Semivectorial Bilevel Optimization on Riemannian Manifolds

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Outline

Section

Semivectorial bilevel problem (SVB$_\sigma$): *optimistic case*

Scalar upper level (leader's objective) $f : M_1 \times M_2 \to \mathbb{R}$

$$\min_x \min_y f(x, y),$$

subject to

Vector lower level (follower's objective) $F : M_1 \times M_2 \to \mathbb{R}^r$,

for each fixed $x \in M_1$, y is a σ-efficient (Pareto) solution for

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M_1, M_2 connected Riemannian manifolds verifying Hopf-Rinow theorem; C is a closed, convex, pointed cone, $\text{int}\,(C) \neq \emptyset$, and $\sigma \in \{w, p\}$.

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Particular cases

Classical bilevel optimization (optimistic or pessimistic case)

We obtain the problem

$$\min_x \min_y f(x,y) \quad \text{(resp. } \min_x \sup_y f(x,y)) \quad \text{subject to}$$

$$y \in \psi(x),$$

where

$$\psi(x) = \operatorname*{argmin}_{y'} \Big(F(x,y') \quad \text{s.t.} \quad h(y') = 0 \Big),$$

considering $Z = \mathbb{R}$, $M_1 = \mathbb{R}^m$, , $M_2 = \{y \in \mathbb{R}^{n+p} | \; h(y) = 0\}$, where $h : \mathbb{R}^{n+p} \to \mathbb{R}^p$ is smooth and regular.

Particular cases

Optimization over the σ-efficient set

We obtain the problem

$$\min f(y) \quad \text{subject to}$$

y is a σ-efficient solution for the vector optimization problem

$$\text{MIN}_C\, F(y') \quad \text{subject to} \quad y' \in S_0,$$

considering $M_1 = \{x_0\}$, $f(\cdot) = f(x_0, \cdot)$, $F(\cdot) = F(x_0, \cdot)$, $M_2 = S_0 \subset \mathbb{R}^n$.

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In my talk I consider the Riemannian setting case.

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Why Riemannian ?

- Constrained optimization problems can be seen as unconstrained ones from the Riemannian geometry viewpoint;
- Moreover some nonconvex optimization problems in the Euclidean setting may become convex introducing an appropriate Riemannian metric.
- In the last years researchers began the study of optimization problems in the Riemannain setting.
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Section

Hopf-Rinow Theorem

Let M be a connected Riemannian manifold. The following statements are equivalent:

1. M is complete as a metric space.
2. M is geodesically complete (i.e. all the geodesics are defined on $\mathbb{R}$).
3. Closed and bounded sets on M are compact.

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3. Closed and bounded sets on M are compact.

Moreover, each of the statements (1-3) implies that any two points of M can be joined by a minimizing geodesic.

Vector optimization on Riemannian manifolds

Let M be a connected RM verifying the Hopf-Rinow theorem.

Let $C \subset \mathbb{R}^r$ be a convex pointed cone[a], closed with $\operatorname{int}(C) \neq \emptyset$.

For any $y, y' \in \mathbb{R}^r$ we denote

- $y \preccurlyeq y' \iff y' - y \in C$
- $y \prec y' \iff y' - y \in \operatorname{int}(C)$
- $y \precneqq y' \iff y' - y \in C \setminus \{0\}$.

- We have

$$y \prec y' \Longrightarrow y \precneqq y' \Longrightarrow y \preccurlyeq y'.$$

$\preccurlyeq$ is a partial order relation on $\mathbb{R}^r$.

$\prec$ and $\precneqq$ are transitive relations.

