About Uniform Regularity of Collections of Sets in Hilbert Spaces

Alexander Kruger

Centre for Informatics and Applied Optimization,
Faculty of Science & Technology
Federation University Australia
a.kruger@federation.edu.au

South Pacific Continuous Optimization Meeting
Adelaide, 8–12 February, 2015

Outline

Outline

1. Uniform regularity
2. Metric and dual characterizations
3. Uniform regularity in Hilbert spaces
4. Alternating projections

Uniform regularity

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ $(m > 1)$ $\quad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

Definition

$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ if $\exists \alpha, \delta > 0$ such that

$$\bigcap_{i=1}^{m} (\Omega_i - \omega_i - x_i) \bigcap (\rho \mathbb{B}) \neq \emptyset \qquad \forall \rho \in (0, \delta)$$

$\forall \omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, m)$ with $\max_{1 \le i \le m} \|x_i\| < \alpha \rho$

Uniform regularity

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ $(m > 1)$ $\quad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

Definition

$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ if $\exists \alpha, \delta > 0$ such that

$$\bigcap_{i=1}^{m} (\Omega_i - \omega_i - x_i) \bigcap (\rho\mathbb{B}) \neq \emptyset \qquad \forall \rho \in (0, \delta)$$

$\forall \omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, m)$ with $\max\limits_{1 \le i \le m} \|x_i\| < \alpha\rho$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\substack{\omega_i \xrightarrow{\Omega_i} \bar{x} \\ \rho \downarrow 0}} \frac{\sup\left\{ r \ge 0 \mid \bigcap\limits_{i=1}^{m} (\Omega_i - \omega_i - x_i) \bigcap (\rho\mathbb{B}) \neq \emptyset, \forall x_i \in r\mathbb{B} \right\}}{\rho} > 0$$

Outline

Metric characterizations

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ $(m > 1)$ $\quad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \liminf_{\substack{x \to \bar{x} \\ x_i \to 0 \ (1 \le i \le m) \\ x \notin \bigcap_{i=1}^m (\Omega_i - x_i)}} \frac{\max_{1 \le i \le m} d(x, \Omega_i - x_i)}{d\left(x, \bigcap_{i=1}^m (\Omega_i - x_i)\right)}$$

Metric characterizations

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ $(m > 1)$ $\quad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \liminf_{\substack{x \to \bar{x} \\ x_i \to 0\ (1 \le i \le m) \\ x \notin \bigcap_{i=1}^m (\Omega_i - x_i)}} \frac{\max_{1 \le i \le m} d(x, \Omega_i - x_i)}{d\left(x, \bigcap_{i=1}^m (\Omega_i - x_i)\right)}$$

Uniform regularity

$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\exists \gamma, \delta > 0$ such that

$$\gamma d\left(x, \bigcap_{i=1}^m (\Omega_i - x_i)\right) \le \max_{1 \le i \le m} d(x, \Omega_i - x_i)$$

for any $x \in B_\delta(\bar{x})$, $x_i \in \delta\mathbb{B}$ $(i = 1, \ldots, m)$

Collections of sets vs set-valued mappings

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ $(m > 1)$ $\quad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

$F : X \rightrightarrows X^m$: $\quad F(x) := (\Omega_1 - x) \times \ldots \times (\Omega_m - x)$ (Ioffe, 2000)

Proposition

$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ F is metrically regular at $(\bar{x}, 0)$, i.e., $\exists \gamma, \delta > 0$ such that

$$\gamma d\left(x, F^{-1}(y)\right) \leq d\left(y, F(x)\right) \quad \forall x \in B_\delta(\bar{x}),\ y \in \delta\mathbb{B}^m$$

Collections of sets vs set-valued mappings

X, Y – Banach spaces
$F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$

$\Omega_1 = \operatorname{gph} F$, $\Omega_2 = X \times \{\bar{y}\} \in X \times Y$, $\boldsymbol{\Omega} := \{\Omega_1, \Omega_2\}$

Collections of sets vs set-valued mappings

X, Y – Banach spaces
$F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$

$\Omega_1 = \operatorname{gph} F$, $\Omega_2 = X \times \{\bar{y}\} \in X \times Y$, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\}$

