Semitopological groups versus topological groups

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Semitopological groups

A triple $(G, \cdot, \tau)$ is called a semitopological group if:

(i) $(G, \cdot)$ is a group and (G, τ) is a topological space;

(ii) multiplication, $(x, y) \mapsto x \cdot y$, from $G \times G$ into G is separately continuous.

A triple $(G, \cdot, \tau)$ is a topological group if:

(i) $(G, \cdot)$ is a group and (G, τ) is a topological space;

(ii) multiplication, $(x, y) \mapsto x \cdot y$, from $G \times G$ into G is jointly continuous;

(iii) inversion, $x \mapsto x^{-1}$, from G onto G is continuous.

Clearly, every topological group is a semitopological group.

However, there are some other interesting/natural examples that are not topological groups.

Example 1. $(\mathbb{R}, +, \tau_S)$, where τ_S (the Sorgenfrey topology) is the topology on $\mathbb{R}$ generated by the sets

$$\{[a, b) : a, b \in \mathbb{R} \text{ and } a < b\}.$$

Note that in this example $(x, y) \mapsto x \cdot y$ is continuous but $x \mapsto x^{-1}$ is not.

Example 2. Let (X, τ) be a nonempty topological space and let G be a nonempty subset of X^X. If $(G, \circ)$ is a group (where "$\circ$" denotes the binary relation of function composition) and τ_p denotes the topology on X^X of pointwise convergence on X then $(G, \circ, \tau_p)$ is a semitopological group provided the members of G are continuous functions.

Example 3. Let G denote the set of all homeomorphisms on $(\mathbb{R}, \tau_S)$. From Example 2 we see that $(G, \circ, \tau_p)$ is a semi topological group. However, $(G, \circ, \tau_p)$ is not a topological group.

To see this, define $g_n : \mathbb{R} \to \mathbb{R}$ by, $g_n(x) := [1 + 1/(n+1)]x$, $a_n := 1 + 1/(2n)$ and

$$f_n(x) := \begin{cases} x & \text{if } x \notin [a_n, a_n + 1/(2n)) \cup [n, n + 1/(2n)) \\ n + (x - a_n) & \text{if } x \in [a_n, a_n + 1/(2n)) \\ a_n + (x - n) & \text{if } x \in [n, n + 1/(2n)). \end{cases}$$

Then both f_n and g_n converge pointwise to id - the identity map, however, $\lim_{n\to\infty} (f_n \circ g_n)(1) = \infty \neq (id \circ id)(1) = id(1) = 1$.

On the other hand sometimes Example 2 does give rise to topological groups.

Example 4. Let (M, d) be a metric space and let G be the set of all isometries on (M, d). Then $(G, \circ, \tau_p)$ is a topological group.

To prove this, one just needs to apply the triangle inequality a few times. Indeed, if $f_n, g_n \in G$, f_n converges pointwise to $f \in G$ and g_n converges pointwise to $g \in G$ then for each $x \in M$,

$$\begin{aligned} 0 &\leq d(f_n(g_n(x)), f(g(x))) \\ &\leq d(f_n(g_n(x)), f_n(g(x)) + d(f_n(g(x)), f(g(x))) \\ &= d(g_n(x), g(x)) + d(f_n(g(x)), f(g(x)). \end{aligned}$$

Semitopological groups also naturally arise in the study of group actions (topological dynamics).

Example 5. Let $(G, \cdot)$ be a group and let (X, τ) be a topological space. Further, let $\pi : G \times X \to X$ be a mapping (group action) such that:

(i) $\pi(e, x) = x$ for all $x \in X$, where e denotes the identity element of G;

(ii) $\pi(g \cdot h, x) = \pi(g, \pi(h, x))$ for all $g, h \in G$ and $x \in X$;

(iii) for each $g \in G$, the mapping, $x \mapsto \pi(g, x)$, is a continuous function on X.