[a] i.e. $\mathbb{R}_+ C + C \subset C$, $C \cap (-C) = \{0\}$

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Consider a vector function $G = (G_1, \dots, G_r) : M \to \mathbb{R}^r$, and the multiobjective optimization problem

$$\text{(MOP)} \qquad \text{MIN}_C G(x) \quad \text{s.t. } x \in M.$$

For (MOP) the point $a \in M$ is called:

- *Pareto solution* if there is no $x \in M$ such that $G(x) \lneqq G(a)$, ;
- *weakly Pareto solution* if there is no $x \in M$ such that $G(x) \prec G(a)$;
- *properly Pareto solution* if a is a Pareto solution, and there exists a pointed convex cone K such that $C \setminus \{0\} \subset \operatorname{int}(K)$ and a is a Pareto solution for the problem $\text{MIN}_K G(x) \quad \text{s.t. } x \in M$, in other words $G(M) \cap (G(a) - K) = \{G(a)\}$.

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In the particular case $C = \mathbb{R}^r_+$ (the Pareto cone), the previous definitions can be stated as follows.

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- *properly Pareto solution* (provided that $G(M) + \mathbb{R}^r_+$ is convex) if a is a Pareto solution, and there is a real number $\mu > 0$ such that for each $i \in \{1, \ldots, r\}$ and every $x \in M$ with $G_i(x) < G_i(a)$ at least one $j \in \{1, \ldots, r\}$ exists with $G_j(x) > G_j(a)$ and

$$\frac{G_i(a) - G_i(x)}{G_j(x) - G_j(a)} \leq \mu.$$

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$$\frac{G_i(a) - G_i(x)}{G_j(x) - G_j(a)} \leq \mu.$$

Vector optimization on Riemannian manifolds

Let the symbol $\sigma \in \{w, p\}$ stands for :

weak if $\sigma = w$,

or

proper if $\sigma = p$.

We denote by

$$\sigma\text{-}\mathrm{ARGMIN}_{x \in M} {}_C G(x)$$

the set of all σ-Pareto solutions.

Definition

A real function $h : M \to \mathbb{R}$ is called *convex* if for any two distinct points a and b in M, and for any minimizing geodesic segment $\gamma : [0,1] \to M$ with $\gamma(0) = a$, $\gamma(1) = b$, the function $h \circ \gamma$ is convex in the usual way, i.e. for all $t \in]0,1[$

$$h(\gamma(t)) \leq (1-t)h(a) + th(b).$$

If the last inequality is strict, we say that h is *strictly convex*.

Definition

The *vector valued function* $G = (G_1, \ldots, G_r) : M \to \mathbb{R}^r$ is called *C-convex* (resp. *w-strictly C-convex* or *p-strictly C-convex*) if for any two distinct points a and b in M, and for any geodesic segment *gamma* $: [0, 1] \to M$ with $\gamma(0) = a$, $\gamma(1) = b$, we have respectively

$$\forall t \in]0,1[\qquad G(\gamma(t)) \preccurlyeq (1-t)G(a)+tG(b),$$

$$\forall t \in]0,1[\qquad G(\gamma(t)) \prec (1-t)G(a)+tG(b),$$

$$\forall t \in]0,1[\qquad G(\gamma(t)) \precneqq (1-t)G(a)+tG(b).$$

Vector optimization on Riemannian manifolds

Remark

In the case $C = \mathbb{R}^r_+$ we have

- G is $\mathbb{R}^r_+$-convex $\iff$ G_i is convex for all $i = 1, \ldots, r$;
- G is w-strictly $\mathbb{R}^r_+$-convex $\iff$ G_i is strictly convex for all $i = 1, \ldots, r$;
- G is $\mathbb{R}^r_+$-convex and there exists $i \in \{1, \ldots, r\}$ such that G_i is strictly convex $\implies$ G is p-strictly $\mathbb{R}^r_+$-convex.

