Computational NSO

(NonSmooth Optimization)

a tutorial focusing on bundle methods

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SPCOM Tutorial, Adelaide, Feb 8th, 2015

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Computational NSO: what does it mean?

For the unconstrained problem

$$\min f(x),$$

where f is convex but not differentiable at some points

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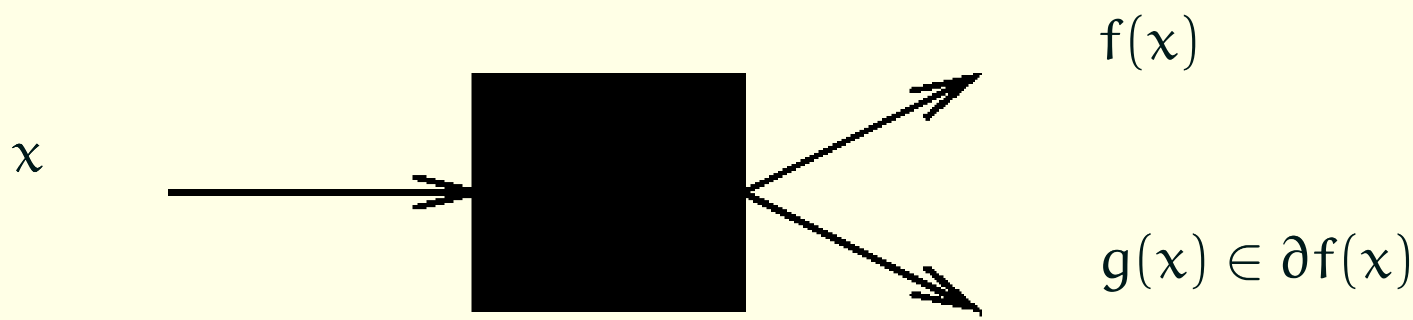

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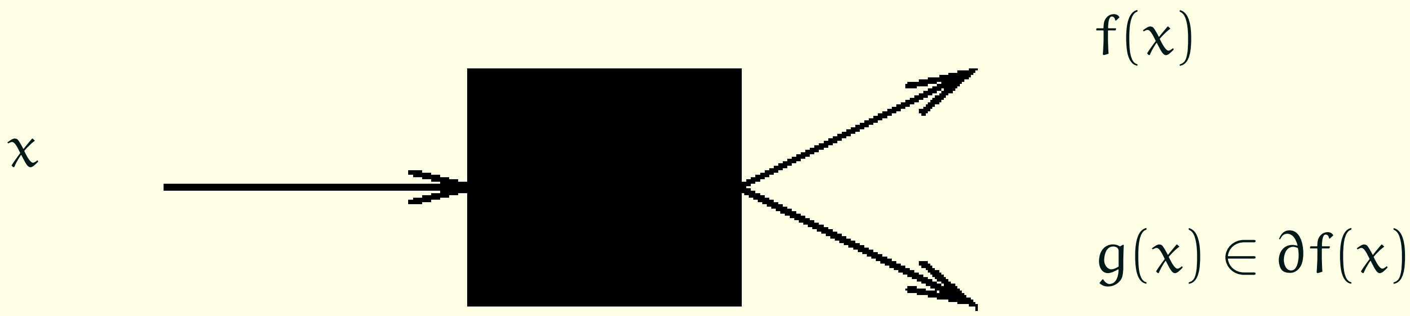
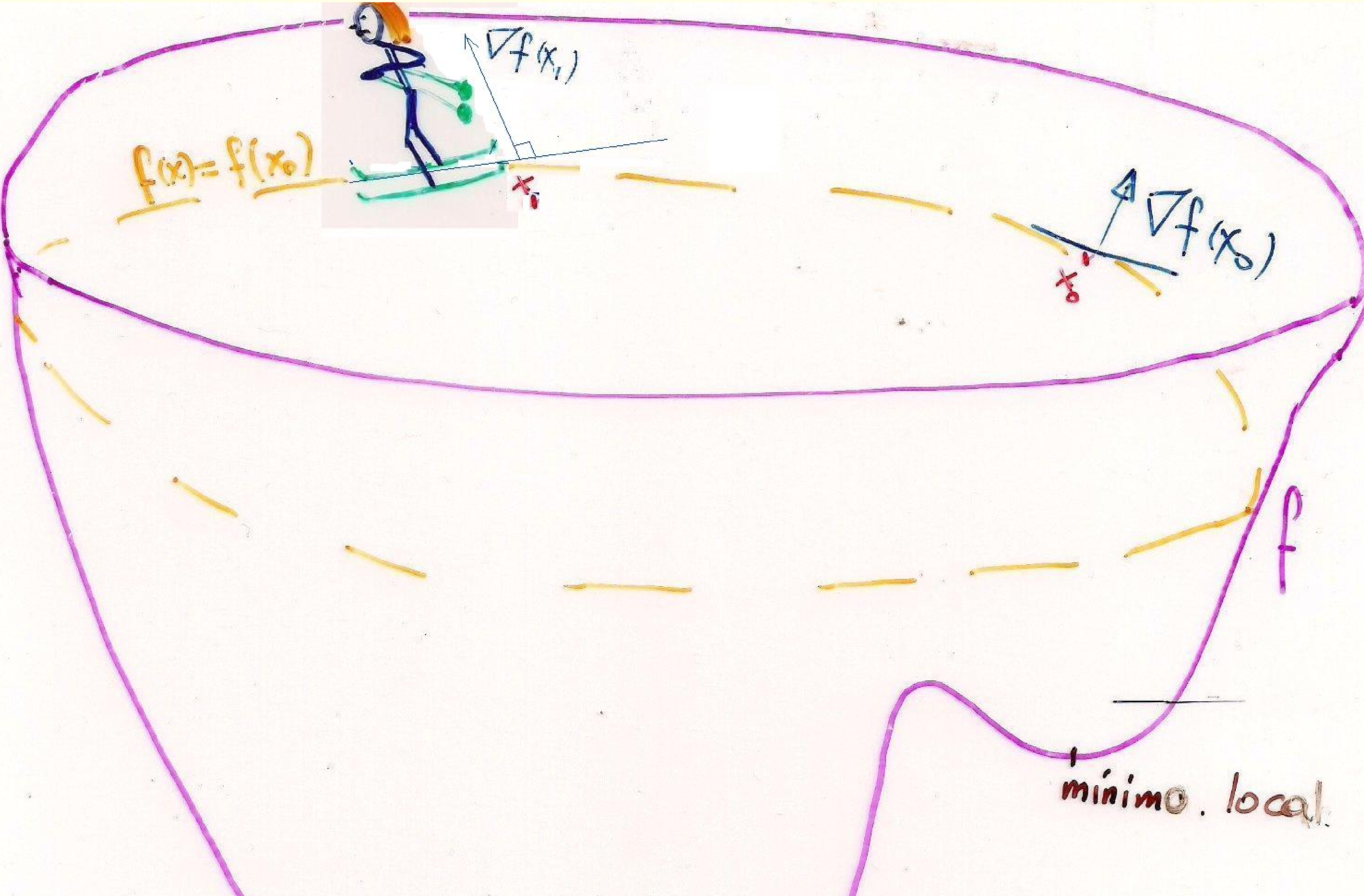
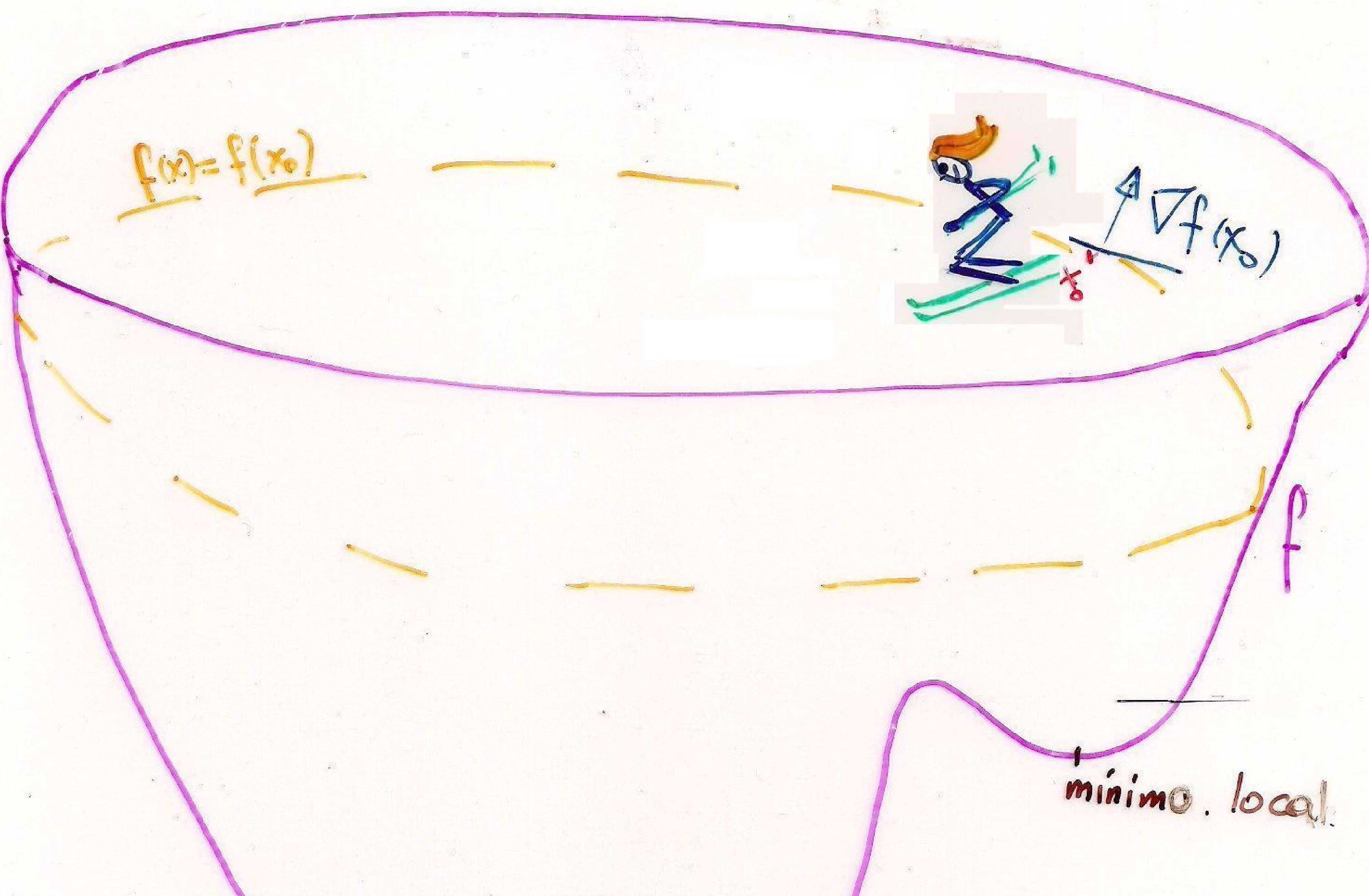


Relation with this morning tutorial?



$\nabla f(x_1)$
$f(x)=f(x_0)$
x_0
$\nabla f(x_0)$
x_0
f
mínimo. local.



f(x) = f(x0)
∇f(x0)
x0
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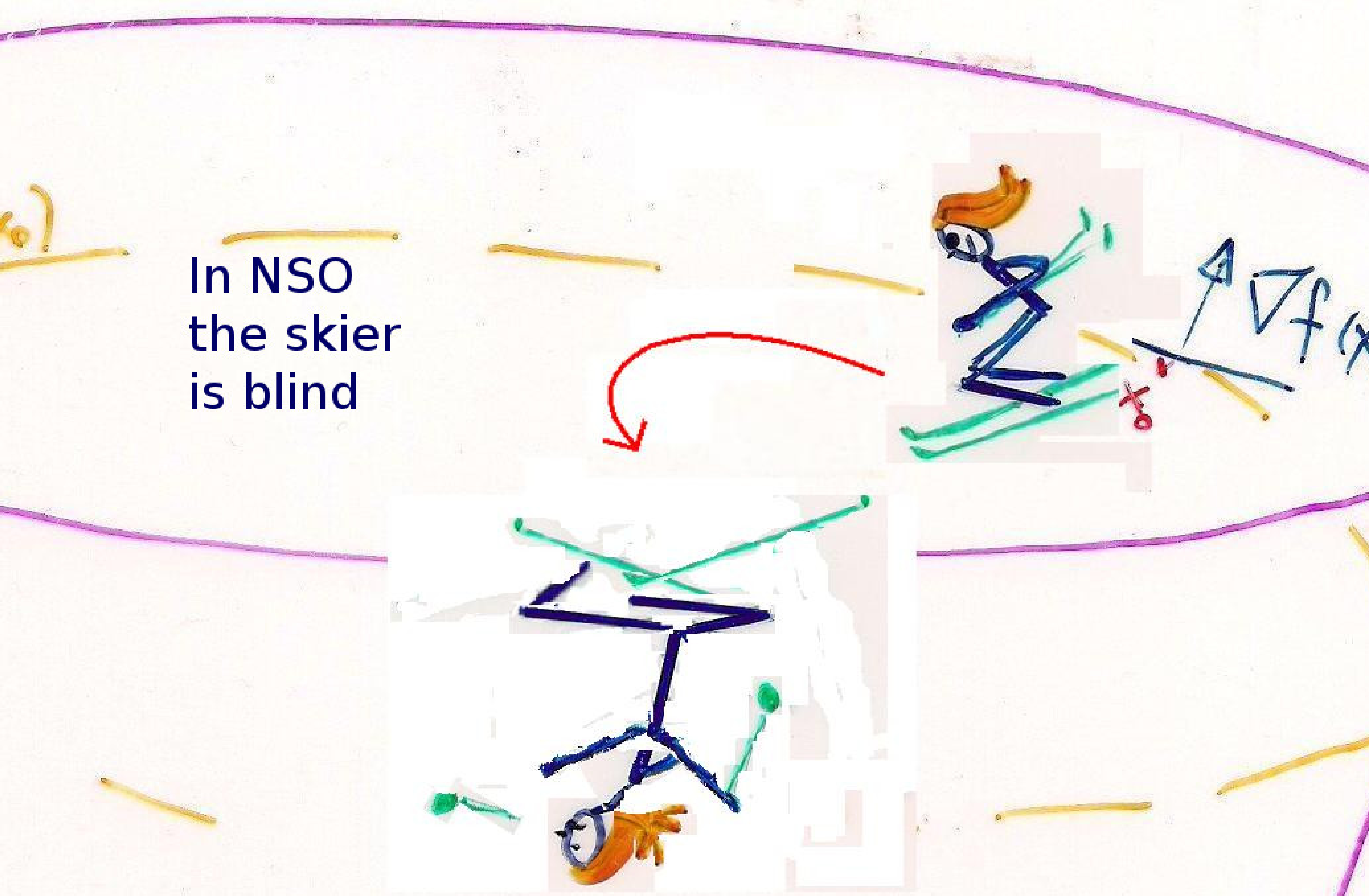
In NSO
the skier
is blind



