Extraordinary transmission, symmetry and the Blaschke Product

Michael Meylan[a], Mahmood-ul- Hassan[b], and Amna Bashir[b]

School of Mathematical and Physical Sciences,
The University of Newcastle[a] and Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan[b]

Extraordinary optical transmission

Extraordinary optical transmission (EOT) is the phenomenon of greatly enhanced transmission of light. It goes back to Ebbesen, et. al. ''Extraordinary optical transmisison through sub-wavelength hole arrays,'' *Nature* 391 (1998), 667. It is thought to be caused by surface plasmons.

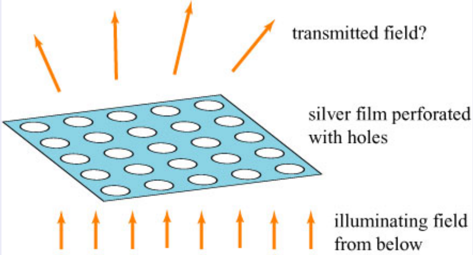

Figure: Schematic of Extraordinary optical transmission

Extraordinary acoustic transmission

Extraordinary acoustic transmission is the acoustic analogy of extraordinary optical transmission.

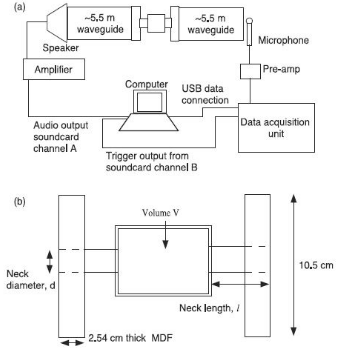

FIG. 1. (a) Schematic arrangement of the experimental configuration. (b) Close-up of the Helmoltz resonator and barrier dimensions.

Figure: Schematic of Extraordinary acoustic transmission from Crow et. al. *AIP Advances* 5, 027114 (2015)

This figure from Crow et. al. shows the key results of the experiment

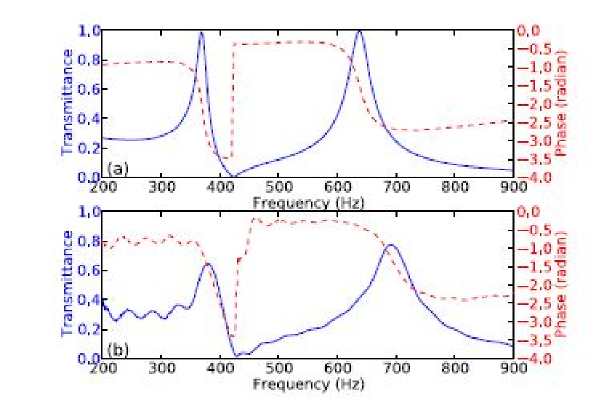

Figure: Figure 6 from Crow et. al.

Symmetry

We want to explain this extraordinary transmission and to give conditions for it to exist. Symmetry plays a key role. We assume we have one mode of propagation and that the problem is symmetric.

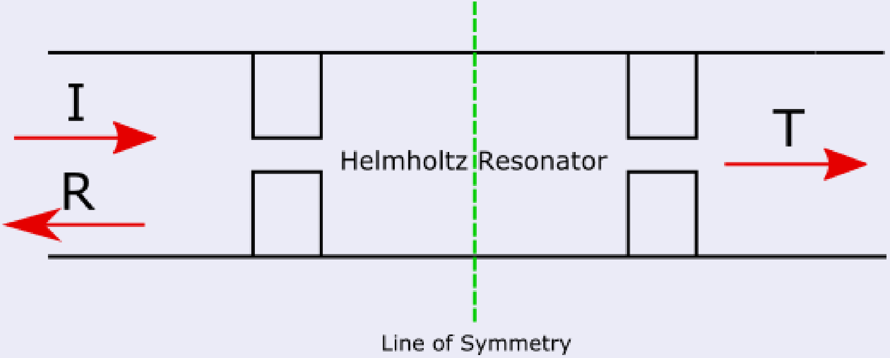

Figure: Schematic diagram of our problem

We can decompose the problem into the following two subproblems.

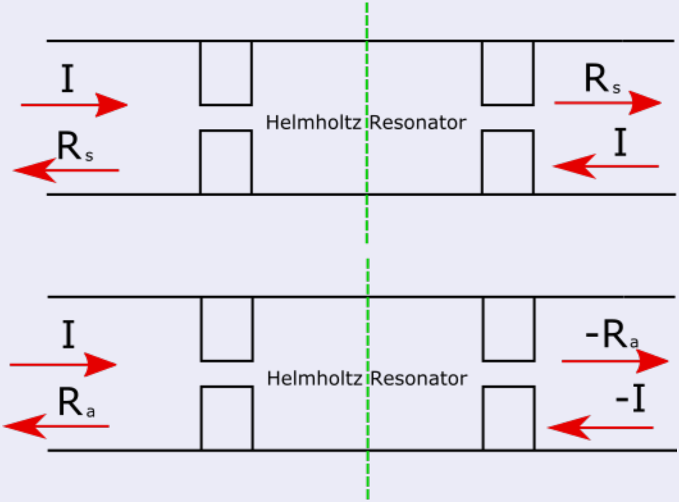

Figure: Symmetric and antisymmetric problem

If we compare the three problems

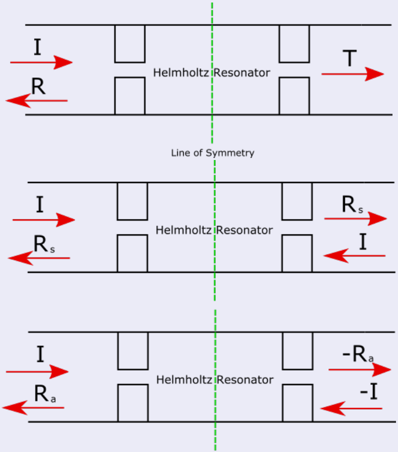

Figure: It follows that $R = \frac{1}{2}(R_s + R_a)$ and $T = \frac{1}{2}(R_s - R_a)$

Decomposition of the Reflection

The reflection coefficients R_s and R_a are analytic functions of the frequency k. For a wave guide they are meromorphic with poles in the lower half plane.