The *dual cone of C* (or positive polar cone) is the set

$$C^* := \{\lambda \in \mathbb{R}^r \,|\, \langle \lambda, y \rangle \geq 0 \quad \forall y \in C\},$$

and its *quasi-interior* is given by

$$C^*_\sharp := \{\lambda \in \mathbb{R}^r \,|\, \langle \lambda, y \rangle > 0 \quad \forall y \in C \setminus \{0\}\}.$$

Notice that

$$(\mathbb{R}^r_+)^* = \mathbb{R}^r_+, \quad (\mathbb{R}^r_+)^*_\sharp = \mathrm{int}\,(\mathbb{R}^r_+) = \{\lambda \in \mathbb{R}^r \,|\, \lambda_i > 0 \quad i = 1, \ldots, r\}.$$

Let us denote

$$\Lambda_\sigma = \begin{cases} \{\lambda \in C^* \,|\, \|\lambda\|_1 = 1\} & \text{if } \sigma = w \\ C^*_\sharp & \text{if } \sigma = p. \end{cases}$$

Proposition

A. The dual cone C^* is a closed set in $\mathbb{R}^r$.

B. The set $C^*_\sharp$ (the quasi-interior of C^*) is a nonempty open set,[a] and it is in fact the topological interior of C^*.

C. The set Λ_w is compact.

[a] This fact it is not true in general, i.e. when C is a cone in a topological vector space, but in our setting we take advantage of the finite dimension of $\mathbb{R}^r$.

Vector optimization on Riemannian manifolds

Theorem (Scalarization)

For each $\sigma \in \{w, p\}$, *we have*

$$\bigcup_{\lambda \in \Lambda_\sigma} \arg\min_{x \in M} \langle \lambda, G(x) \rangle \subset \sigma\text{-}\mathrm{ARGMIN}_{x \in M}{}_C G(x).$$

Moreover, if G *is* C*-convex on* M*, then the previous inclusion becomes an equality, i.e.*

$$\sigma\text{-}\mathrm{ARGMIN}_{x \in M}{}_C G(x) = \bigcup_{\lambda \in \Lambda_\sigma} \arg\min_{x \in M} \langle \lambda, G(x) \rangle.$$

Section

Hypotheses

- Leader decision space : (M_1, g_1) RM verifying HR
- Follower(s) decision space : (M_2, g_2) RM verifying HR
- Leader scalar objective function : $f : M_1 \times M_2 \to \mathbb{R}$
- Follower(s) multiobjective function :
  $F = (F_1, \ldots, F_r) : M_1 \times M_2 \to \mathbb{R}^r$
- For each $x \in M_1$, $\psi(x)$ stands for the *weakly or properly Pareto solution set of the follower multiobjective optimization problem,* i.e.

$$\psi(x) := \sigma\text{-}\underset{y \in M_2}{\mathrm{ARGMIN}_C} F(x,.y)$$

Thus $\psi : M_1 \rightrightarrows M_2$ is a set valued function.

Hypotheses

(HC)$_\sigma$ *For each* $x \in M_1$,

the function $F(x,\cdot)$ *is* σ*-strictly* C*-convex on* M_2, $\sigma \in \{w,p\}$.

(HCC)$_\sigma$ *For all* $x \in M_1$ *and* $\lambda \in \Lambda_\sigma$, *the function* $y \mapsto \langle \lambda, F(x,y)\rangle$ *has bounded sublevel sets, i.e, for all reals* α, *the set*

$$\{y \in M_2 | \langle \lambda, F(x,y)\rangle \leq \alpha\}$$

is bounded.

The problems

- The *optimistic semivectorial bilevel problem*

$$(OSB) \qquad \min_{x \in M_1} \min_{y \in \psi(x)} f(x,y).$$

The follower cooperates with the leader, i.e., for each $x \in M_1$, the follower chooses amongst all its σ-Pareto solutions (his best responses) one which is the best for the leader (assuming that such a solution exists).

- The *pessimistic semivectorial bilevel problem*

$$(PSB) \qquad \min_{x \in M_1} \sup_{y \in \psi(x)} f(x,y).$$

There is no cooperation between the leader and the follower, and the leader expects the worst scenario, i.e., for each $x \in M_1$, the follower may choose amongst all its σ-Pareto solutions (his best responses) one which is unfavorable for the leader (in this case we prefer to use "sup" instead of "max").

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An equivalent problem

Proposition

For any $x \in M_1$ and any $\lambda \in \Lambda_\sigma$, the real valued function

$$M_2 \ni y \mapsto \langle \lambda, F(x,y) \rangle$$

is strictly convex.