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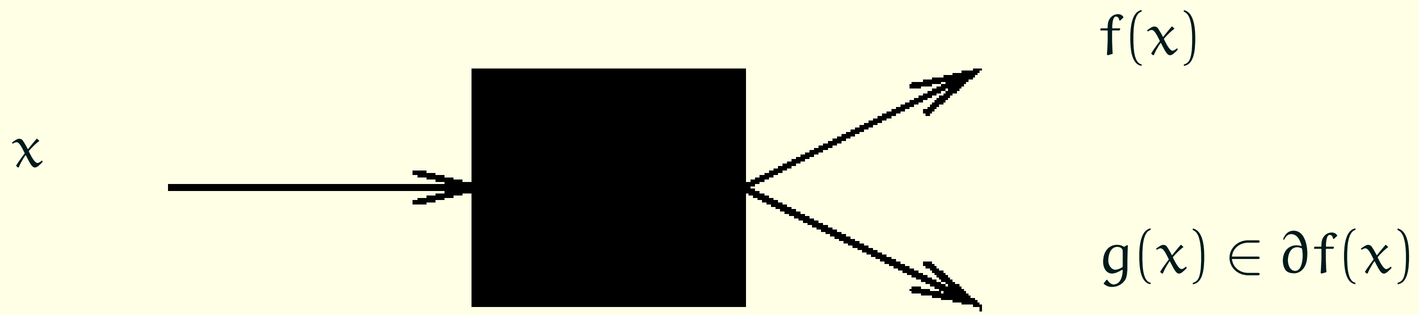

What do we mean by an algorithm?

An example

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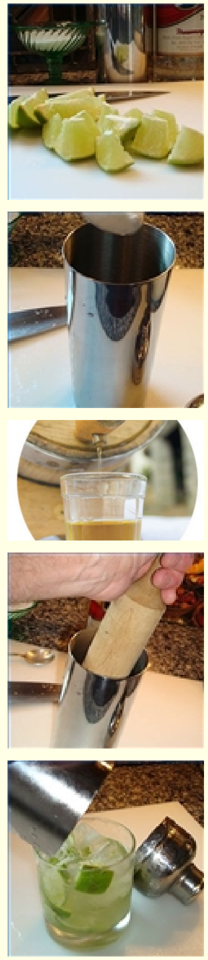

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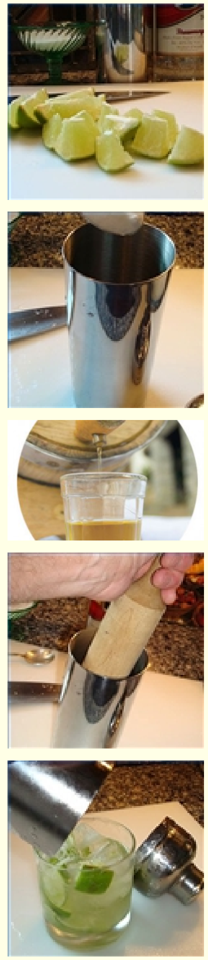

repeat until . . . ??

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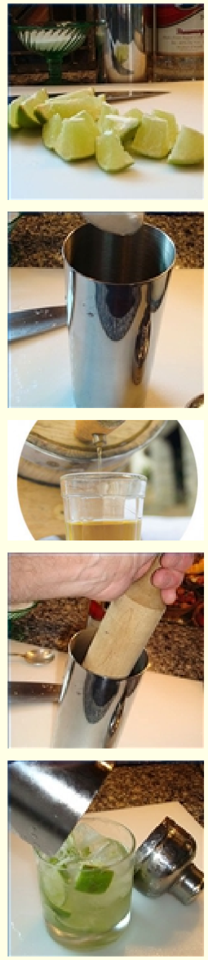

An algorithm

is a sequence of steps

that are repeated

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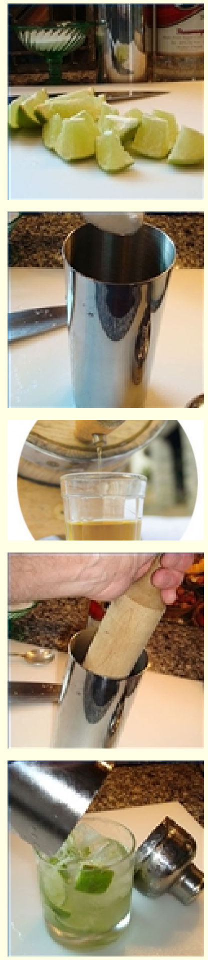

An algorithm

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Back to Computational NSO

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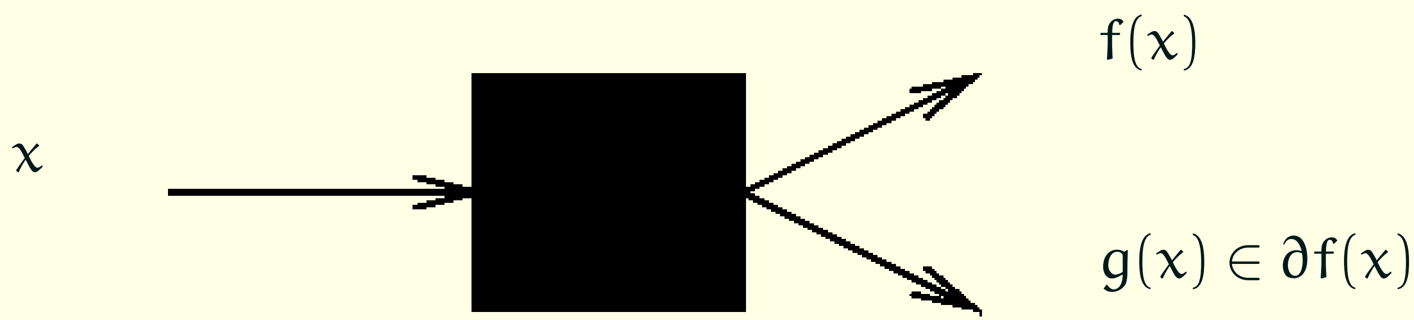
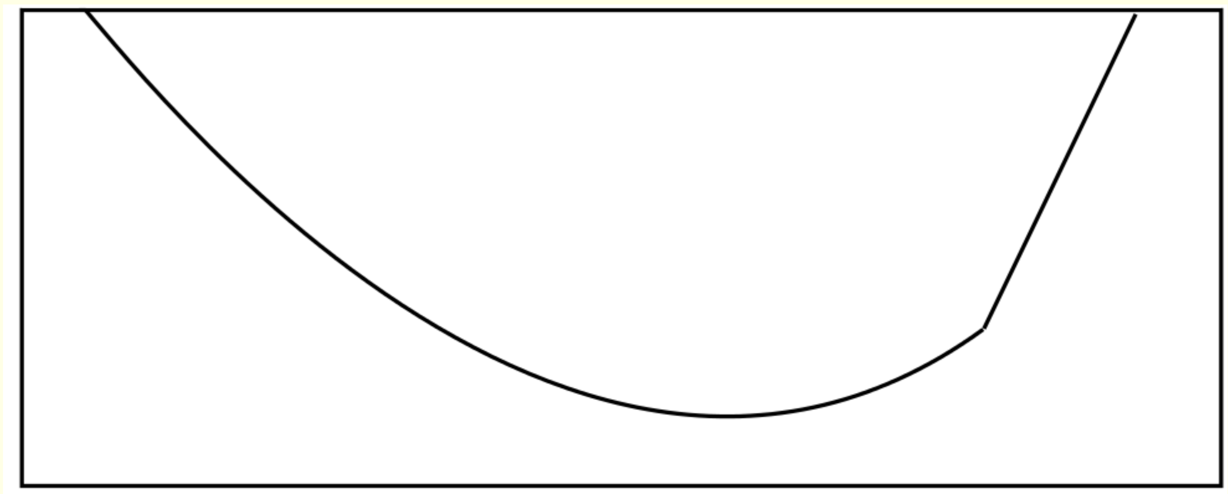


endowed with reliable **stopping tests**

A quick overview of Convex Analysis

An example of a convex nonsmooth function



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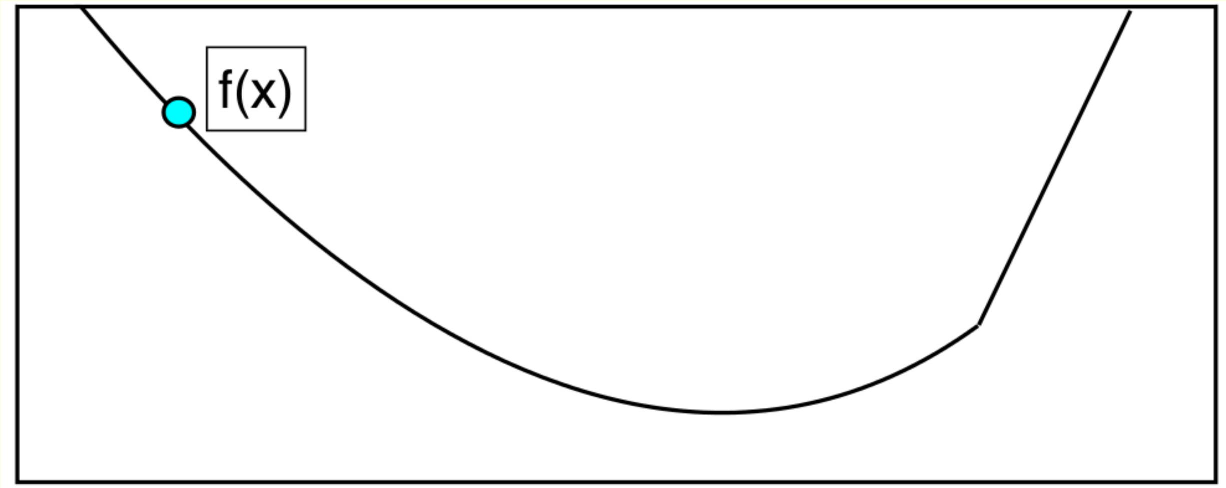

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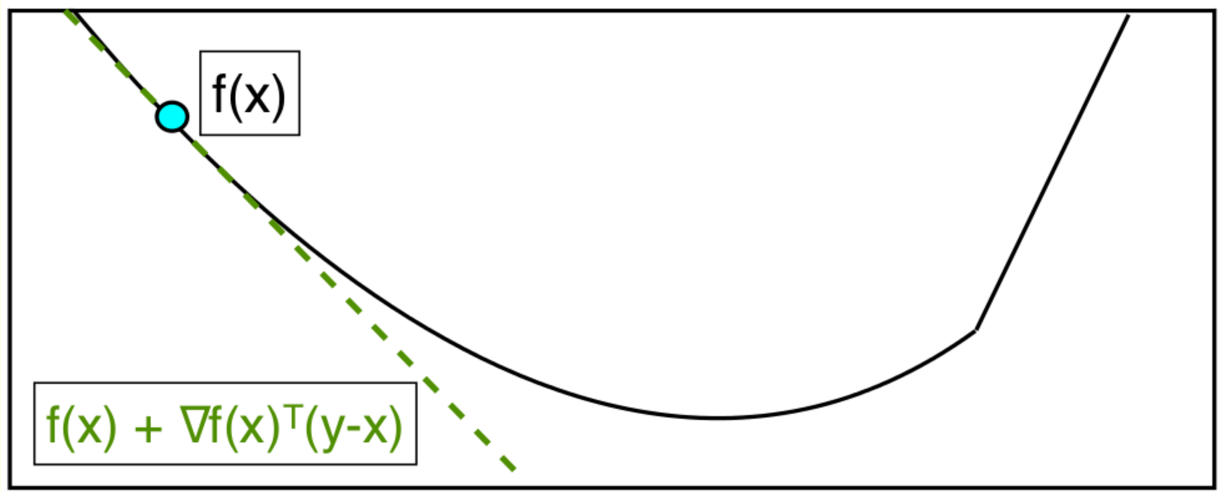

$$\begin{aligned} \partial f(x) &= \{\nabla f(x)\} \\ &= \{\text{slopes of linearizations supporting } f\text{, tangent at } x\} \end{aligned}$$

