\|^\sim$ on X such that for all closed bounded convex sets $C \subseteq X$ and for all mappings $T : C \longrightarrow C$ of $\|\cdot\|^\sim$-cascading nonexpansive type, T has a fixed point in C.

5. Part (2) of my talk

In the recent paper of Burns, Lennard and Sivek (Studia Mathematica **223(3) (2014), 275-283)**, we prove the existence of a contractive and fixed point free mapping on a weakly compact convex subset of the Banach space $L^1[0,1]$ (with its usual norm), which answers a long-standing open question. This work constitutes part of the doctoral dissertation of the third author [Siv].

In 1965 Kirk [K] proved that every nonexpansive mapping U on a weakly compact convex subset C of a Banach space X with *normal structure* has a fixed point, extending the analogous results of Browder [B1, B2] and Göhde [G] for uniformly convex spaces.

For a long time it was unknown if every nonexpansive mapping U on a weakly compact convex subset C of a Banach space X has a fixed point. In 1981 Alspach [A] settled this open question by inventing the first example of a nonexpansive mapping T on a weakly compact convex C in a Banach space X for which T is fixed point free. Alspach's mapping is an isometry, and $X = L^1[0,1]$, with its usual norm. Soon after, Sine [Si] and Schechtman [Sc] invented more of these interesting fixed point free isometries T (again on a weakly compact convex $C \subseteq X =$ an L^1-space, with its usual norm).

It is easy to check that for Alspach's mapping T, $S := (I + T)/2$ is another nonexpansive fixed point free

map on C. Moreover, S contracts the distance between some pairs of unequal points and preserves the distance between other such pairs. Further, this fact is true for S when T is Sine's map. We thank Brailey Sims for pointing out to us that this is also true for S when T is any one of Schechtman's mappings.

The question as to whether there exists a *contractive* mapping U (i.e., U contracts the distances between *all* pairs of unequal points) that is fixed point free on a weakly compact convex subset of a Banach space was still open. This question remained open until the authors recently resolved it (see Theorem 5.1 below).

We now describe this solution. First, we define the set

$$C_{1/2} = \left\{ f : [0,1] \to [0,1] \ : \ f \in L^1[0,1] \textbf{ and } \int_0^1 f = \frac{1}{2} \right\}.$$

This set is a weakly compact convex subset of the Lebesgue function space $L^1[0,1]$, with its usual norm $\|\cdot\|_1$. For the rest of Part (2) of this talk, T will stand for Alspach's map as defined in $[A]$.

Alspach's mapping T is given by: for all integrable functions $f : [0,1] \longrightarrow [0,1]$,

$$(Tf)(x) = \begin{cases} 2f(2x) \wedge 1 & , \ 0 \le x < \frac{1}{2} \\ (2f(2x-1) \vee 1) - 1 & , \ \frac{1}{2} \le x < 1. \end{cases}$$

Here, for all $\alpha, \beta \in \mathbb{R}$, $\alpha \wedge \beta := \min\{\alpha, \beta\}$ and $\alpha \vee \beta := \max\{\alpha, \beta\}$.

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Alspach's map preserves areas in the sense that $\|Tf - Tg\|_1 = \|f - g\|_1$ for all integrable functions $f, g : [0,1] \to [0,1]$. In particular $T : C_{1/2} \longrightarrow C_{1/2}$. This and other facts about Alspach's mapping were discussed in Alspach [A]; and also in, for example, Day and Lennard [DL] (where the *minimal invariant sets* of T are characterized). In the above-mentioned paper of Burns, Lennard and Sivek, we proved the following theorem.

Theorem 5.1. *The mapping*

$$R : C_{1/2} \to C_{1/2} : f \mapsto \sum_{n=0}^{\infty} \frac{T^n f}{2^{n+1}} = \left(\frac{I}{2} + \frac{T}{4} + \frac{T^2}{8} + \cdots \right)(f)$$

is contractive [i.e., $\|Rf - Rg\|_1 < \|f - g\|_1$, for all $f \neq g$ in $C_{1/2}$] and fixed point free on $C_{1/2}$.