Proposition

F is metrically regular at $(\bar{x}, \bar{y})$ $\iff$ $\mathbf{\Omega}$ is uniformly regular at $(\bar{x}, \bar{y})$

Dual characterizations: Fréchet normals

$x \in \Omega$
Fréchet normal cone to Ω at x:

$$N_\Omega(x) := \left\{ x^* \in X^* \middle| \limsup_{u \to x,\, u \in \Omega \setminus \{x\}} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq 0 \right\}$$

Dual characterizations

X – Asplund space, $\Omega_1, \ldots, \Omega_m$ – closed

$$\hat{\theta}[\boldsymbol{\Omega}](\bar{x}) = \lim_{\delta \downarrow 0} \inf_{\substack{\sum_{i=1}^m \|x_i^*\|=1 \\ \omega_i \in \Omega_i \cap B_\delta(\bar{x}),\ x_i^* \in N_{\Omega_i}(\omega_i)\ (i=1,\ldots,m)}} \left\| \sum_{i=1}^m x_i^* \right\|$$

Dual characterizations

X – Asplund space, $\Omega_1, \ldots, \Omega_m$ – closed

$$\hat{\theta}[\boldsymbol{\Omega}](\bar{x}) = \lim_{\delta \downarrow 0} \inf_{\substack{\sum_{i=1}^m \|x_i^*\|=1 \\ \omega_i \in \Omega_i \cap B_\delta(\bar{x}),\ x_i^* \in N_{\Omega_i}(\omega_i)\ (i=1,\ldots,m)}} \left\| \sum_{i=1}^m x_i^* \right\|$$

Uniform regularity

$\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\exists \alpha, \delta > 0$ such that

$$\left\| \sum_{i=1}^m x_i^* \right\| \geq \alpha$$

$\forall \omega_i \in \Omega_i \cap B_\delta(\bar{x})$, $x_i^* \in N_{\Omega_i}(\omega_i)$ $(i = 1, \ldots, m)$ with $\sum_{i=1}^m \|x_i^*\| = 1$

Outline

Uniform regularity in Hilbert spaces

X – Hilbert space, Ω_1, Ω_2 – closed, $\boldsymbol{\Omega} := \{\Omega_1, \Omega_2\}$,
$\bar{x} \in \mathrm{bd}\,\Omega_1 \cap \mathrm{bd}\,\Omega_2$

Uniform regularity in Hilbert spaces

X – Hilbert space, Ω_1, Ω_2 – closed, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\}$,
$\bar{x} \in \mathrm{bd}\,\Omega_1 \cap \mathrm{bd}\,\Omega_2$

$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\hat{\theta}[\mathbf{\Omega}](\bar{x}) > 0$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \lim_{\delta\downarrow 0} \inf_{\substack{\|v_1\|+\|v_2\|=1 \\ \omega_i\in\Omega_i\cap B_\delta(\bar{x}),\ v_i\in N_{\Omega_i}(\omega_i)\ (i=1,2)}} \|v_1 + v_2\|$$

Uniform regularity in Hilbert spaces

X – Hilbert space, Ω_1, Ω_2 – closed, $\boldsymbol{\Omega} := \{\Omega_1, \Omega_2\}$,
$\bar{x} \in \mathrm{bd}\,\Omega_1 \cap \mathrm{bd}\,\Omega_2$

$\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\hat{\theta}[\boldsymbol{\Omega}](\bar{x}) > 0$

$$\hat{\theta}[\boldsymbol{\Omega}](\bar{x}) = \lim_{\delta\downarrow 0} \inf_{\substack{\|v_1\|+\|v_2\|=1 \\ \omega_i\in\Omega_i\cap B_\delta(\bar{x}),\, v_i\in N_{\Omega_i}(\omega_i)\,(i=1,2)}} \|v_1 + v_2\|$$

$$= \frac{1}{2} \lim_{\delta\downarrow 0} \inf_{\omega_i\in\Omega_i\cap B_\delta(\bar{x}),\, v_i\in N_{\Omega_i}(\omega_i)\cap\mathbb{S}\,(i=1,2)} \|v_1 + v_2\|$$