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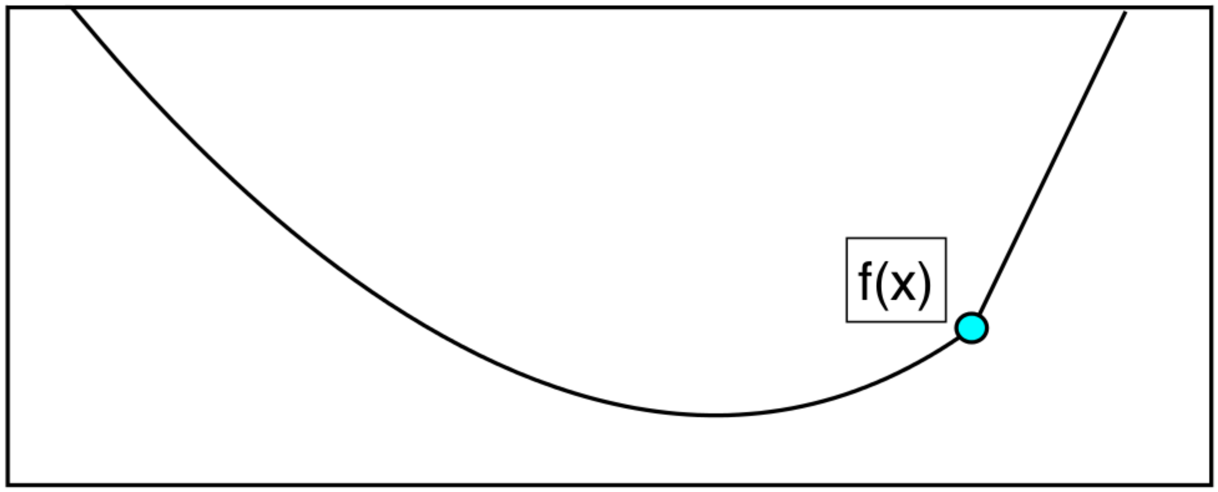

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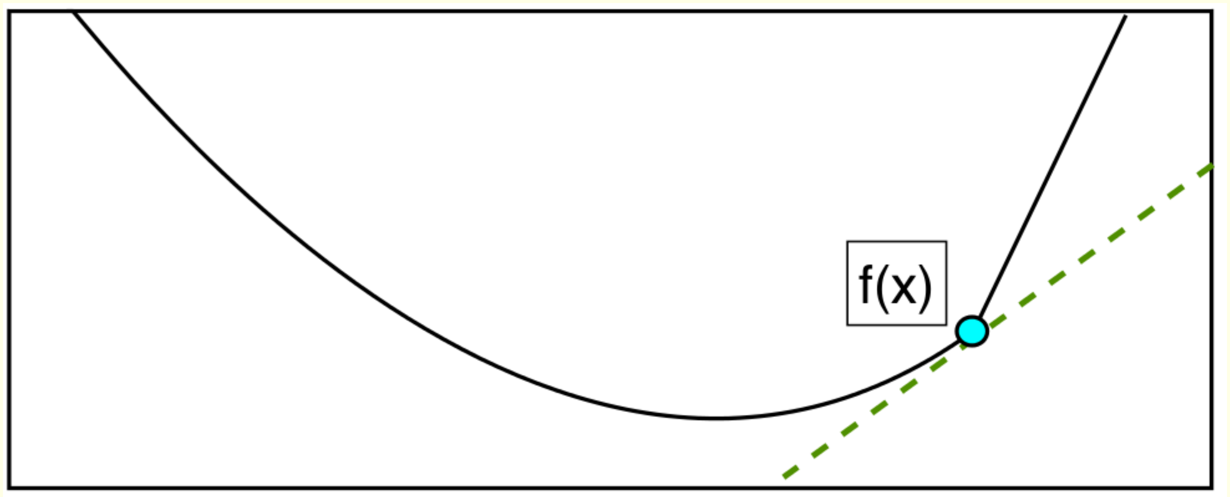

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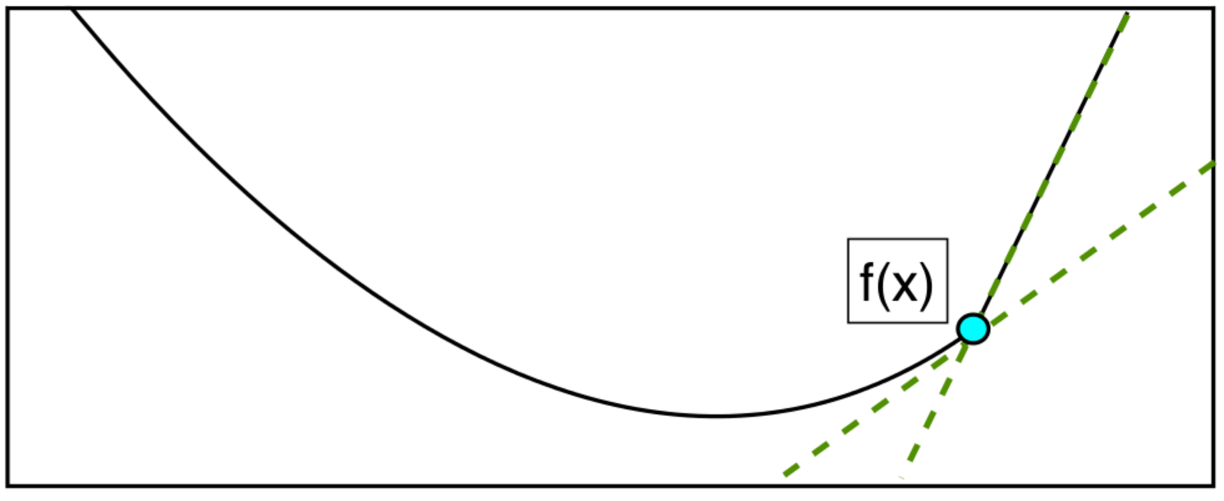

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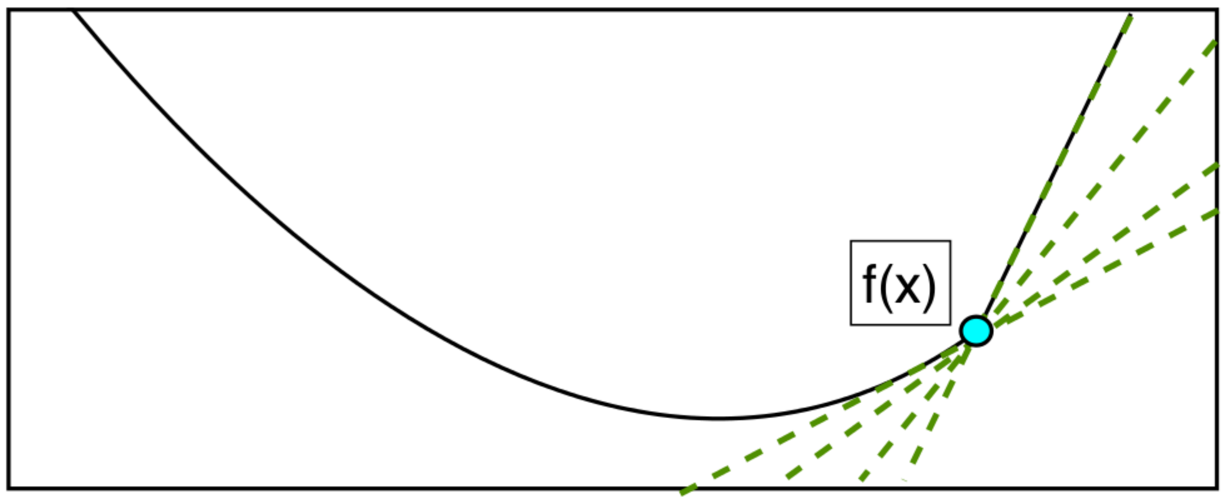

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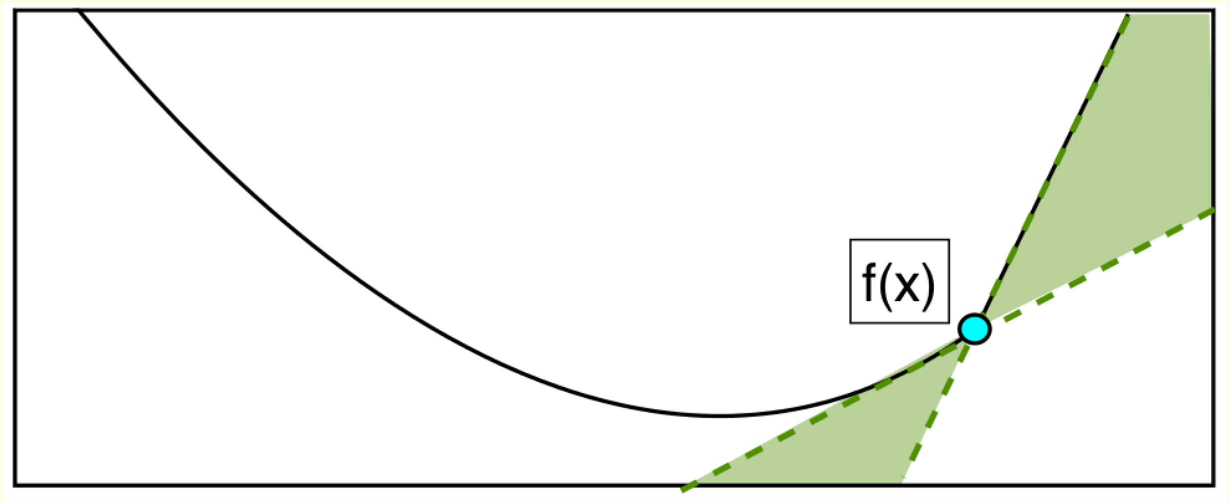

$$\partial f(x) \quad = \quad \{g \in \mathbb{R}^n : f(y) \geq f(x) + g^\top (y - x) \text{ for all } y\}$$

f x

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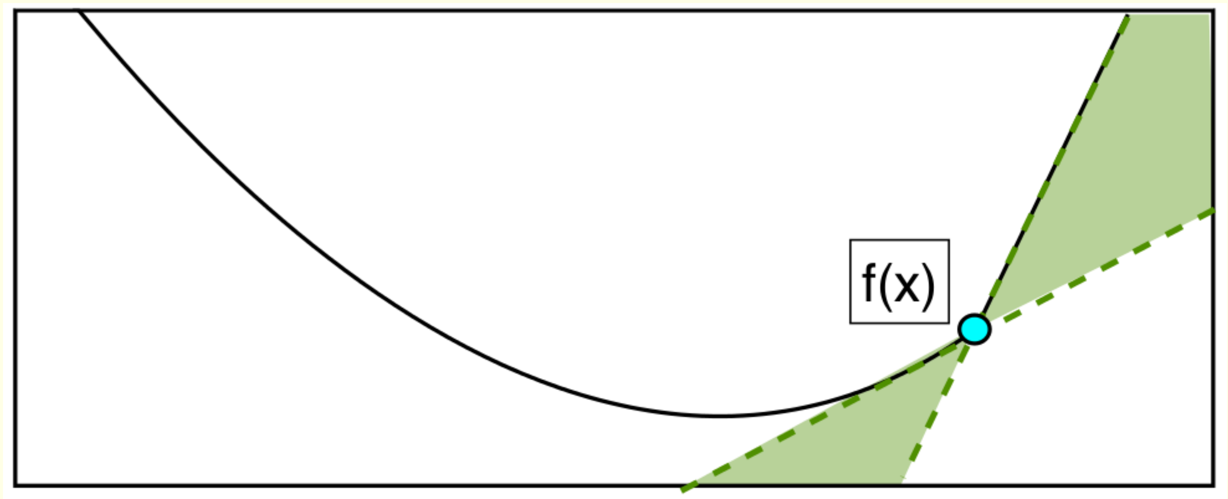

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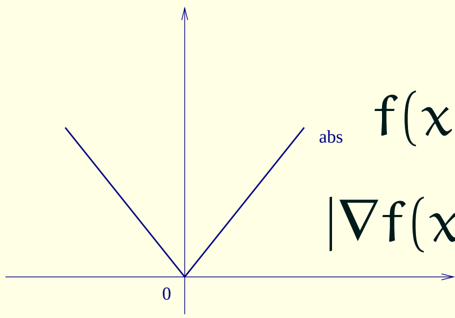
Why special NSO methods?