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6. PART(3) OF MY TALK: INTRODUCTION.

Recently, T. Gallagher, C. Lennard and R. Popescu (J. Mathematical Analysis and Applications 431 (2015), 471-481) showed that there exists a non-weakly compact, closed, bounded, convex (c.b.c.) subset W of the Banach space of convergent scalar sequences $(c, \|\cdot\|_\infty)$, such that every nonexpansive mapping $T : W \longrightarrow W$ has a fixed point. This answers a question left open in the 2003 and 2004 papers of Dowling, Lennard and Turett [DLT1], [DLT2].

Consider the related Banach space of all scalar sequences that converge to zero, $(c_0, \|\cdot\|_\infty)$. In [DLT2] Theorem 1, the authors show that for a c.b.c. subset K of $(c_0, \|\cdot\|_\infty)$ to be weakly compact, it is sufficient for every nonexpansive mapping $U : K \longrightarrow K$ to have

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a fixed point. Earlier, Maurey [Maur] had shown that this condition is also necessary; and consequently it characterizes weak compactness among the c.b.c. subsets of $(c_0, \|\cdot\|_\infty)$.

In [DLT1] Theorem 4, the authors prove the following preliminary theorem: *[Every non-weakly compact, closed, bounded, convex subset K of $(c_0, \|\cdot\|_\infty)$ contains a further subset C of the same type for which there exists a nonexpansive mapping $V : C \longrightarrow C$ that is fixed point free].* In [DLT1] Corollary 7, they proved the analogous result in the space $(c, \|\cdot\|_\infty)$: *[Every non-weakly compact, closed, bounded, convex subset K of $(c, \|\cdot\|_\infty)$ contains a further subset C of the same type for which there exists a nonexpansive mapping $V : C \longrightarrow C$ that is fixed point free].* It was left open as to

whether this result could be extended to: (‡) *[On every non-weakly compact, closed, bounded, convex subset K of $(c, \|\cdot\|_\infty)$ there exists a nonexpansive mapping $U: K \longrightarrow K$ that is fixed point free].* That the analogous strengthening is true in $(c_0, \|\cdot\|_\infty)$ was established in [DLT2], as mentioned above.

The main purpose of the above-mentioned paper is to show via appropriate examples that statement (‡) is *false.* (See Theorem 8.1, and many other examples in our paper...) These are also the first examples of non-weakly compact, closed, bounded, convex subsets D of a Banach space X isomorphic to c_0, such that D has the fixed point property for nonexpansive mappings.

It is an interesting related fact, proven by Maurey [Maur] (also see Borwein and Sims [Borw-Sims]), that the

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analogue of Maurey's theorem in c_0 is true in $(c, \|\cdot\|_\infty)$: *[For all weakly compact, convex subsets E of $(c, \|\cdot\|_\infty)$, every nonexpansive mapping $T : E \longrightarrow E$ has a fixed point].*

In Section 9 we use Theorem 8.1 and the fact that $(c, \|\cdot\|_\infty)$ is isomorphic to $(c_0, \|\cdot\|_\infty)$, to define an equivalent norm $\|\cdot\|^\sim$ on c_0 for which there exist non-weakly compact c.b.c. subsets that *have* the fixed point property (FPP) for $\|\cdot\|^\sim$-nonexpansive mappings. (See Theorem 9.1.)

Finally, as we remarked above, Lin [Lin] proved that for a certain equivalent norm $\|\|\cdot\|\|$ on ℓ^1, *every c.b.c. subset C of ℓ^1 has the FPP.* Whether an analogous such equivalent norm exists on c (or equivalently, c_0) remains an open question.