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For each $x \in M_1$ and $\lambda \in \Lambda_\sigma$, the minimization problem

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admits a unique solution which will be denoted hereafter $y(x, \lambda)$.

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An equivalent problem

Corollary

For each fixed $x \in M_1$, the map $\lambda \mapsto y(x,\lambda)$ is a surjection from Λ_σ to $\psi(x)$, hence

$$\psi(x) = \bigcup_{\lambda \in \Lambda_\sigma} \{y(x,\lambda)\}.$$

An equivalent problem

Theorem

Problem (OSB) is equivalent to the following problem[a]

$$\min_{x\in M_1} \min_{\lambda\in\Lambda_\sigma} f(x, y(x,\lambda))$$

Problem (PSB) is equivalent to the following problem

$$\min_{x\in M_1} \sup_{\lambda\in\Lambda_\sigma} f(x, y(x,\lambda))$$

[a] $y(x,\lambda)$ is the unique solution to the problem $\min_{y\in M_2} \langle \lambda, F(x,y)\rangle$

Section

Optimality conditions for $\min_{y \in M_2} \langle \lambda, F(x,y) \rangle$

From now on we suppose that $f(\cdot,\cdot)$ and $F(\cdot,\cdot)$ are smooth fonctions.

grad_i stands for the gradient operator on (M_i, g_i), $i = 1, 2$. F^a (resp. λ_a), $a = 1, \cdots, r$, are the components functions of the map $F : M_1 \times M_2 \longrightarrow \mathbb{R}^r$ (resp. the canonical coordinates of the vector $\lambda \in \mathbb{R}^r$).

Proposition (Necessary and sufficient conditions for $y(x, \lambda)$)

Let $\lambda \in \Lambda_\sigma$ and $x \in M_1$ be given. Let $y \in M_2$. Then

$$y = y(x, \lambda) \Longleftrightarrow \lambda_a \mathrm{grad}_2 F^a(x, y) = 0.$$

More about the map $(x, \lambda) \mapsto y(x, \lambda)$

Consider the map $G : \mathbb{R}^r \times M_1 \times M_2 \longrightarrow TM_2$ defined by

$$G(\lambda, x, y) = \lambda_a \mathrm{grad}_2 F^a(x, y).$$

For each $(\lambda, x) \in \Lambda_\sigma \times M_1$, the solution $y = y(x, \lambda)$ to the problem $\min_{y \in M_2} \langle \lambda, F(x, y) \rangle$ satisfies the equation

$$G(\lambda, x, y) = 0.$$

Denote by $\delta_2 G(\lambda, x, y) : T_y M_2 \longrightarrow T_y M_2$ the partial differential of G w.r.t. to y at the point (λ, x, y).

More about the map $(x, \lambda) \mapsto y(x, \lambda)$

Proposition

Let $(\lambda_0, x_0) \in \Lambda_\sigma \times M_1$, and let $y_0 = y(x_0, \lambda_0)$ be the unique solution of the problem $\min_{y \in M_2} \langle \lambda_0, F(x_0, y) \rangle$.

Suppose that $\delta_2 G(\lambda_0, x_0, y_0)$ is an isomorphism[a]. Then, in a neighborhood of (λ_0, x_0) the function $y(\cdot, \cdot)$ is smooth and

$$\frac{\partial}{\partial \lambda} y(\lambda, x) = -(\delta_2 G(\lambda, x, y))^{-1} \circ \frac{\partial G}{\partial \lambda}(\lambda, x, y)$$

and

$$\delta_1 y(\lambda, x) = -(\delta_2 G(\lambda, x, y))^{-1} \circ \delta_1 G(\lambda, x, y),$$

where δ_1 denotes the partial differential operator w.r.t. $x \in M_1$.

[a] This hypothesis holds for example if we assume that G is a conformal map, or for example if there exists a real number $c > 0$ such that $g_2(\delta_2 G(\lambda_0, x_0, y_0)(v), v) \geq c g_2(v, v)$, $\forall v \in T_y M_2$.