Smooth optimization methods **do not work**



$$f(x) = |x|$$

$$|\nabla f(x^k)| = 1,\ \forall x \neq 0 \quad \partial f(0) = [-1, 1]$$

Smooth stopping test **fails**: $|\nabla f(x^k)| \leq$TOL $\qquad (\leftrightarrow |g(x^k)| \leq$TOL$)$

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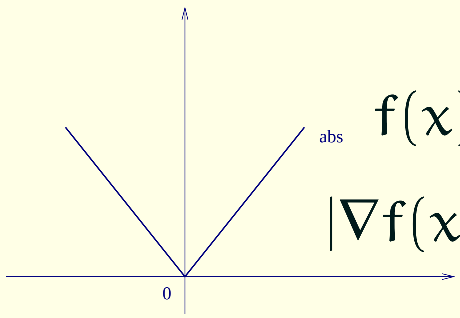

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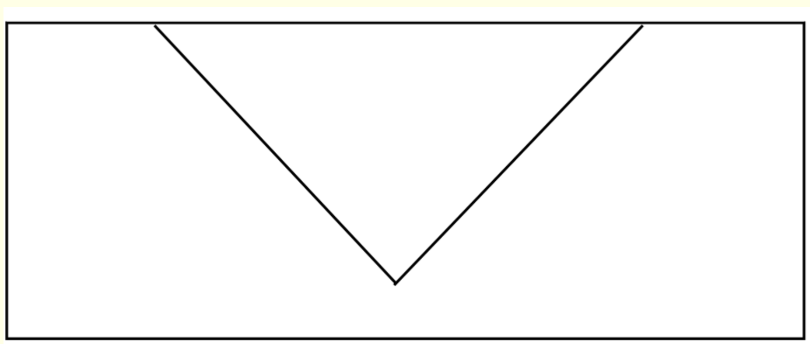

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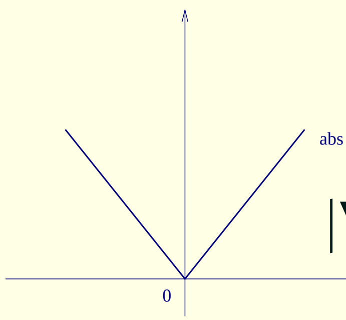

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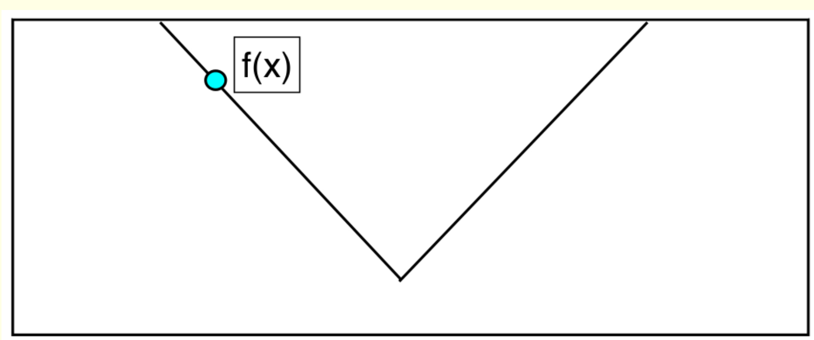

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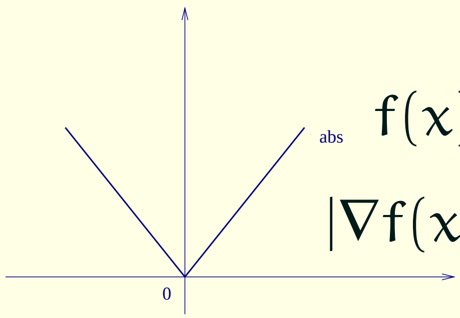

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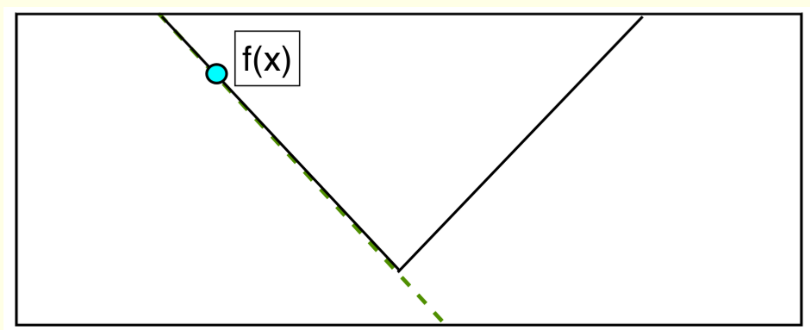

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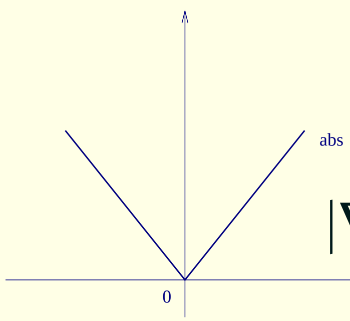

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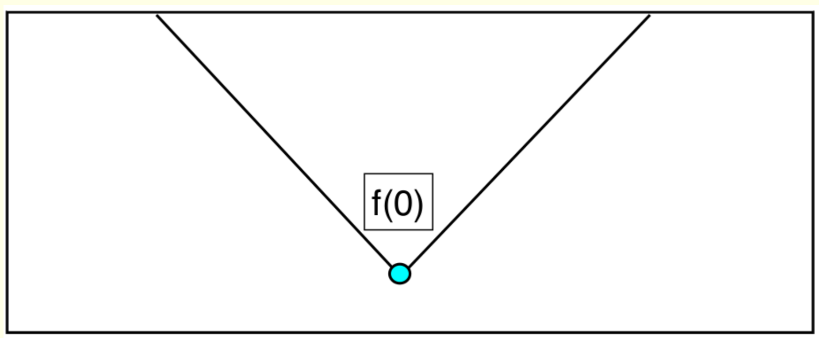

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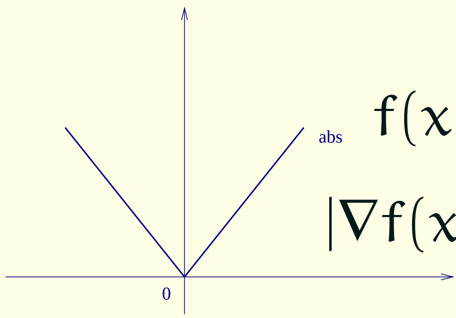

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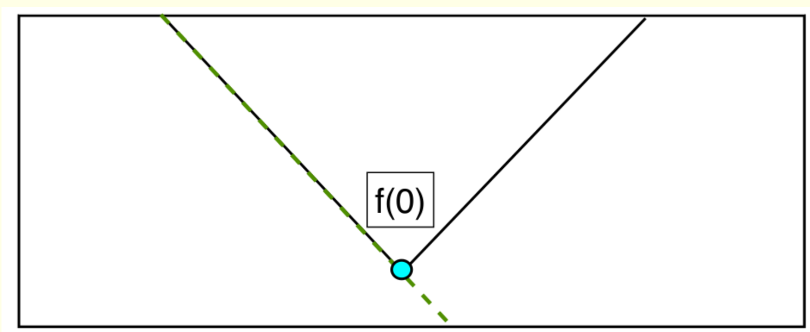

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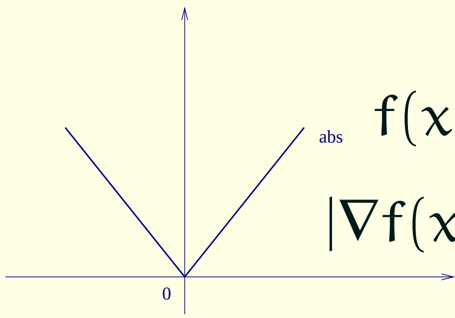

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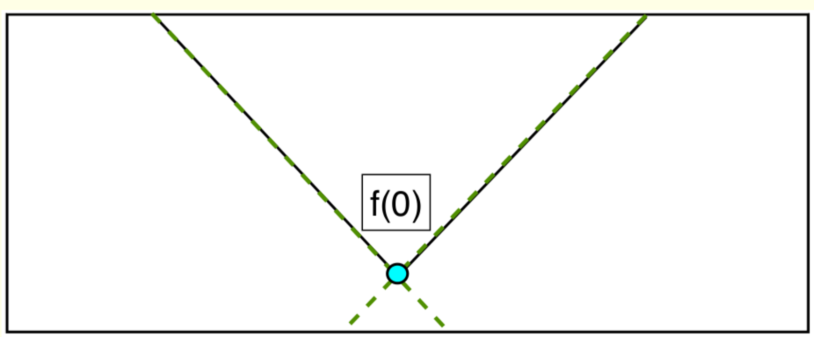

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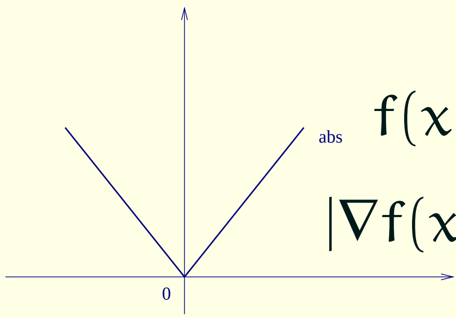

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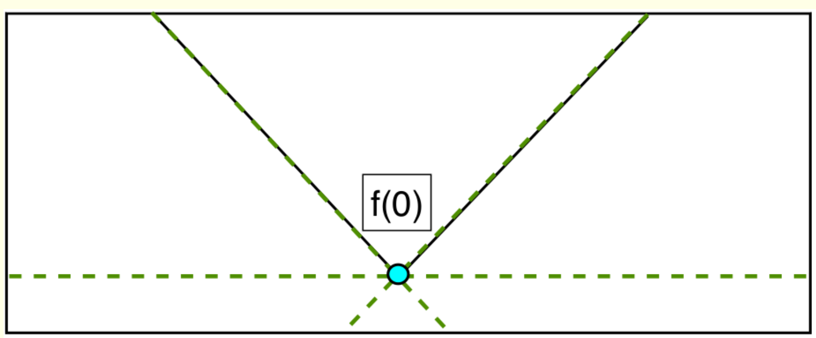

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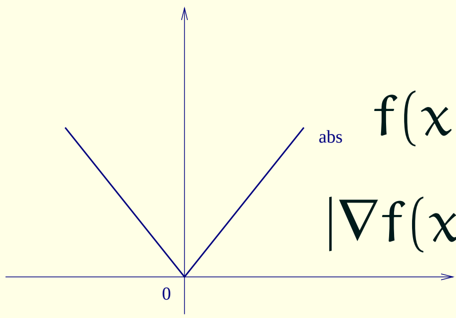

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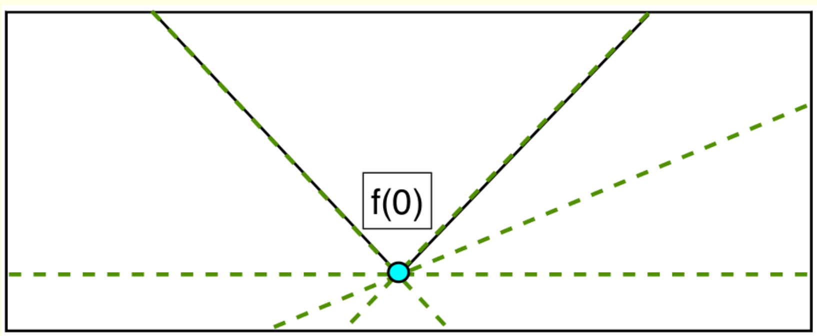

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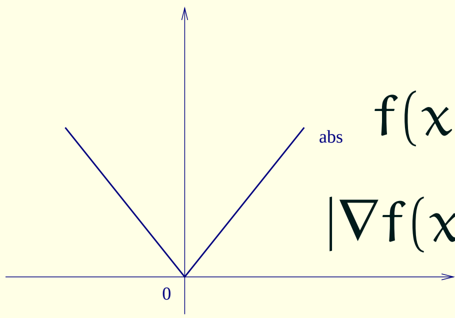

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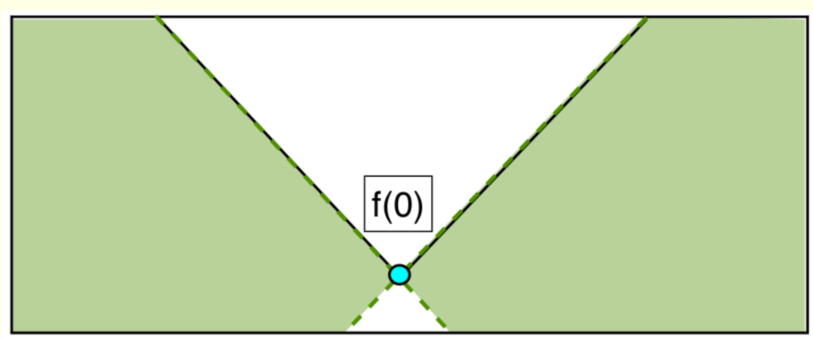

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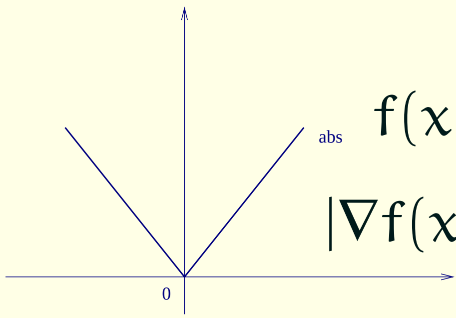

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Finite difference approximations **fail** **(no automatic differentiation)**

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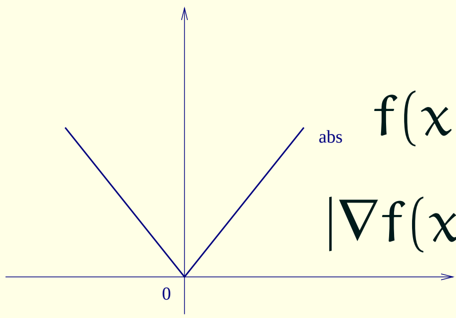

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Linesearches get trapped in kinks and **fail**

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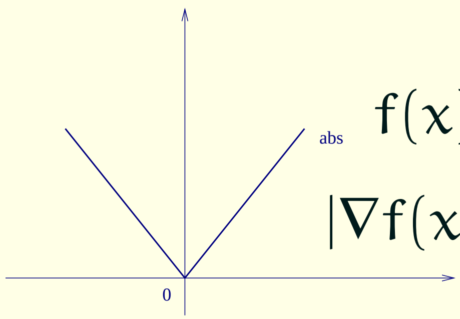

