CODERIVATIVE CHARACTERIZATIONS OF MAXIMAL MONOTONICITY

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MONOTONICITY AND HYPOMONOTONICITY

Let $T\colon X \rightrightarrows X$ be a set-valued operator in a Hilbert space

DEFINITION We say that

(i) T is globally monotone on X if

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \quad \text{for all} \quad (u_1, v_1),\ (u_2, v_2) \in \operatorname{gph} T$$

T is said to be globally maximal monotone on X if in addition we have $\operatorname{gph} T = \operatorname{gph} S$ whenever S is monotone with $\operatorname{gph} T \subset \operatorname{gph} S$

(ii) T globally hypomonotone on X if there is $r > 0$ such that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2 \quad \text{for all} \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T$$

(iii) T semilocally hypomonotone at $\bar{x} \in \operatorname{dom} T$ if there exist a neighborhood U of $\bar{x}$ and a number $r > 0$ such that

$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2$ for $(u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times X)$

T is semilocally hypomonotone on a set Ω if it has this property at every point $\bar{x} \in \Omega$

Hypomonotonicity properties are not restrictive. In particular, semilocal hypomonotonicity holds for Lipschitzian single-valued mappings, for subdifferential mappings generated by the so-called lower-$\mathcal{C}^2$ (subsmooth) functions on open sets, etc. The local hypomonotonicity considered below holds for subdifferential mappings generated by any prox-regular and subdifferentially continuous extended-real-valued function

CODERIVATIVES

Given $T\colon X \rightrightarrows X$ and $(\bar{x},\bar{y}) \in \operatorname{gph} T$, the regular coderivative of T at $(\bar{x},\bar{y})$ is defined by

$$\widehat{D}^*T(\bar{x},\bar{y})(u) := \left\{ v \in X \,\middle|\, \limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\ y\in T(x)}} \frac{\langle u, x-\bar{x}\rangle - \langle v, y-\bar{y}\rangle}{\|x-\bar{x}\| + \|y-\bar{y}\|} \le 0 \right\}$$

The mixed limiting coderivative of T at $(\bar{x},\bar{y})$ is

$$D^*_M T(\bar{x},\bar{y})(\bar{u}) := w - \operatorname*{Lim\,sup}_{\substack{(x,y)\to(\bar{x},\bar{y})\\ u\to\bar{u}}} \widehat{D}^*T(x,y)(u)$$

$$= \left\{ v \,\middle|\, \exists \text{ seqs. } (x_k,y_k)\to(\bar{x},\bar{y}),\ v_k\in\widehat{D}^*T(x_k,y_k)(u_k),\ u_k\to\bar{u},\ v_k \overset{w}{\to} v \right\}$$

The mixed coderivative $D^*_M T$ enjoys full pointwise calculus

REG. CODERIVATIVE CHARACT. OF MAX MONOTONICITY

THEOREM Let T be a set-valued mapping with closed graph. The following assertions are equivalent

(i) T is globally maximal monotone on X

(ii) T is globally hypomonotone on X and for any $(u,v) \in \mathrm{gph}\, T$

$$\langle z, w\rangle \geq 0 \quad \text{whenever} \quad z \in \widehat{D}^*T(u,v)(w)$$

If the domain of T is convex, then the global hypomonotonicity in **(ii)** can be replaced by the semilocal one

MIXED CODERIVATIVE CHARACT. OF MAX MONOTONICITY

THEOREM Let T be a set-valued mapping with closed graph. The following assertions are equivalent

(i) T is globally maximal monotone on X

(ii) T is globally hypomonotone on X and for any $(u,v)\in\mathrm{gph}\,T$

$$\langle z,w\rangle\ge 0 \quad\text{whenever}\quad z\in D^*_M T(u,v)(w)$$

If the domain of T is convex, then the global hypomonotonicity in **(ii)** can be replaced by the semilocal one

Examples shows that the hypomonotonicity conditions are essential for coderivative characterizations of maximal monotonicity