Then (G, X) is called a flow on X. If we consider the mapping $\rho : G \to X^X$ defined by, $\rho(g)(x) = \pi(g, x)$ for all $x \in X$.

Then $(\rho(G), \circ, \tau_p)$ is a semitopological group.

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Research on the problem of which topological conditions on a semitopological group imply that it is a topological group possibly began in

[D. Montgomery,“Continuity in topological groups" *Bull. Amer. Math. Soc.* **42** (1936)]

when the author showed that each completely metrizable semitopological group has jointly continuous multiplication. Later, in

[R. Ellis,“A note on the continuity of the inverse" *Proc. Amer. Math. Soc.* **8** (1957) and “Locally compact transformation groups" *Duke Math. J.* **24** (1957)]

Ellis showed that each locally compact semitopological group is in fact a topological group. This answered a question raised by A. D. Wallace in

[A. D. Wallace,“The structure of topological semigroups" *Bull. Amer. Math.* **61** (1955)].

Next in

[W. Zelazko,“A theorem on B_0 division algebras" *Bull. Acad. Pol. Sci.* **8** (1960)]

Zelazko used Montgomery’s result from 1936 to show that each completely metrizable semitopological group is a topological group. Much later, in

[A. Bouziad,“Every Čech-analytic Baire semitopological group is a topological group" *Proc. Amer. Math. Soc.* **124** (1996)]

Bouziad improved both of these results and answered a question raised by Pfister in

[H. Pfister,“Continuity of the inverse" *Proc. Amer. Math. Soc.* **95** (1985)] by showing that each Čech-complete semitopological group is a topological group.

(Recall that a topological space (X, τ) is called Cech-complete if it is a G_δ subset of a compact Hausdorff space.)

It is well-known that both locally compact and completely metrizable topological spaces are Čech-complete).

To do this, it was sufficient for Bouziad to show that every Čech-complete semitopological group has jointly continuous multiplication since earlier, Brand

[N. Brand,"Another note on the continuity of the inverse" *Arch. Math.* **39** (1982)]

had proven that every Čech-complete semitopological group with jointly continuous multiplication is a topological group. Brand's proof of this was later improved and simplified in

[H. Pfister,"Continuity of the inverse" *Proc. Amer. Math. Soc.* **95** (1985)].

Apart from those named above there have been many other contributors to the question of when a semitopological group is in fact a topological group.

For example, Arhangel'skii, Brown, Cao, Choban, Drozdowski, Guran, Hola, Kenderov, Korovin, Kortezov, Lawson, Moors, Namioka, Piotrowski, Ravsky, Reznichenko, Romaguera, Sanchis and Tkachenko.

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Topological Games

In [P. S. Kenderov, I. Kortezov and W.B. Moors,"Topological games and topological groups" *Topology Appl.* **109** (2001)]

the authors used a two player topological game to determine some conditions on a semitopological group that imply it is a topological group. Using this game they were able to prove a theorem considerably more general than the following.

Theorem 1. Let $(G, \cdot, \tau)$ be a semitopological group such that (G, τ) is a regular Baire space. If any of the following conditions hold, then $(G, \cdot, \tau)$ is a topological group.

(i) (G, τ) is metrizable (or more generally, (G, τ) is a p-space);
(ii) (G, τ) is Čech-analytic (or more generally, has countable separation);
(iii) (G, τ) is locally countably compact.

Recall that a topological space (X, τ) is called

(i) regular if every closed subset of X and every point outside of this set, can be separated by disjoint open sets and

(ii) Baire if the intersection of every countable family of dense open sets is dense in X.

An advantage to the “game" approach used in [KKM] is that it covers many different situations at once. However, there is also a disadvantage to the game approach. Namely, people find the use of games unappealing, artificial and hard to understand.

Hence some of the consequences of the paper [KKM] have gone unnoticed until quite recently. However, there is now a cottage industry showing that certain topological spaces satisfy the game hypotheses given in [KKM]. Unfortunately, most of these results were already known to the authors of [KKM] back in 2001.