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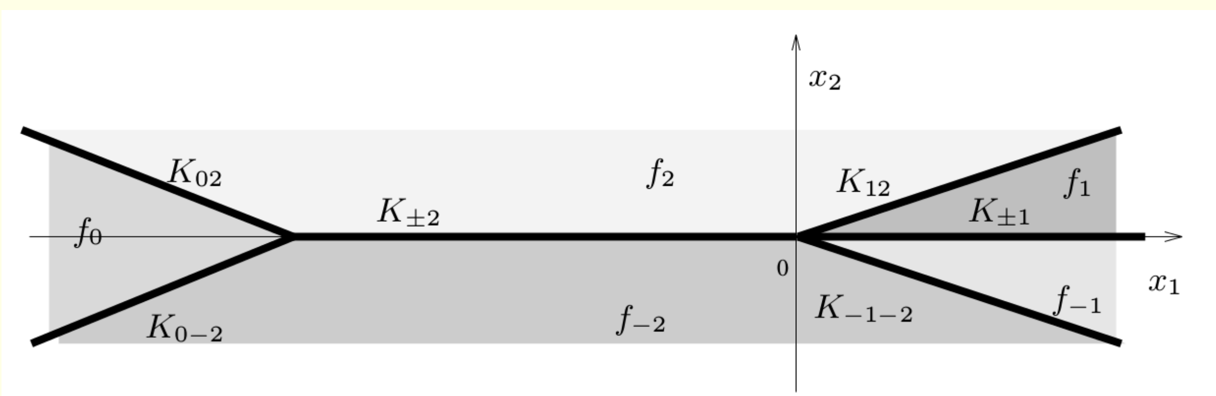

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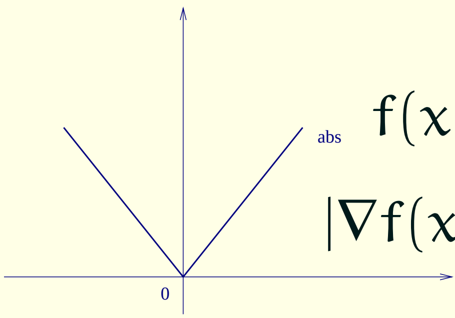

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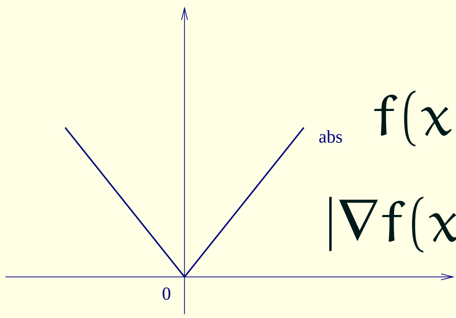

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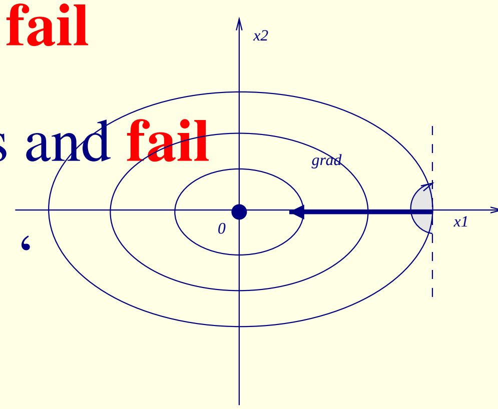
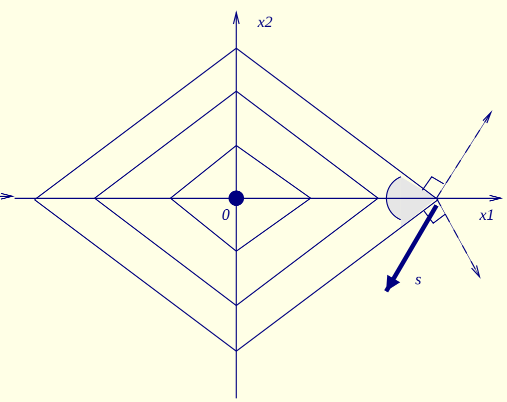
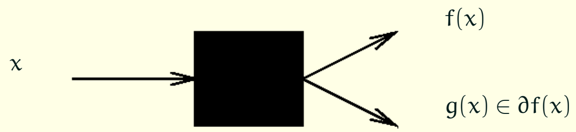




How is the oracle information used?

We look for algorithms based on information provided by an oracle



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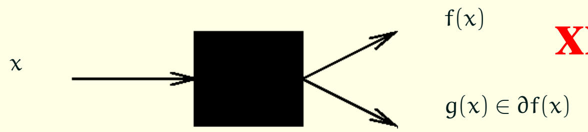

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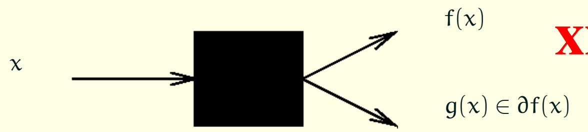
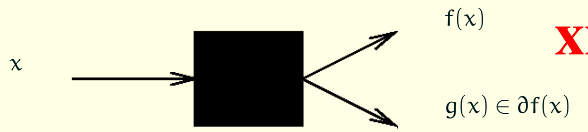


~~endowed with reliable stopping tests~~

Subgradient Methods

How is the oracle information used?

We look for algorithms based on information provided by an oracle



~~endowed with reliable stopping tests~~

Subgradient Methods

0 Choose x^1 and set $k = 1$.

1 Call the oracle at x^k.

2 Compute $x^{k+1} = x^k - t_k g(x^k)$ for a suitable stepsize $t_k > 0$.

3 Make $k = k + 1$ and loop to 1.

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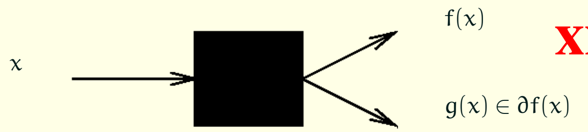

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Is this a good “recipe”?

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SG methods are
the algorithmic version
of this road sign

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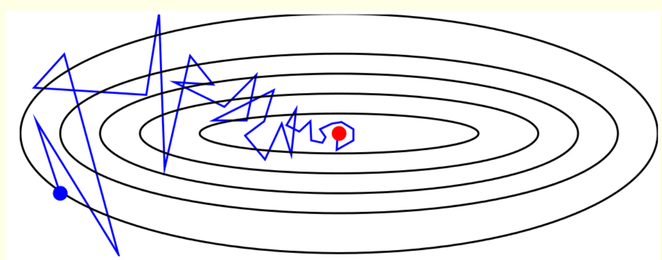

Non-monotone!

Subgradient Methods: why a “not-good” recipe

- Non-monotone functional values, but converges
- because distance to solution set decreases for t_k sufficiently small
- Lacks a stopping test

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…does not use all available information

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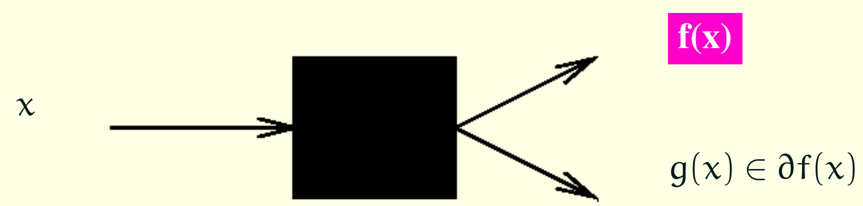

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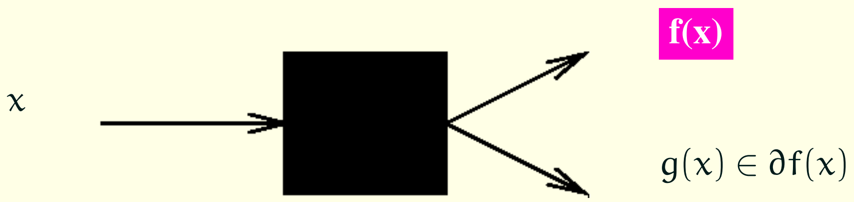

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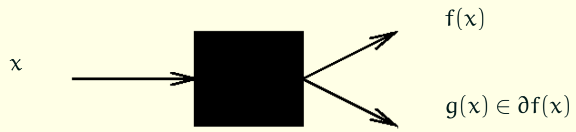

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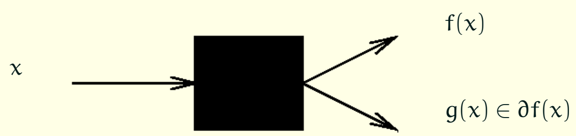

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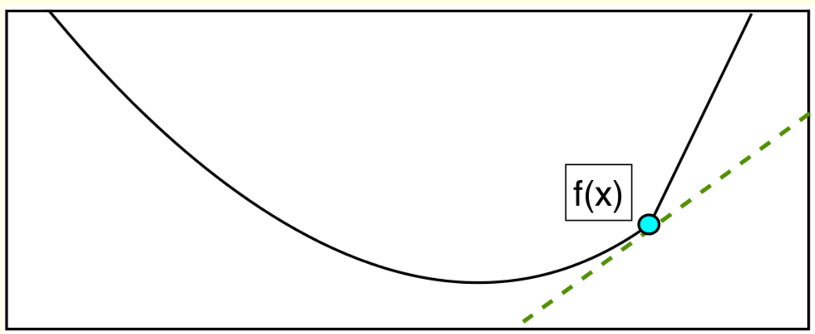
Black box information defines linearizations



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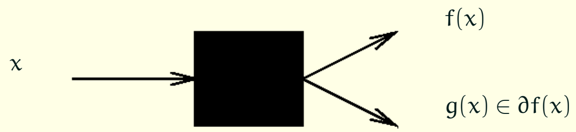

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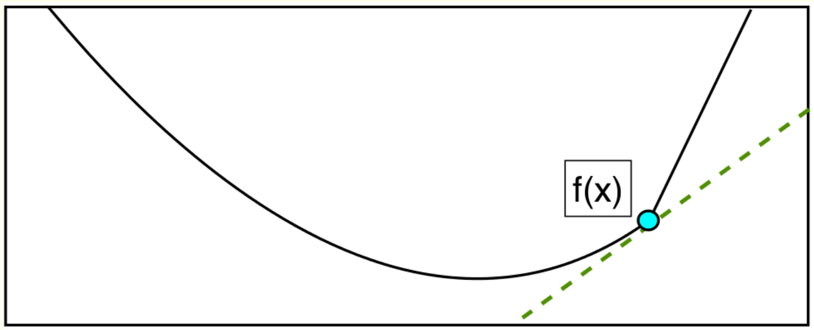

that put together create a **model M** of the function f.

The model is used to define iterates and to put in place a reliable stopping test

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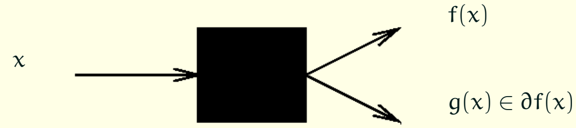

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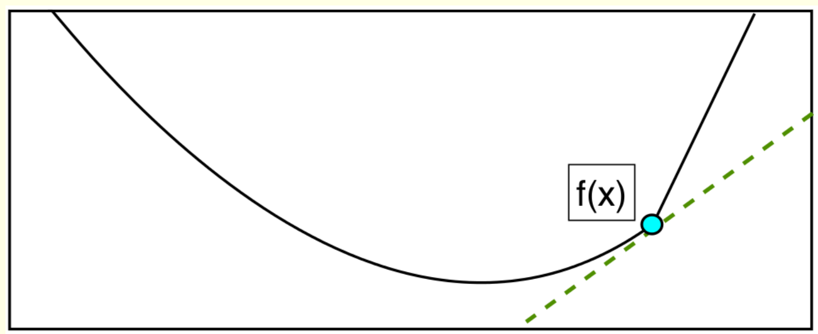

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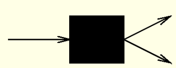
x^i



$$f^i = f(x^i)$$
$$g^i = g(x^i)$$
$$\Longrightarrow \qquad f^i + g^{i\top}(x - x^i)$$

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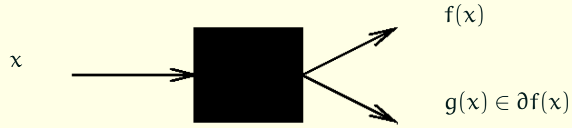

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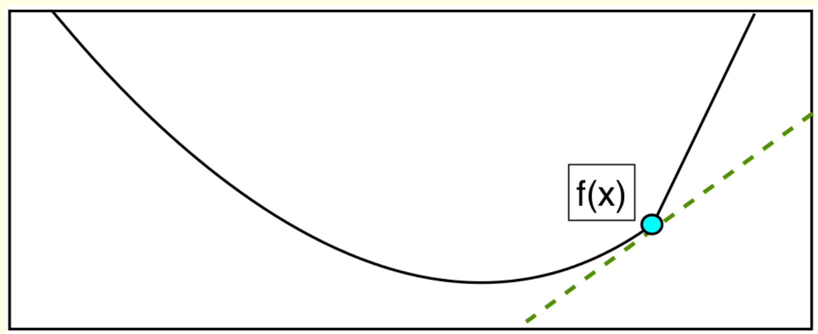

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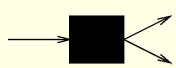

$$f^i = f(x^i)$$
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$$\Longrightarrow \mathbf{M}(x) = \max_i \{ f^i + g^{i\top}(x - x^i) \}$$

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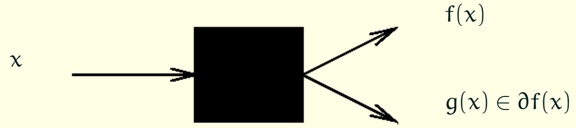

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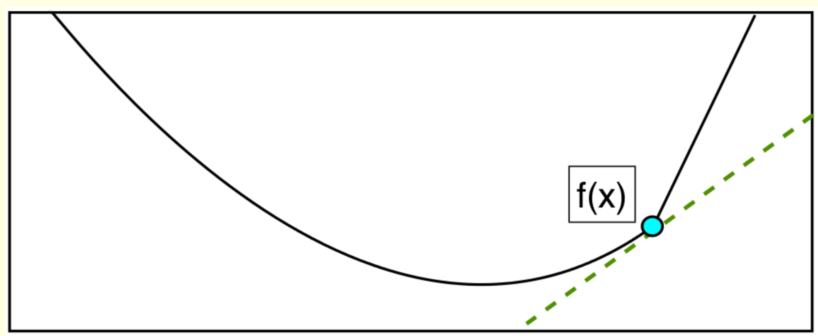