Uniform regularity in Hilbert spaces

X – Hilbert space, Ω_1, Ω_2 – closed, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\}$,
$\bar{x} \in \mathrm{bd}\,\Omega_1 \cap \mathrm{bd}\,\Omega_2$

$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\hat{\theta}[\mathbf{\Omega}](\bar{x}) > 0$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \lim_{\delta\downarrow 0} \inf_{\substack{\|v_1\|+\|v_2\|=1 \\ \omega_i\in\Omega_i\cap B_\delta(\bar{x}),\, v_i\in N_{\Omega_i}(\omega_i)\,(i=1,2)}} \|v_1 + v_2\|$$

$$= \frac{1}{2} \lim_{\delta\downarrow 0} \inf_{\omega_i\in\Omega_i\cap B_\delta(\bar{x}),\, v_i\in N_{\Omega_i}(\omega_i)\cap\mathbb{S}\,(i=1,2)} \|v_1 + v_2\|$$

$$\hat{c}[\mathbf{\Omega}](\bar{x}) = -\lim_{\delta\downarrow 0} \inf_{\omega_i\in\Omega_i\cap B_\delta(\bar{x}),\, v_i\in N_{\Omega_i}(\omega_i)\cap\mathbb{S}\,(i=1,2)} \langle v_1, v_2\rangle$$

Uniform regularity in Hilbert spaces

X – Hilbert space, Ω_1, Ω_2 – closed, $\boldsymbol{\Omega} := \{\Omega_1, \Omega_2\}$,
$\bar{x} \in \mathrm{bd}\,\Omega_1 \cap \mathrm{bd}\,\Omega_2$

$\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\hat{\theta}[\boldsymbol{\Omega}](\bar{x}) > 0$

$$\hat{\theta}[\boldsymbol{\Omega}](\bar{x}) = \lim_{\delta\downarrow 0} \inf_{\substack{\|v_1\|+\|v_2\|=1 \\ \omega_i\in\Omega_i\cap B_\delta(\bar{x}),\, v_i\in N_{\Omega_i}(\omega_i)\ (i=1,2)}} \|v_1 + v_2\|$$

$$= \frac{1}{2} \lim_{\delta\downarrow 0} \inf_{\omega_i\in\Omega_i\cap B_\delta(\bar{x}),\, v_i\in N_{\Omega_i}(\omega_i)\cap\mathbb{S}\ (i=1,2)} \|v_1 + v_2\|$$

$$\hat{c}[\boldsymbol{\Omega}](\bar{x}) = -\lim_{\delta\downarrow 0} \inf_{\omega_i\in\Omega_i\cap B_\delta(\bar{x}),\, v_i\in N_{\Omega_i}(\omega_i)\cap\mathbb{S}\ (i=1,2)} \langle v_1, v_2\rangle$$

$$2(\hat{\theta}[\boldsymbol{\Omega}](\bar{x}))^2 + \hat{c}[\boldsymbol{\Omega}](\bar{x}) = 1$$

Uniform regularity in Hilbert spaces

X – Hilbert space, Ω_1, Ω_2 – closed, $\boldsymbol{\Omega} := \{\Omega_1, \Omega_2\}$,
$\bar{x} \in \mathrm{bd}\,\Omega_1 \cap \mathrm{bd}\,\Omega_2$

$\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\hat{c}[\boldsymbol{\Omega}](\bar{x}) < 1$

Uniform regularity in Hilbert spaces

X – Hilbert space, Ω_1, Ω_2 – closed, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\}$,
$\bar{x} \in \operatorname{bd}\Omega_1 \cap \operatorname{bd}\Omega_2$

$\mathbf{\Omega}$ is uniformly regular at $\bar{x} \iff \hat{c}[\mathbf{\Omega}](\bar{x}) < 1$

Proposition

$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\exists c < 1,\ \delta > 0$ *s.t.*

$$-\langle v_1, v_2 \rangle < c \quad \forall v_1 \in N_{\Omega_1}(\omega_1) \cap \mathbb{S},\ v_2 \in N_{\Omega_2}(\omega_2) \cap \mathbb{S}$$