7. Part (3): Preliminaries.

The Banach space of all bounded sequences $(\ell^\infty, \|\cdot\|_\infty)$ is given by

$$\ell^\infty := \left\{ x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R} \ , \ \textbf{and } \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty \right\} .$$

The Banach space of all convergent sequences $(c, \|\cdot\|_\infty)$ is defined by

$$c := \left\{ x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \textbf{ and } \lambda(x) := \lim_{n\longrightarrow\infty} x_n \textbf{ exists in } \mathbb{R} \right\} .$$

Moreover, the Banach space of all convergent to zero sequences $(c_0, \|\cdot\|_\infty)$ is defined by

$$c_0 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \ \textbf{each } x_n \in \mathbb{R} \ , \ \textbf{and } \lim_{n\longrightarrow\infty} x_n = 0 \right\} .$$

Clearly, each of the above Banach spaces contains the next as a proper closed subspace. We also define $c_0^+ := \{x \in c_0 : \text{ each } x_j \geq 0\}$.

Following Aronszajn and Panitchpakdi [AP] Definition 1, p. 410 (also see, for example, Goebel and Kirk [Goeb-Kirk] Definition 4.4, pp. 46-47), we say that a metric space (M, ρ) is *hyperconvex* if for all families $\big(B(x_a; r_a)\big)_{a \in A}$ of closed balls in M for which

$$\rho(x_a, x_b) \leq r_a + r_b \text{ , } \text{ for all } a, b \in A \text{ ,}$$

it follows that

$$\bigcap_{a \in A} B(x_a; r_a) \neq \emptyset \text{ .}$$

Examples of hyperconvex metric spaces include the real line with its usual metric, $(\mathbb{R}, d_{|\cdot|})$; $(\mathbb{R}^n, d_{\|\cdot\|_\infty})$, for

all $n \in \mathbb{N}$; and $(\ell^\infty, d_{\|\cdot\|_\infty})$. **Also, all** *closed balls* **and** *order intervals* **in these metric spaces are hyperconvex (see, for example,** [Goeb-Kirk]**, p. 47 and p. 49).**

We will use the following interesting theorem of Soardi [Soardi]**. (Also see, for example,** [Goeb-Kirk] **p. 48.)**

Theorem 7.1 (Soardi, 1979)**.** *Let (M, ρ) be a bounded, hyperconvex metric space. Then every nonexpansive mapping $T : M \longrightarrow M$ has a fixed point.*

We will also use the interesting characterization theorem below that is a corollary of Aronszajn and Panitchpakdi [AP] **Theorem 9, p. 423. (Also see, for example, Espínola and Khamsi** [Kirk-Sims] **Corollary 4.8, p. 401.) A mapping R from a metric space (M, ρ) into**

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a metric subspace S is called a *retraction of M onto S*, if R is continuous on M and $R(s) = s$, for all $s \in S$.

Theorem 7.2 (Aronszajn and Panitchpakdi, 1956). *Let (H, d) be a metric space. Then the following are equivalent.*
(1) H is hyperconvex.
(2) There exists a hyperconvex metric space (M, ρ) such that $H \subseteq M$ and $\rho|_{H \times H} = d$, and a nonexpansive retraction R of M onto H.

8. A non-weakly compact c.b.c. subset of c with the fixed point property.

Theorem 8.1. *There exists a non-weakly compact, closed, bounded, convex subset W of the Banach space $(c, \|\cdot\|_\infty)$ such that every nonexpansive mapping $T : W \longrightarrow W$ has a fixed point.*

Proof. Let us define

$$W := \{y = (y_n)_{n \in \mathbb{N}} \in \ell^\infty : 1 \geq y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0\}\,.$$

Clearly, by the monotone convergence theorem, W is a subset of c. It is easy to check that W is a closed, bounded, convex subset of $(c, \|\cdot\|_\infty)$. That W is not weakly compact may be seen by using the criterion for weak convergence of sequences in c given in Banach [Ban] Chapter IX, p. 83.