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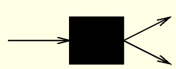
x^i  $f^i = f(x^i)$, $g^i = g(x^i)$ $\Longrightarrow \mathbf{M}(x) = \max_i \{ f^i + {g^i}^\top (x - x^i) \}$

(just an example, many other models are possible)

Cutting-plane methods

To minimize f (unavailable in an explicit manner), minimize its model $\mathbf{M}(x) = \max_i \left\{ f^i + g^{i\top}(x - x^i) \right\}$

Improve the model at each iteration

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Instead of $x^* \in \arg\min f(x)$ at one shot,

$x^{k+1} \in \arg\min \mathbf{M}_k(x)$ **iteratively**

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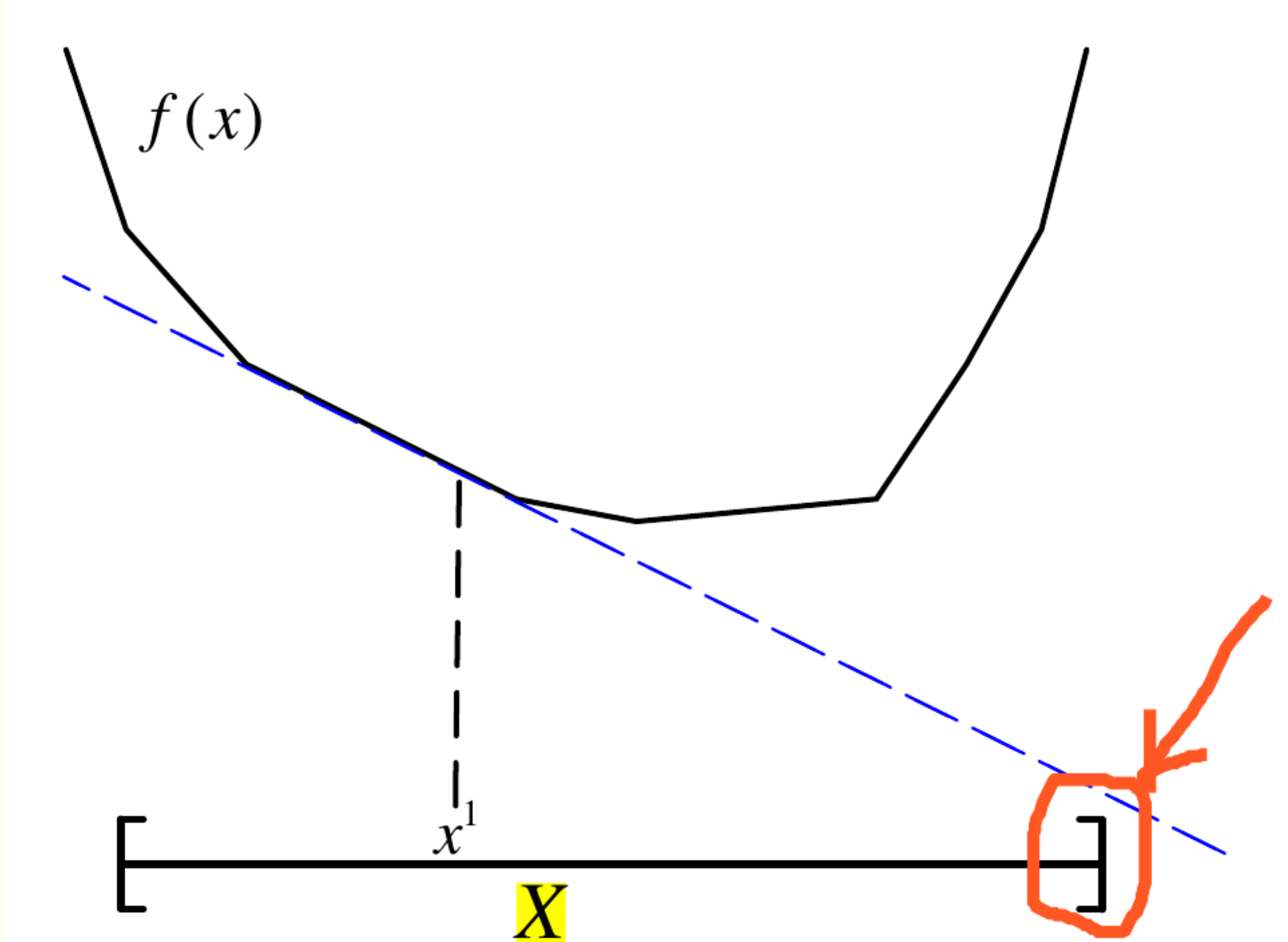

Artificial bounding at least for the first iterations

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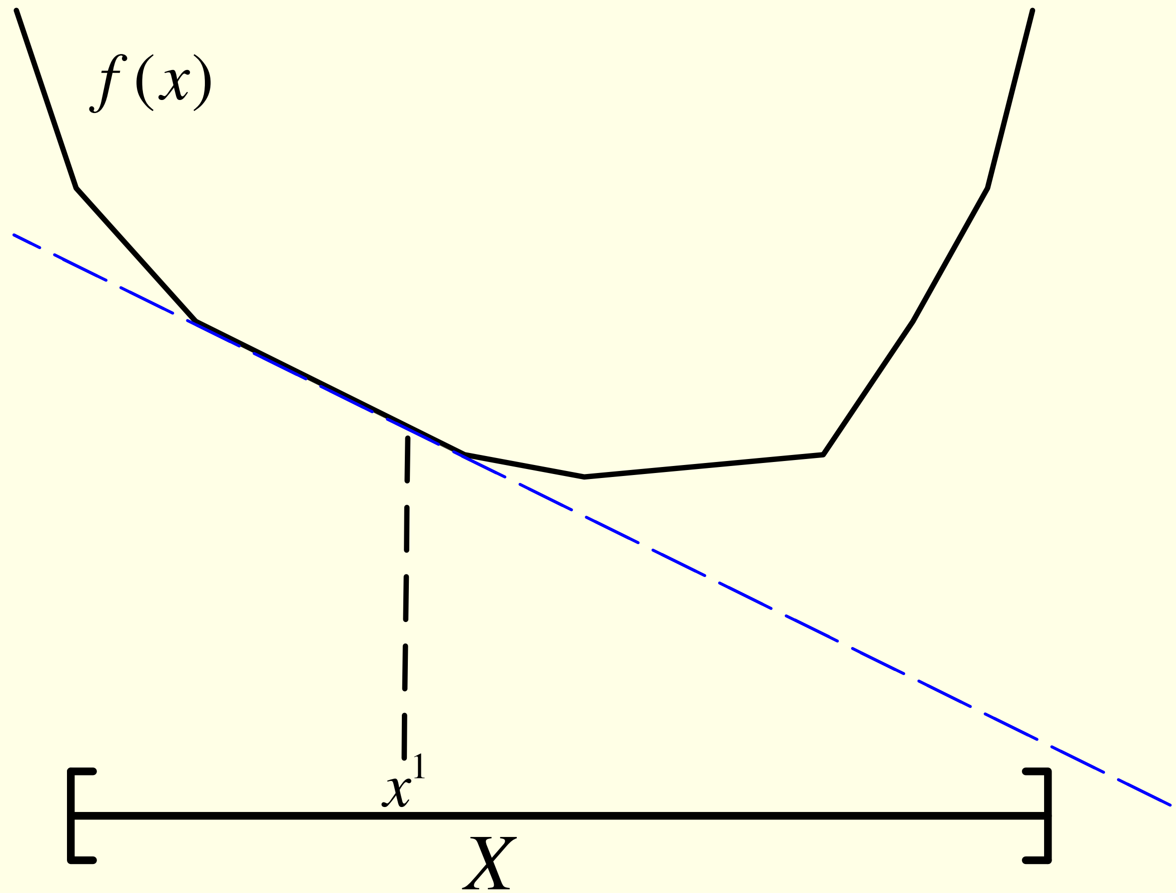

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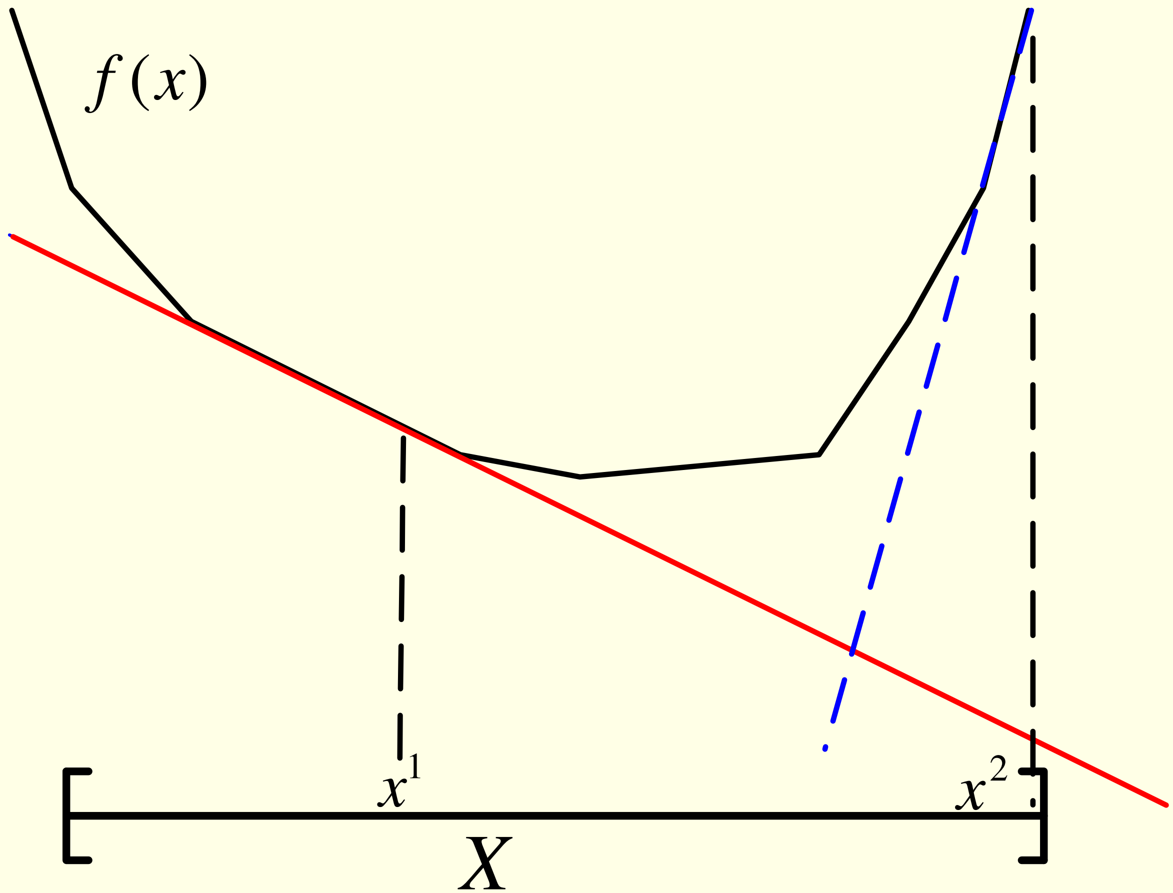

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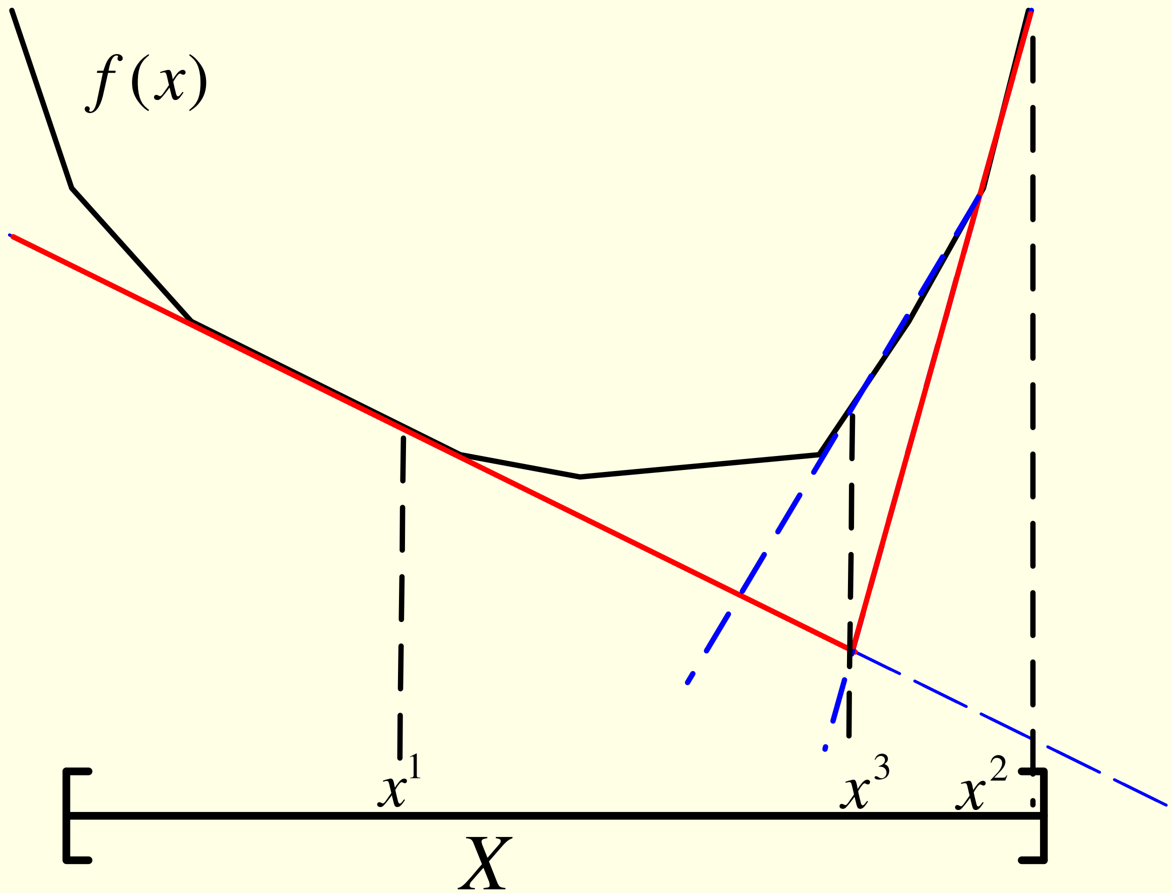