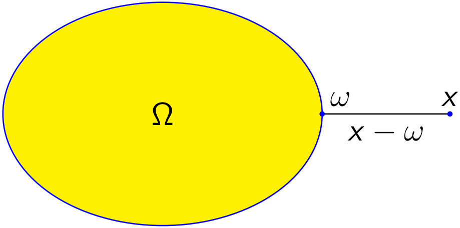
when $\omega_1 \in \Omega_1 \cap B_\delta(\bar{x}),\ \omega_2 \in \Omega_2 \cap B_\delta(\bar{x})$

Outline

Projections

X – Hilbert space, $\Omega \neq \emptyset$ – closed,

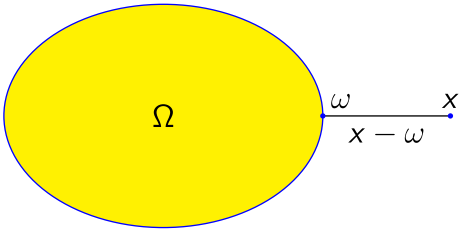
$$P_\Omega(x) := \{\omega \in \Omega \mid \|x - \omega\| = d(x, \Omega)\}$$



Projections

X – Hilbert space, $\Omega \neq \emptyset$ – closed,

$$P_\Omega(x) := \{\omega \in \Omega \mid \|x - \omega\| = d(x, \Omega)\}$$

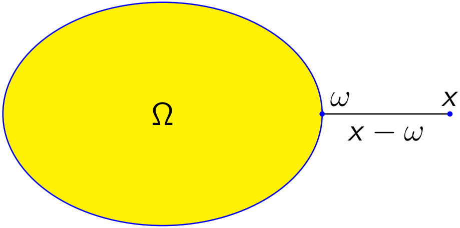


1. If $\dim X < \infty$, then $P_\Omega(x) \neq \emptyset$

Projections

X – Hilbert space, $\Omega \neq \emptyset$ – closed,

$$P_\Omega(x) := \{\omega \in \Omega \mid \|x - \omega\| = d(x, \Omega)\}$$

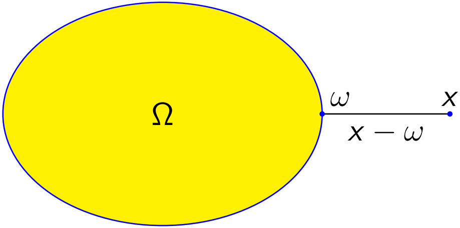


1. If $\dim X < \infty$, then $P_\Omega(x) \neq \emptyset$
2. If $\dim X < \infty$ and Ω is convex, then $P_\Omega(x)$ is a singleton

Projections

X – Hilbert space, $\Omega \neq \emptyset$ – closed,

$$P_\Omega(x) := \{\omega \in \Omega \mid \|x - \omega\| = d(x, \Omega)\}$$

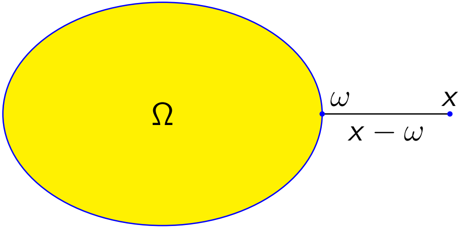


1. If $\dim X < \infty$, then $P_\Omega(x) \neq \emptyset$
2. If $\dim X < \infty$ and Ω is convex, then $P_\Omega(x)$ is a singleton
3. $\omega \in P_\Omega(x) \quad \Longrightarrow \quad x - \omega \in N_\Omega(\omega)$

Projections

X – Hilbert space, $\Omega \neq \emptyset$ – closed,

$$P_\Omega(x) := \{\omega \in \Omega \mid \|x - \omega\| = d(x, \Omega)\}$$

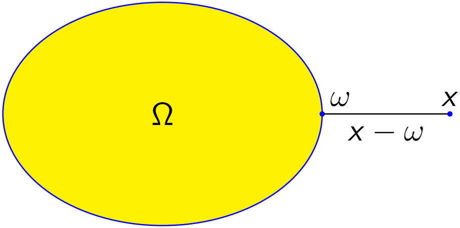


Proximal normal cone to Ω at $\omega \in \Omega$: $N^p_\Omega(\omega) := \mathrm{cone}\left(P_\Omega^{-1}(\omega) - \omega\right)$