The game that we shall consider involves two players which we will call player α and player β. The "field/court" that the game is played on is a fixed topological space (X, τ) with a fixed dense subset D. The name of the game is the "$G_S(D)$-game".

After naming the game we need to describe how to "play" the $G_S(D)$-game.

The player labeled β starts the game every time (life is not always fair). For their first move the player β must select a nonempty open subset B_1 of X.

Next, α gets a turn. For α's first move he/she must select a nonempty open subset A_1 of B_1. This ends the first round of the game.

In the second round, β goes first (again) and selects a nonempty open subset B_2 of A_1.

α then gets to respond by choosing a nonempty open subset A_2 of B_2. This ends the second round of the game. At this stage we have

$$A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

In general, after α and β have played the first n-rounds of the $G_S(D)$-game, β will have selected nonempty open sets $B_1, B_2, \ldots, B_n$ and α will have selected nonempty open sets $A_1, A_2, \ldots, A_n$ such that:

$$A_n \subseteq B_n \subseteq A_{n-1} \subseteq B_{n-1} \subseteq \cdots \subseteq A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

At the start of the $(n+1)$-round of the game, β goes first (again!) and selects a nonempty open subset B_{n+1} of A_n.

As with the previous n-rounds, player α gets to respond to this move by selecting a nonempty open subset A_{n+1} of B_{n+1}.

Continuing this process indefinitely (i.e., continuing-on forever) the players produce an infinite sequence, (called a play of the $G_S(D)$-game)

$$\{(A_n, B_n) : n \in \mathbb{N}\}$$

of pairs of nonempty open subsets of X such that

$$A_{n+1} \subseteq B_{n+1} \subseteq A_n \subseteq B_n \quad \text{for all } n \in \mathbb{N}.$$

As with any game, we need a rule to determine who wins
– otherwise it is a very boring game. ☺

We shall declare that α wins a play $\{(A_n, B_n) : n \in \mathbb{N}\}$ of the $G_S(D)$-game if:

(i) $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and
(ii) each sequence $(x_n : n \in \mathbb{N})$ in D with $x_n \in A_n$ for all $n \in \mathbb{N}$, has a cluster-point in X.

If α does not win a play of the $G_S(D)$-game then we declare that β wins that play of the $G_S(D)$-game.

So every play is won by either α or β and no play is won by both players.

Continuing further into game theory we need to introduce the notion of a strategy.

A strategy for the player β (player α) is a “rule" that specifies how the player β (player α) must respond/move in every possible situation that may occur during the course of the game. [A more precise mathematical description of a strategy is possible, but we shall not give it here.]

We may now finally define a “strongly Baire" space.

We shall say that a topological space (X, τ) is strongly Baire if it is regular and there exists a dense subset D of X such that the player β (i.e., the player with the privilege of going first) does NOT have a winning strategy in the $G_S(D)$-game played on X (that is to say, that no matter what strategy player β adopts there is always at least one play of the $G_S(D)$-game where α wins.).

Clearly, if α actually possesses a winning strategy himself/ herself then β cannot possibly possess a winning strategy as well and so all spaces (X, τ) in which α has a winning strategy in the $G_S(D)$-game are strongly Baire.

Note: significantly there are some strongly Baire spaces in which the player α does not possess a winning strategy.

Structure of Game Proofs

Using the notion of a strongly Baire space the authors in [KKM] proved the following very general result.

Theorem 2. [KKM, 2001] Let $(G, \cdot, \tau)$ be a semitopological space. If (G, τ) is a strongly Baire space then $(G, \cdot, \tau)$ is a topological group.

The use of games can often simplify the presentation of certain inductive arguments. One can design a game that exactly suits/fits the particular inductive argument under consideration. That is, the game can be tailor made to fit the situation.