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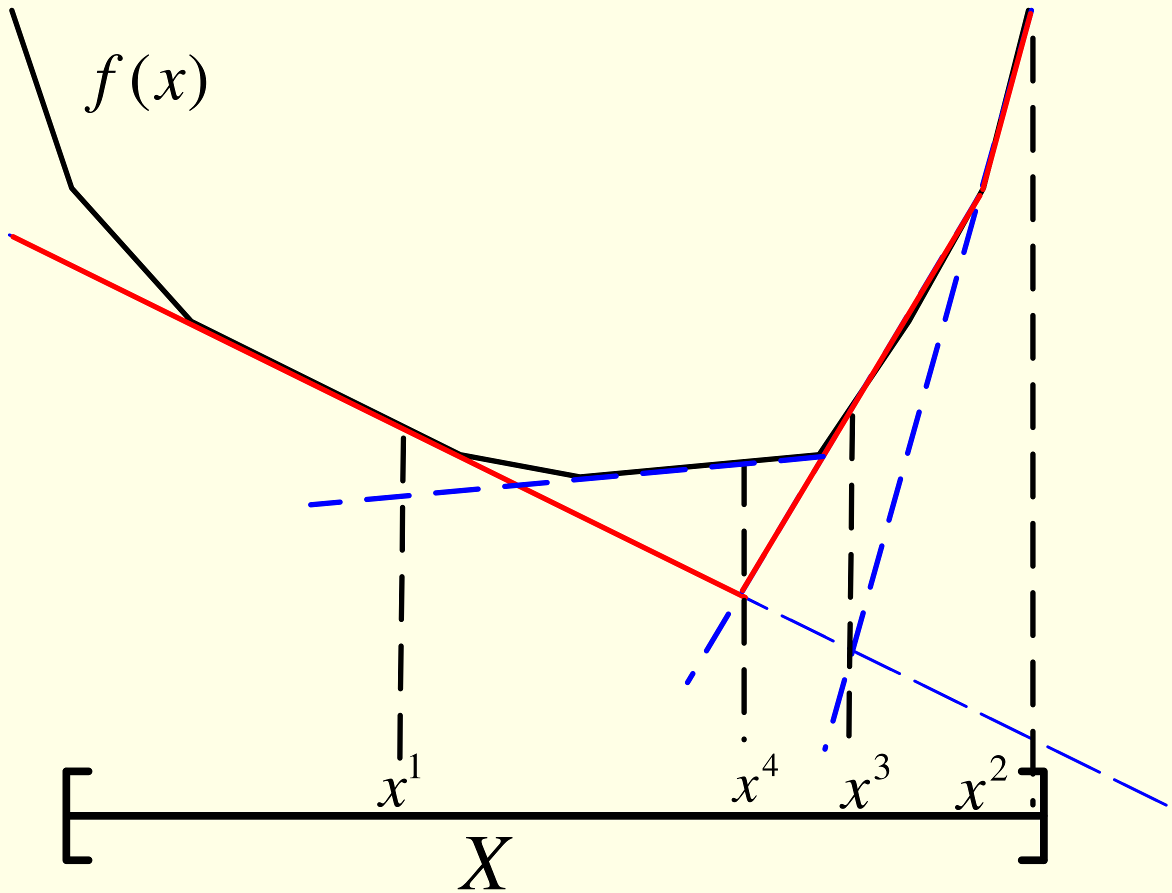

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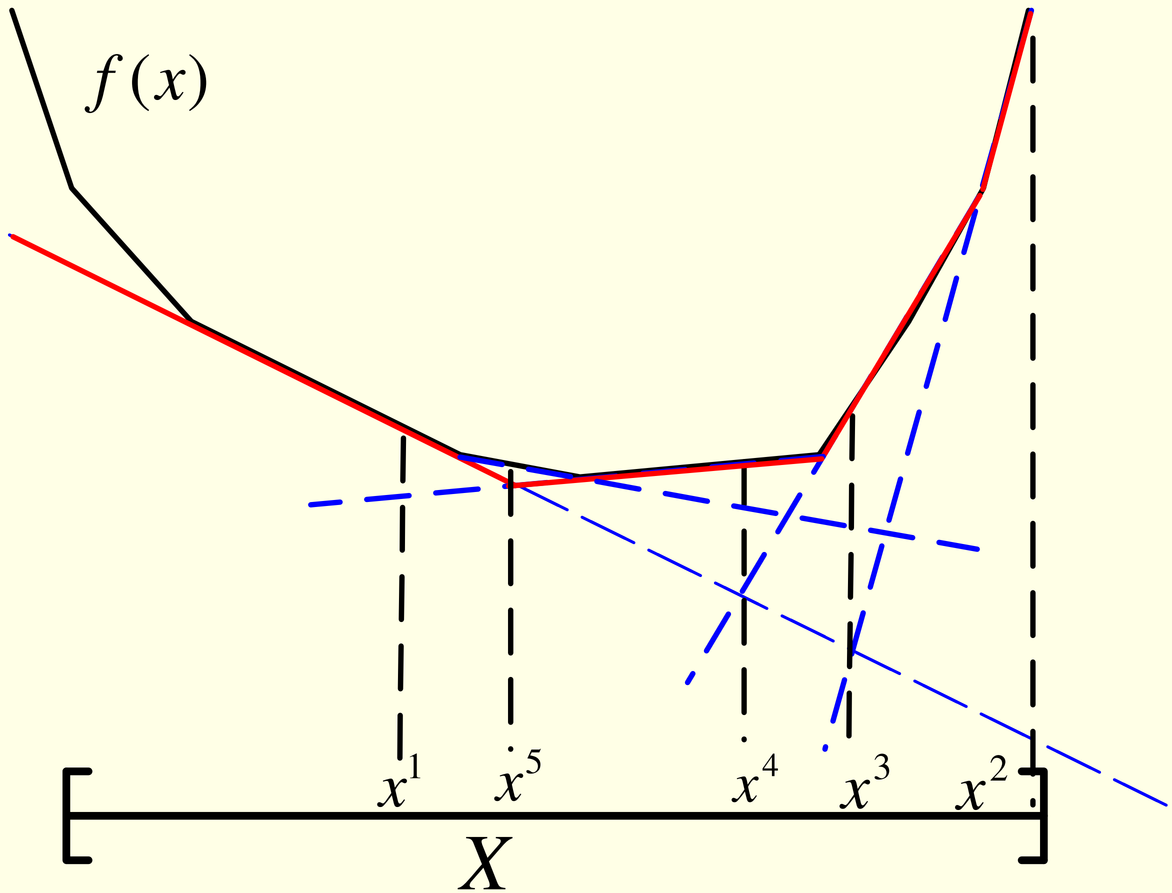

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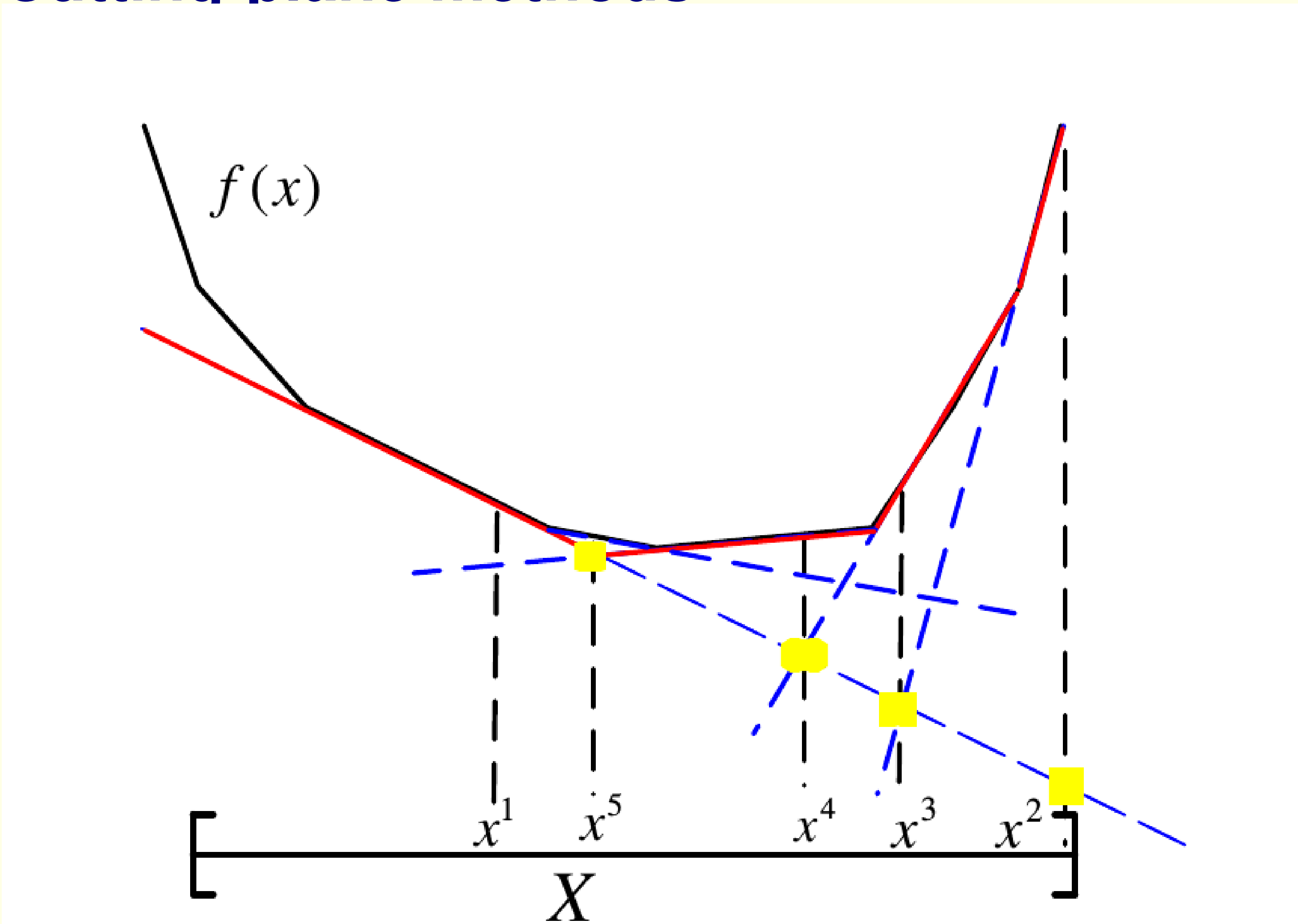

$\{\mathbf{M}_k(x^{k+1})\}$ increases

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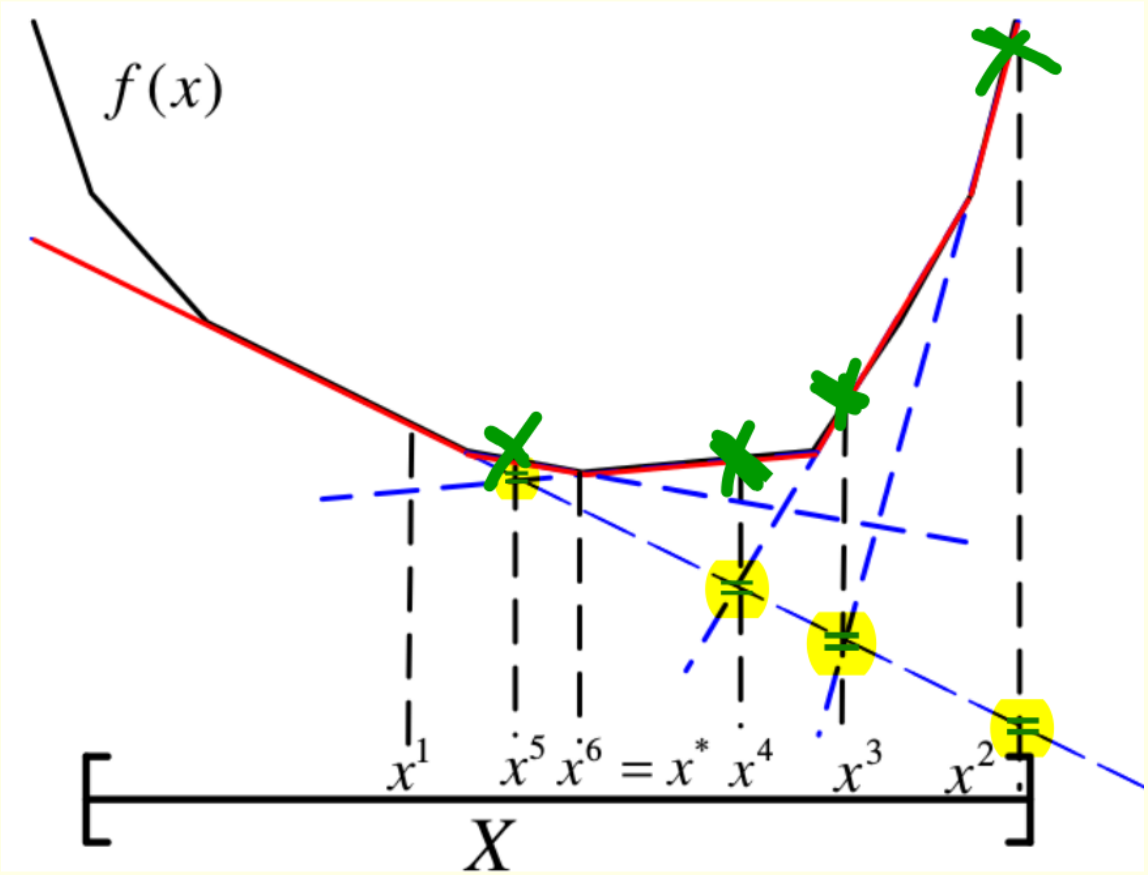