We know that, from conservation of energy

$$|R_s(k)| = |R_a(k)| = 1, k \in \mathbb{R}$$

We can then conclude that R_s is basically a Blaschke Product

$$R_s(k) = \mathrm{e}^{\mathrm{i}f(k)} \prod_i \left(\frac{k - \overline{a_m}}{k - a_m} \right)$$

where $f(k)$ takes real values for real k.

Blaschke Product

$$R_s(k) = \mathrm{e}^{\mathrm{i}f(k)} \prod_i \left(\frac{k - \overline{a_m}}{k - a_m} \right)$$

- The values a_m are the points where the analytic extension of the scattering matrix is singular .
- The a_m are known as resonances.
- They are the points where the resolvent is singular and they are almost eigenvalues.
- Without symmetry the scattering matrix has dimension two and we cannot apply this decomposition.

Resonances and Helmholtz Resonators

- A Helmholtz resonator is a physical object - a bottle which you blow over for example.
- Mathematically we have perturbed the eigenvalues which have become resonances.
- Resonances and eigenvalues are **not** the same thing. For example the mode associated with a resonances grows with distance away.

From the Blaschke Product to Extraordinary Transmission

The form of the transmission is

$$T = \frac{1}{2}(R_s - R_a)$$

and we know that

$$|R_s(k)| = |R_a(k)| = 1, k \in \mathbb{R}$$

and

$$R_s(k) = \mathrm{e}^{\mathrm{i}f(k)} \prod_i \left(\frac{k - \overline{a_m}}{k - a_m} \right)$$

Therefore near a_m there is a change of phase of 2π. This in turn implies somewhere near a_m, $|T| = 1$. We are assuming that $f(k)$ is slowly varying.

Example Calculations

The governing equation is

$$\nabla^2\phi(x,y) + k^2\phi(x,y) = 0$$

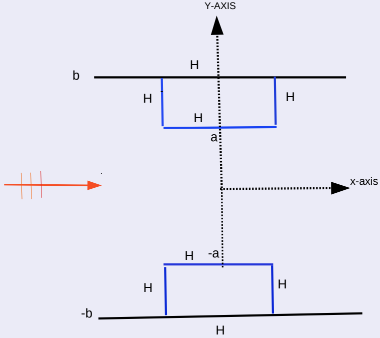

Figure: Our geometry

Example Calculations

The solutions is found by mode matching

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n^{(s)} \mathrm{e}^{-\mathrm{i}\bar{\alpha}_n(x+l)} \psi_n(y) + \mathrm{e}^{\mathrm{i}\bar{\alpha}_1(x+l)} \psi_1(y)$$

where

$$\alpha_n = \frac{(n-1)\pi}{2b}, \qquad \bar{\alpha}_n = \sqrt{(k^2 - \alpha_n^2)},$$

and

$$\psi_n(y) = \begin{cases} \sqrt{\frac{1}{b}} \cos \alpha_n (y-b), & n \neq 1, \\ \sqrt{\frac{1}{2b}}, & n = 1, \end{cases}$$

Analytic Extension

There are two challenges to this solution method.

1. How to find the analytic extension of the solution for complex k?
2. How to find the singularities a_m?

These are overcome by

1. A homotopy method. The roots should not jump across branch cuts.
2. Newton's method combined with a complex variable bisection method.

Results

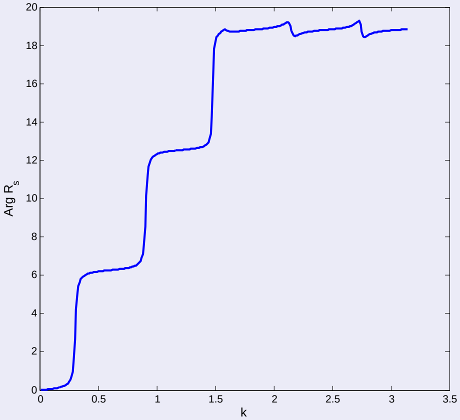

Figure: Reflection coefficient against Arg(Rs)

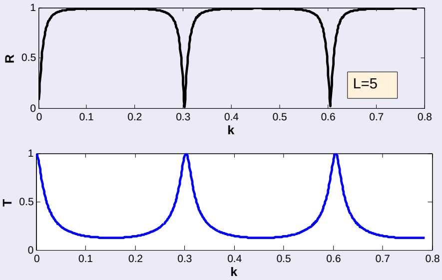

Figure: Reflection and Transmission against wave number for $L = 5$

Conclusions

- If there is a symmetry and a single mode of propagation we will get extraordinary transmission, for each resonance.
- We have tested this theory for a simple case in acoustic scattering similar to a recent experimental verification.
- There are many interesting questions raised by this research, ranging from proof of analyticity to possible applications in wave splitting or even an acoustic prism.