CHARACTERIZATIONS OF STRONG MAX MONOTONICITY

$T\colon X \rightrightarrows X$ is globally strongly maximal monotone with modulus $\kappa > 0$ if it is maximal monotone and $T - \kappa I$ is globally monotone

COROLLARY Let T be of closed graph. Then the following assertions are equivalent

(i) T is globally strongly maximal monotone with modulus $\kappa > 0$

(ii) T is globally hypomonotone on X and for any $(u, v) \in \text{gph}\, T$

$$\langle z, w\rangle \geq \kappa \|w\|^2 \ \text{ whenever } \ z \in \widehat{D}^* T(u, v)(w),\ w \in X$$

(iii) T is globally hypomonotone on X and for any $(u, v) \in \text{gph}\, T$

$$\langle z, w\rangle \geq \kappa \|w\|^2 \ \text{ whenever } \ z \in D^*_M T(u, v)(w),\ w \in X$$

If the $\operatorname{dom} T$ is convex, the global hypomonotonicity in assertions **(ii)** and **(iii)** can be equivalently replaced by the semilocal one

LOWER-$\mathcal{C}^2$ FUNCTIONS

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower-$\mathcal{C}^2$ if for each $\bar{x} \in \mathbb{R}^n$ there is a neighborhood V of $\bar{x}$ on which f admits the representation

$$f(x) = \max_{t \in T} f_t(x), \quad x \in V$$

where f_t are of class $\mathcal{C}^2$ on V, T is compact, and $f_t(x)$ and all their partial derivatives in x through the second order depend continuously on $(t, x) \in T \times V$

This class of subsmooth functions is among the most favorable classes of functions in variational analysis and optimization. In particular, it includes maximum functions of the type

$$f(x) := \max\Big\{f_1(x), \ldots, f_m(x)\Big\}$$

where each function f_i is of class $\mathcal{C}^2$

SUBDIFFERENTIALS

Let $f\colon I\!R^n \to \overline{I\!R} := (-\infty, \infty]$ with $\bar{x} \in \text{dom}\, f$

(i) The (basic, limiting) first-order subdifferential of f at $\bar{x}$ is

$$\partial f(\bar{x}) := \Big\{ v \in I\!R^n \Big|\ \exists x_k \to \bar{x},\ f(x_k) \to f(\bar{x}),\ v_k \to v \ \text{ s.t. } \liminf_{x \to x_k} \frac{f(x) - f(x_k) - \langle v, x - x_k \rangle}{\|x - x_k\|} \ge 0 \Big\}$$

(ii) The basic second-order subdifferential of f at $\bar{x}$ relative to the subgradient $\bar{v} \in \partial f(\bar{x})$ is

$$\partial^2 f(\bar{u}, \bar{v})(w) := \Big(D^* \partial f \Big)(\bar{u}, \bar{v})(w), \quad w \in I\!R^n$$

(iii) The modified or combined second-order subdifferential of f at $\bar{x}$ relative to the subgradient $\bar{v}$ is

$$\widetilde{\partial}^2 f(\bar{x},\bar{v})(w) := \left(\widehat{D}^*\partial f\right)(\bar{x},\bar{v})(w), \quad w \in I\!R^n$$

For $\mathcal{C}^2$ function we have

$$\partial^2 f(\bar{x},\bar{v})(w) = \widetilde{\partial}^2 f(\bar{x},\bar{v})(w) = \left\{\nabla^2 f(\bar{x})^* w\right\} = \left\{\nabla^2 f(\bar{x}) w\right\}$$

in terms of the classical (symmetric) Hessian.