The proof then divides into two parts.

In one part we use the tailor made game to expedite the proof of the inductive argument. Strategies offering an effect way of recording the inductive hypotheses.

The other part of the proof is then to determine those spaces/situations where the game conditions are satisfied.

This dividing of the proof into two parts is an important feature of the game approach.

Another feature of the game formalism is the possibility of considering spaces where neither player possesses a winning strategy.

Initially, it is not at all clear how one can use the assumption:

"I do not possess a winning strategy."

The way in which one usually exploits the hypothesis/condition that β does not possess a winning strategy is the following.

One uses a proof by contradiction. That is, assume that the conclusion of the statement (that one wants to prove) is false.

Then use this additional information to construct a strategy t for the player β.

The fact that t is not a winning strategy for the player β then yields the existence of a play $\{(A_n, B_n) : n \in \mathbb{N}\}$ where α wins.

This play $\{(A_n, B_n) : n \in \mathbb{N}\}$ is then used to obtain the required contradiction.

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As mentioned earlier, the difficulty with the game formulation is that many people that are not familiar with game theory are intimidated.

The net result is that many of the consequences of the paper [KKM] have gone unnoticed - though this is starting to change now.

As an example: one can easily show that every Baire metric space is a strongly Baire space. Thus, we have that every Baire metric semitopological group is a topological group.

However, even in [M. Tkachenko, “Paratopological and semitopological groups versus topological groups" Recent Progress in General Topology. III, Atlantis Press, Paris, 2014] the author says:

"Further, Solecki and Srivastava establish in [91] a general fact implying that every separable metrizable Baire semitopological group is a topological group and mention there that, by an unpublished result of Reznichenko, one can drop the separability restriction in this corollary. Let us show, using an argument from [80] - (an unpublished manuscript), that this is certainly the case."

By modifying the game considered in [KKM] one can further generalise the class of strongly Baire spaces. This was done in both

W. Moors, "Semitopological groups, Bouziad spaces and topological groups" Topology Appl. 160 (2013).

and

W. Moors, “Any semitopological group that is homeomorphic to a product of Cech-complete spaces is a topological group” Set-Valued Var. Anal. 21 (2013).

to solve some open questions. In particular, the above papers solve most of the open problems in

[M. Tkachenko, “Paratopological and semitopological groups versus topological groups" Recent Progress in General Topology. III, Atlantis Press, Paris, 2014] (concerning when a semitopological group is a to topological group)

as well as ALL the open problems in

[A. Arhangel’skii, M. Choban, and P. Kenderov, “Topological games and topologies on groups” Math. Maced. 8 (2010)].

Thank you for your attention and for the opportunity to present my work.

A PDF version of this talk is available at:

www.math.auckland.ac.nz/$\sim$moors/

The End

But wait, there is more...

Although the class of strongly Baire spaces provided a convenient framework for our theorems these spaces are, unfortunately, not readily identifiable. So in this part of the talk we will introduce a related class of spaces whose membership properties are more readily determined. [This is the second part of the game approach that was mentioned earlier].

A topological space (Y, τ) is said to be cover semi-complete if there exists a pseudo-metric d on Y such that;

(i) each d-convergent sequence in Y has a cluster-point in Y;

(ii) (ii) Y is fragmented by d, that is, for each $\varepsilon > 0$ and nonempty subset A of Y there exists a nonempty relatively open subset B of A such that d-diam $B < \varepsilon$.

Clearly, all metric spaces are cover semi-complete, as are, all regular countably compact spaces.

With a little more work one can show that all Cech-analytic spaces (which includes all Cech-complete spaces) are cover semi-complete.

THEOREM 3. [KKM, 2001] Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) contains, as a second category subset, a cover semi-complete space Y, then $(G, \cdot, \tau)$ is a topological group. In particular, if (G, τ) is a cover semi-complete Baire space then $(G, \cdot, \tau)$ is a topological group.

The End