Optimality conditions for the optimistic problem

Theorem (Necessary optimality conditions for OSB$_p$)

Suppose that $\operatorname{loc-arg\,min}_{\lambda\in\Lambda_p} f(x, y(x,\lambda)) \neq \emptyset$ *for each* $x \in M_1$. *Let* $(x^*, \lambda^*) \in M_1 \times \Lambda_p$ *be a (local) solution of the problem*

$$\min_{x\in M_1} \min_{\lambda\in\Lambda_p} f\Big(x, y(x,\lambda)\Big).$$

Let $y^* = y(x^*, \lambda^*)$. *Then*

$$\begin{aligned} \operatorname{grad}_1 f(x^*, y^*) + \operatorname{grad}_2 f(x^*, y^*) \circ \delta_1 y(x^*, \lambda^*) &= 0 \\ \operatorname{grad}_2 f(x^*, y^*) \circ \frac{\partial y}{\partial \lambda}(x^*, \lambda^*) &= 0. \end{aligned}$$

Moreover, the bilinear form $\operatorname{Hess}(\varphi)(x^*, \lambda^*)$ *is positive semidefinite, where* $(x,\lambda) \mapsto \varphi(x,\lambda) := f\big(x, y(x,\lambda)\big)$.

Optimality conditions for the optimistic problem

Theorem (Necessary optimality conditions for OSB$_w$)

Let $(x^*, \lambda^*) \in M_1 \times \Lambda_w$ *be a (local) solution of the problem*

$$\min_{x \in M_1} \min_{\lambda \in \Lambda_w} f\Big(x, y(x, \lambda)\Big).$$

Let $y^* = y(x^*, \lambda^*)$. *Then*

$$\begin{aligned} \mathrm{grad}_1 f(x^*, y^*) + \mathrm{grad}_2 f(x^*, y^*) \circ \delta_1 y(x^*, \lambda^*) &= 0 \\ \mathrm{grad}_2 f(x^*, y^*) \circ \frac{\partial y}{\partial \lambda}(x^*, \lambda^*) + N^C_{\Lambda_w}(\lambda^*) &\ni 0, \end{aligned}$$

where $N^C_{\Lambda_w}(\lambda^*)$ *is the Clarke normal cone to the set* Λ_w *at the point* λ^*.

Optimality conditions for the optimistic problem

For $C = \mathbb{R}^r_+$ the previous theorem becomes

Theorem

Let $(x^*, \lambda^*) \in M_1 \times \Lambda_w$ be a (local) solution of the problem

$$\min_{x \in M_1} \min_{\lambda \in \Lambda_w} f\Big(x, y(x, \lambda)\Big).$$

Let $y^* = y(x^*, \lambda^*)$. Then there exists a real ν such that

$$\begin{aligned} \mathrm{grad}_1 f(x^*, y^*) + \mathrm{grad}_2 f(x^*, y^*) \circ \delta_1 y(x^*, \lambda^*) &= 0 \\ \mathrm{grad}_2 f(x^*, y^*) \circ \frac{\partial y}{\partial \lambda_i}(x^*, \lambda^*) &= \nu \quad \forall i \in I_+(\lambda^*) \\ \mathrm{grad}_2 f(x^*, y^*) \circ \frac{\partial y}{\partial \lambda_k}(x^*, \lambda^*) &\geq \nu \quad \forall k \in I_0(\lambda^*), \end{aligned}$$

where $I_+(\lambda^*) = \{i | \lambda_i^* > 0\}$ and $I_0(\lambda^*) = \{k | \lambda_k^* = 0\}$.

Section

Existence result for PSB$_w$

For the more difficult case of the pessimistic problem we will deal only with weakly-Pareto solutions.

Theorem

Suppose moreover that the Riemannian manifold (M_1, g_1) *is compact. Then the pessimistic problem*

$$\min_{x \in M_1} \sup_{\lambda \in \Lambda_w} f\Big(x, y(x, \lambda)\Big)$$

has at least one global solution.

THANK YOU!