Projections

X – Hilbert space, $\Omega \neq \emptyset$ – closed,

$$P_\Omega(x) := \{\omega \in \Omega \mid \|x - \omega\| = d(x, \Omega)\}$$



Proximal normal cone to Ω at $\omega \in \Omega$: $N^p_\Omega(\omega) := \mathrm{cone}\,\left(P^{-1}_\Omega(\omega) - \omega\right)$

$$N^p_\Omega(\omega) \subset N_\Omega(\omega)$$

Super-regularity of a set

X – Hilbert space, $\Omega \neq \emptyset$ – closed,

Definition (Lewis, Luke, Malick, 2009)

Ω is super-regular at $\bar{x} \in \Omega$ if

$$\langle x - \omega, u - \omega \rangle \leq \gamma \|x - \omega\| \, \|u - \omega\|$$

$\forall \gamma > 0$, $x \in X$ and $u \in \Omega$ near $\bar{x}$, $\omega \in P_\Omega(x)$

Super-regularity of a set

X – Hilbert space, $\Omega \neq \emptyset$ – closed,

Definition (Lewis, Luke, Malick, 2009)

Ω is super-regular at $\bar{x} \in \Omega$ if

$$\langle x - \omega, u - \omega \rangle \leq \gamma \|x - \omega\| \, \|u - \omega\|$$

$\forall \gamma > 0$, $x \in X$ and $u \in \Omega$ near $\bar{x}$, $\omega \in P_\Omega(x)$

Convexity $\Rightarrow$ Super-regularity

Alternating projections

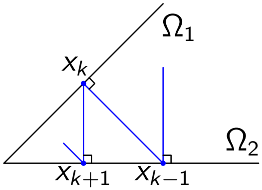
X – Hilbert space, Ω_1, Ω_2 – closed, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\} \subset X$, $\Omega_1 \cap \Omega_2 \neq \emptyset$

Problem: Find a point in $\Omega_1 \cap \Omega_2$

Definition

(x_k) is generated by the alternating projections for $\mathbf{\Omega}$ if

$$x_{2k+1} \in P_{\Omega_1}(x_{2k}) \quad \text{and} \quad x_{2k+2} \in P_{\Omega_2}(x_{2k+1}) \quad (k = 0, 1, \ldots)$$



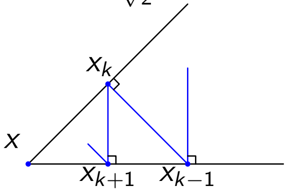
Linear convergence

Definition

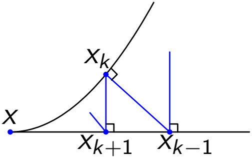
(x_k) linearly converges to x if there is a constant $c \in (0,1)$ s.t.

$$\|x_{k+1} - x\| \le c\|x_k - x\| \quad \forall k \text{ sufficiently large}$$

$c = \frac{1}{\sqrt{2}}$



$\lim_{k\to\infty} \frac{\|x_{k+1}-x\|}{\|x_k-x\|} = 1$



Alternating projections: linear convergence

$\dim X < \infty$, Ω_1, Ω_2 – closed, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\} \subset X$, $\Omega_1 \cap \Omega_2 \neq \emptyset$

Problem: Find a point in $\Omega_1 \cap \Omega_2$

History

- Ω_1, Ω_2 are convex and $\mathbf{\Omega}$ is subregular (Bauschke, Borwein, 1993)

Alternating projections: linear convergence

$\dim X < \infty$, Ω_1, Ω_2 – closed, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\} \subset X$, $\Omega_1 \cap \Omega_2 \neq \emptyset$

Problem: Find a point in $\Omega_1 \cap \Omega_2$

History

- Ω_1, Ω_2 are convex and $\mathbf{\Omega}$ is subregular
  (Bauschke, Borwein, 1993)
- Ω_1 is super-regular and $\mathbf{\Omega}$ is uniformly regular
  (Lewis, Luke, Malick, 2009)