Well-developed calculus is available for $\partial^2 f$ in rather general settings of prox-regular functions

SECOND-ORDER CHARACTERIZATIONS OF CONVEXITY

THEOREM Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a lower-$\mathcal{C}^2$ function. Then the following assertions are equivalent

(i) f is convex on $\mathbb{R}^n$

(ii) For each $(u,v) \in \operatorname{gph} \partial f$ we have

$$\langle z, w\rangle \geq 0 \;\text{ whenever }\; z \in \partial^2 f(u,v)(w),\ w \in \mathbb{R}^n$$

(iii) For each $(u,v) \in \operatorname{gph} \partial f$ we have

$$\langle z, w\rangle \geq 0 \;\text{ whenever }\; z \in \widetilde{\partial}^2 f(u,v)(w),\ w \in \mathbb{R}^n$$

SECOND-ORDER CHARACT. OF STRONG CONVEXITY

f is strongly convex on $I\!R^n$ with modulus $\kappa > 0$ if

$$f\Big(t\lambda x + (1-\lambda)y\Big) \le tf(x) + (1-\lambda)f(y) - \frac{\kappa}{2}\lambda(1-\lambda)\|x-y\|^2,\ x,y \in I\!R^n$$

whenever $\lambda \in (0,1)$

COROLLARY If f is lower $\mathcal{C}^2$, the following are equivalent

(i) f is strongly convex on $I\!R^n$ with modulus κ

(ii) We have the second-order subdifferential condition

$\langle z, w\rangle \ge \kappa\|w\|^2$ for all $z \in \partial^2 f(u,v)(w),\ (u,v) \in \text{gph}\,\partial f,\ w \in I\!R^n$

(iii) We have the modified second-order subdifferential condition

$$\langle z, w\rangle \ge \kappa\|w\|^2 \text{ for all } z \in \widetilde{\partial}^2 f(u,v)(w),\ (u,v) \in \text{gph}\,\partial f,\ w \in I\!R^n$$

LOCAL MONOTONICITY

DEFINITION Let $T\colon X \rightrightarrows X$ be a set-valued operator in a Hilbert space, and let $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. We say that

(i) T is locally strongly monotone around $(\bar{x}, \bar{v})$ with modulus $\kappa > 0$ if there is a neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that

$\langle v_1 - v_2, u_1 - u_2 \rangle \geq \kappa \|u_1 - u_2\|^2$ for all $(u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times V)$

(ii) T is locally strongly maximal monotone around $(\bar{x}, \bar{v})$ with modulus $\kappa > 0$ if there is a neighborhood $U \times V$ such that the above inequality holds and that $\operatorname{gph} T \cap (U \times V) = \operatorname{gph} S \cap (U \times V)$ for any monotone operator S with $\operatorname{gph} T \cap (U \times V) \subset \operatorname{gph} S$

(iii) T is locally hypomonotone around $(\bar{x}, \bar{v})$ if there is a neighborhood $U \times V$ of this point and $r > 0$ such that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2, \ (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times V)$$

NEIGH. CHARACT. OF LOCAL STRONG MAX MONOTONICITY

Theorem Let $T\colon X \rightrightarrows X$ be of closed graph around the point $(\bar{x},\bar{v}) \in \text{gph}\, T$. The following are equivalent

(i) T is locally strongly maximal monotone around $(\bar{x},\bar{v})$ with modulus $\kappa > 0$

(ii) T is locally hypomonotone around $(\bar{x},\bar{v})$ and there is $\eta > 0$ such that

$\langle z, w\rangle \geq \kappa\|w\|^2$ for all $z \in \widehat{D}^*T(u,v)(w),\ (u,v) \in \text{gph}\, T \cap B_\eta(\bar{x},\bar{v})$

The conditions in **(ii)** ensure the strong metric regularity of T around $(\bar{x},\bar{v})$ with modulus κ^{-1}

POINTWISE CHARACT. OF LOCAL STRONG MAX MONOTON.

Theorem Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous around $\bar{x}$. The following are equivalent

(i) T is locally strongly monotone around $(\bar{x}, T(\bar{x}))$ with some modulus $\kappa > 0$

(ii) $D^*T(\bar{x})$ is positive-definite in the sense that

$$\langle z, w\rangle > 0 \text{ whenever } z \in D^*T(\bar{x})(w),\ w \neq 0$$

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