We will show that the metric space $(W, d_{\|\cdot\|_\infty})$ is *hyperconvex.* The theorem will then follow from Soardi's Theorem [Soardi] (see Theorem 7.1 above).

Let $\mathbf{0} := (0, 0, 0, \ldots, 0, \ldots)$ and $\mathbf{1} := (1, 1, 1, \ldots, 1, \ldots)$. Consider the *order interval* $L := [[\mathbf{0}, \mathbf{1}]] := \{u = (u_n)_{n \in \mathbb{N}} : 0 \leq u_n \leq 1\,,$ for all $n \in \mathbb{N}\}$. Then $(L, d_{\|\cdot\|_\infty})$ is a hyperconvex metric space. (See, for example, [Goeb-Kirk] Remark 4.1, p. 49.) Also note that W is a subset of L.

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We define the mapping $S : L \longrightarrow W$ by

$$S(y) := (y_1, \, y_1 \wedge y_2, \, y_1 \wedge y_2 \wedge y_3, \, \dots) \, , \text{ for all } y = (y_j)_{j \in \mathbb{N}} \in L \, .$$

The map S is clearly a retraction of L onto W. Moreover, it is easy to check that

$$\|S(x) - S(y)\|_\infty \leq \|x - y\|_\infty \, , \text{ for all } x, y \in L \, .$$

Thus, S is a nonexpansive retraction of the hyperconvex metric space L onto W. Hence $(W, d_{\|\cdot\|_\infty})$ is hyperconvex, and so has the the fixed point property for nonexpansive mappings. $\square$

9. An equivalent norm on c_0 s.t. some non-weakly compact, c.b.c. sets have the fixed point property.

We define the equivalent norm $\|\cdot\|^\sim$ **on** c_0 **by the formula**

$$\|\beta\|^\sim := \max_{k \in \mathbb{N}} |\beta_{k+1} + \beta_1| \ , \ \textbf{for all } \beta = (\beta_j)_{j \in \mathbb{N}} \in c_0 \ .$$

We further define the non-weakly compact, closed, bounded, convex subset K **of** c_0 **by**

$$K := \big\{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_j, \ldots) \in c_0^+ :$$
$$\alpha_1 + \alpha_2 \leq 1 \textbf{ and } \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \ldots \big\} \ .$$

Theorem 9.1. *The non-weakly compact, closed, bounded, convex subset* K *of* c_0 *has the fixed point property for* $\|\cdot\|^\sim$*-nonexpansive mappings.*

Thank you !

Proof. Consider the one-to-one continuous linear mapping $U :$ $c \longrightarrow c_0$ given by

$$U : u = (u_1, u_2, u_3, \dots) \mapsto (\lambda(u), u_1 - \lambda(u), u_2 - \lambda(u), u_3 - \lambda(u), \dots).$$

The map U is clearly *onto* and the inverse mapping $U^{-1} : c_0 \longrightarrow c$ is given by

$$U^{-1} : \beta = (\beta_1, \beta_2, \beta_3, \dots) \mapsto (\beta_2 + \beta_1, \beta_3 + \beta_1, \beta_4 + \beta_1, \dots).$$

Clearly, U is a linear isomorphism of $(c, \|\cdot\|_\infty)$ onto $(c_0, \|\cdot\|_\infty)$. Moreover, by its construction, U is a linear isometric isomorphism of $(c, \|\cdot\|_\infty)$ onto $(c_0, \|\cdot\|^\sim)$. Recall that the following non-weakly compact, closed, bounded, convex subset of $(c, \|\cdot\|_\infty)$ has the fixed point property for nonexpansive mappings:

$$W := \{y = (y_n)_{n \in \mathbb{N}} \in \ell^\infty : 1 \geq y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0\}.$$

It is easy to check that $U(W) = K$. $\square$

Thank you again !

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