Alternating projections: linear convergence

$\dim X < \infty$, Ω_1, Ω_2 – closed, $\boldsymbol{\Omega} := \{\Omega_1, \Omega_2\} \subset X$, $\Omega_1 \cap \Omega_2 \neq \emptyset$

Problem: Find a point in $\Omega_1 \cap \Omega_2$

History

- Ω_1, Ω_2 are convex and $\boldsymbol{\Omega}$ is subregular
  (Bauschke, Borwein, 1993)
- Ω_1 is super-regular and $\boldsymbol{\Omega}$ is uniformly regular
  (Lewis, Luke, Malick, 2009)
- Ω_1 is Ω_2-super-regular and $\boldsymbol{\Omega}$ is inherently transversal
  (Bauschke, Luke, Phan, Wang, 2013)

Alternating projections: linear convergence

$\dim X < \infty$, Ω_1, Ω_2 – closed, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\} \subset X$, $\Omega_1 \cap \Omega_2 \neq \emptyset$

Problem: Find a point in $\Omega_1 \cap \Omega_2$

History

- Ω_1, Ω_2 are convex and $\mathbf{\Omega}$ is subregular (Bauschke, Borwein, 1993)
- Ω_1 is super-regular and $\mathbf{\Omega}$ is uniformly regular (Lewis, Luke, Malick, 2009)
- Ω_1 is Ω_2-super-regular and $\mathbf{\Omega}$ is inherently transversal (Bauschke, Luke, Phan, Wang, 2013)
- $\mathbf{\Omega}$ is intrinsically transversal (Drusvyatskiy, Ioffe, Lewis; preprint 2014)

Alternating projections: linear convergence

X – Hilbert space, Ω_1, Ω_2 – closed, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\}$, $\Omega_1 \cap \Omega_2 \neq \emptyset$

Problem: Find a point in $\Omega_1 \cap \Omega_2$

Theorem (Lewis, Luke, Malick, 2009)

Suppose that $\mathbf{\Omega}$ is uniformly regular at $\bar{x} \in \Omega_1 \cap \Omega_2$ and Ω_1 is super-regular at $\bar{x}$. Then, for any $c \in (\hat{c}[\Omega_1, \Omega_2](\bar{x}), 1)$, a sequence generated by alternating projections for $\mathbf{\Omega}$ linearly converges to a point in $\Omega_1 \cap \Omega_2$ with rate $\sqrt{c}$, provided x_0 is close enough to $\bar{x}$

References

- A. Y. Kruger and B. S. Mordukhovich, *Extremal points and the Euler equation in nonsmooth optimization*, Dokl. Akad. Nauk BSSR **24**:8 (1980), 684–687, in Russian.
- A. Y. Kruger, *Stationarity and regularity of set systems*, Pacif. J. Optimiz. **1** (2005), 101–126.
- A. Y. Kruger, *About regularity of collections of sets*, Set-Valued Anal. **14** (2006), 187–206.
- A. Y. Kruger and N. H. Thao, *About uniform regularity of collections of sets*, Serdica Math. J. **39** (2013), 287–312.
- A. Y. Kruger and N. H. Thao, *Quantitative characterizations of regularity properties of collections of sets*, J. Optim. Theory Appl. **164** (2015), 41–67.
- A. Y. Kruger and N. H. Thao, *Regularity of collections of sets and convergence of inexact alternating projections*, Submitted.

References

- H. H. Bauschke and J. M. Borwein, *On the convergence of von Neumann's alternating projection algorithm for two sets*. Set-Valued Anal. **1** (1993), 185–212.
- A. Lewis, D. Luke, and J. Malick, *Local linear convergence for alternating and averaged nonconvex projections*, Found. Comput. Math. **9** (2009), 485–513.
- H. H. Bauschke, D. R. Luke, M. H. Phan, and X. Wang, *Restricted normal cones and the method of alternating projections: theory*, Set-Valued Var. Anal. **21** (2013), 431–473.
- D. Drusvyatskiy, A. D. Ioffe, and A. S. Lewis, *Alternating projections and coupling slope*, arXiv: 1401.7569 (2014).
- D. Noll and A. Rondepierre, *On local convergence of the method of alternating projections*, arXiv: 1312.5681v2 (2014).


Thank you