$\{\mathbf{M}_k(x^{k+1})\}$ increases but not necessarily the functional values:

$f(x^5) > f(x^4)$

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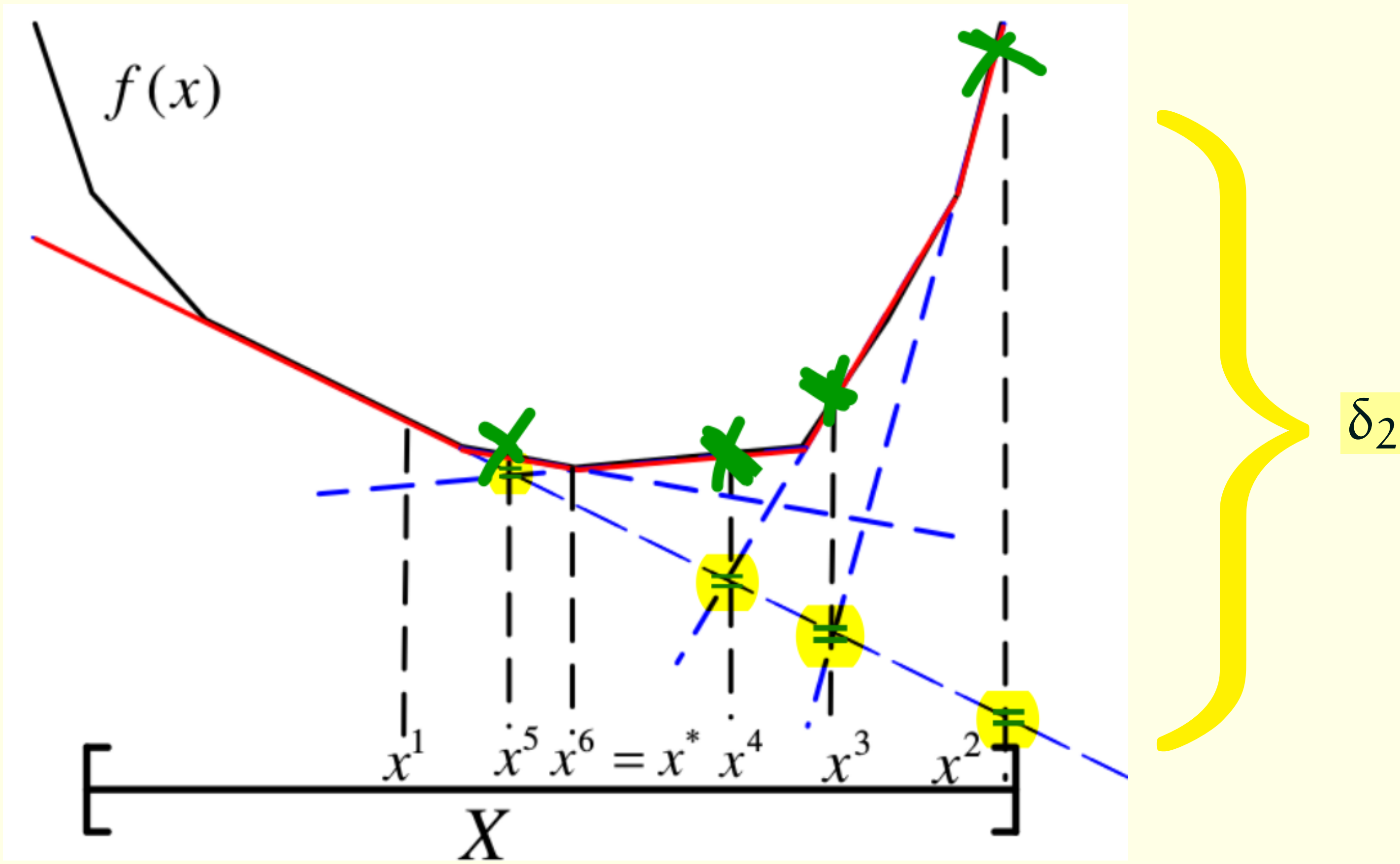

$\{\mathbf{M}_k(x^{k+1})\}$ increases but not necessarily the functional values: $f(x^5) > f(x^4)$. Stopping test measures $\delta_k := f(x^k) - \mathbf{M}_{k-1}(x^k)$

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an improved algorithmic version

of the Aussie sign

a better recipe

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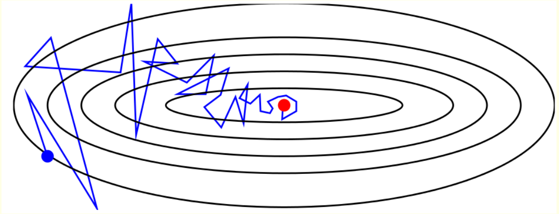
converges, but can stall and



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$$\begin{cases} \text{Non-monotone functional values, but converges} \\ \text{because } \liminf\left(f(x^k) - \mathbf{M}_{k-1}(x^k)\right) \to 0 \\ \text{Has a stopping test, but LP size grows indefinitely} \\ \text{eventually numerical errors prevail.} \end{cases}$$

$x^{k+1} \in \arg\min_X \mathbf{M}_k(x)$ with $\begin{array}{l} \mathbf{M}_k(x) = \max_{i \le k}\{f^i + g^{i\top}(x - x^i)\} \\ \text{and } X \text{ polyhedral} \end{array}$

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is equivalent to solving a linear programming problem

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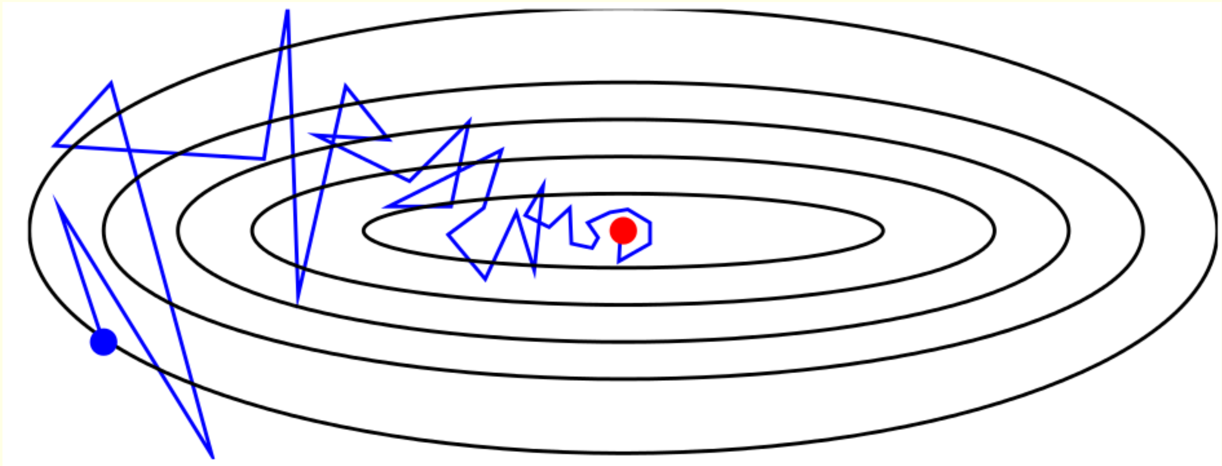
Ingredients for the best recipe

- CP brings in the concept of a model, which gives a stopping test (δ^k)
- CP still non-monotone



Monotonicity defeats instability and oscillations

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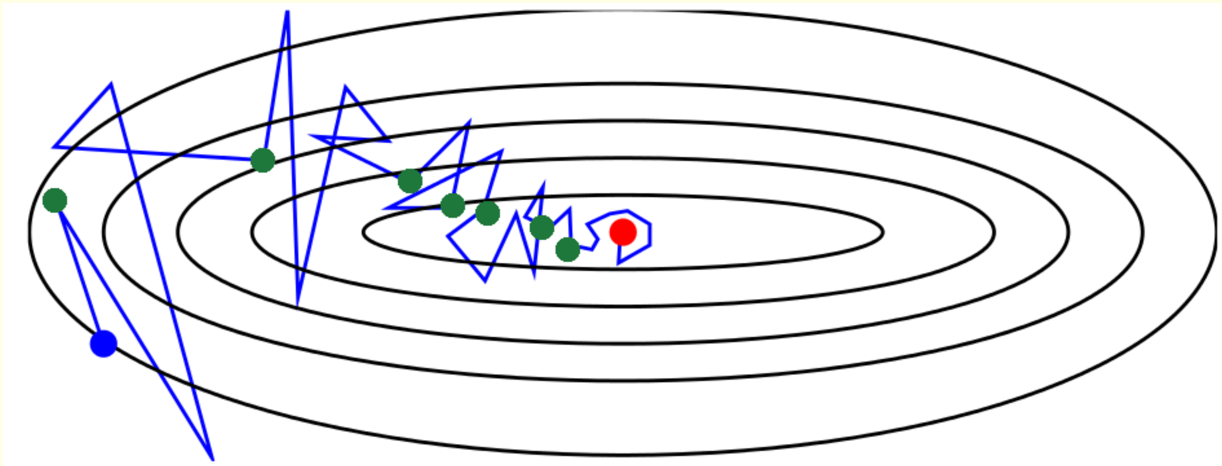

Monotonicity defeats instability and oscillations: the sequence of function values at green-spot iterates converges

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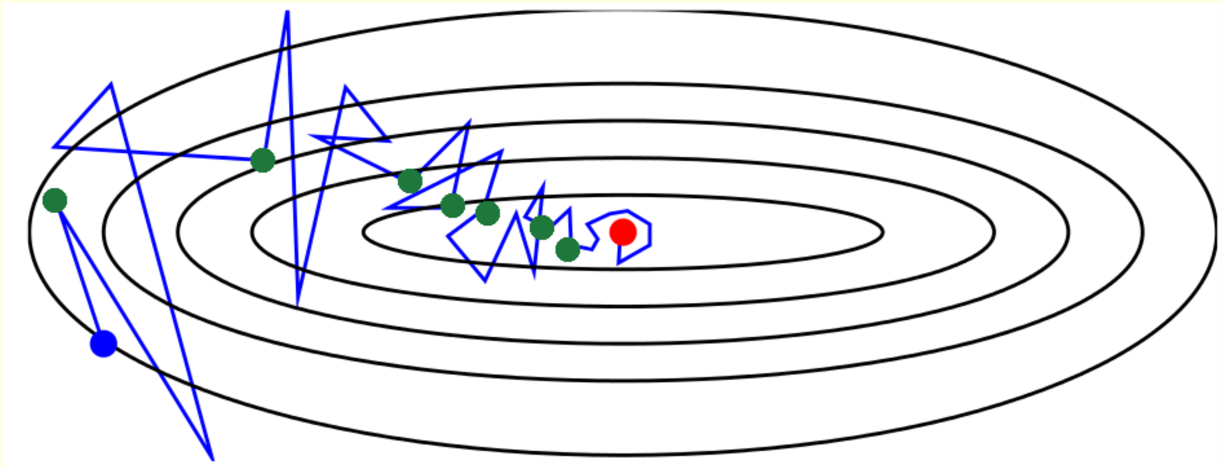

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- Bundle Methods select green-spot iterates using a descent rule

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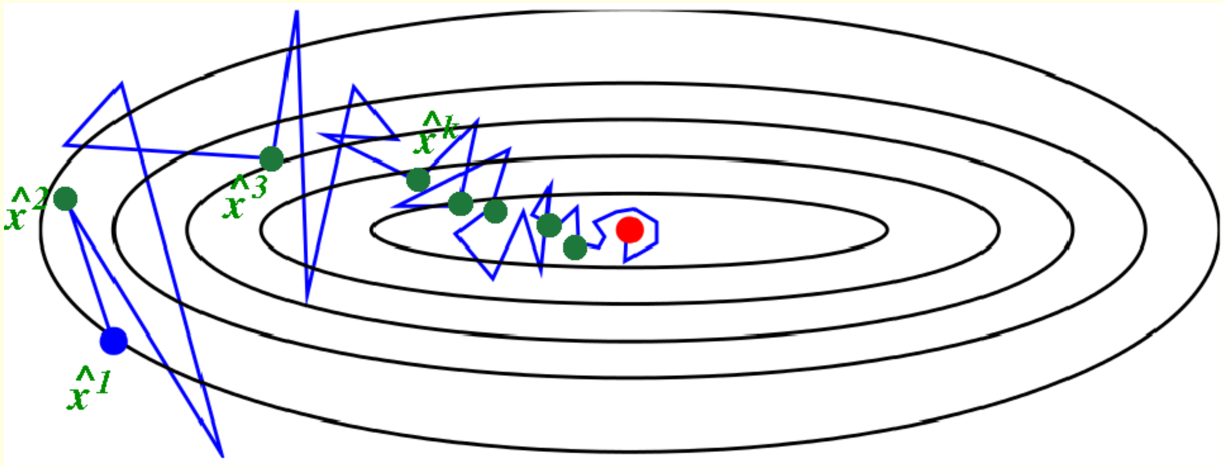


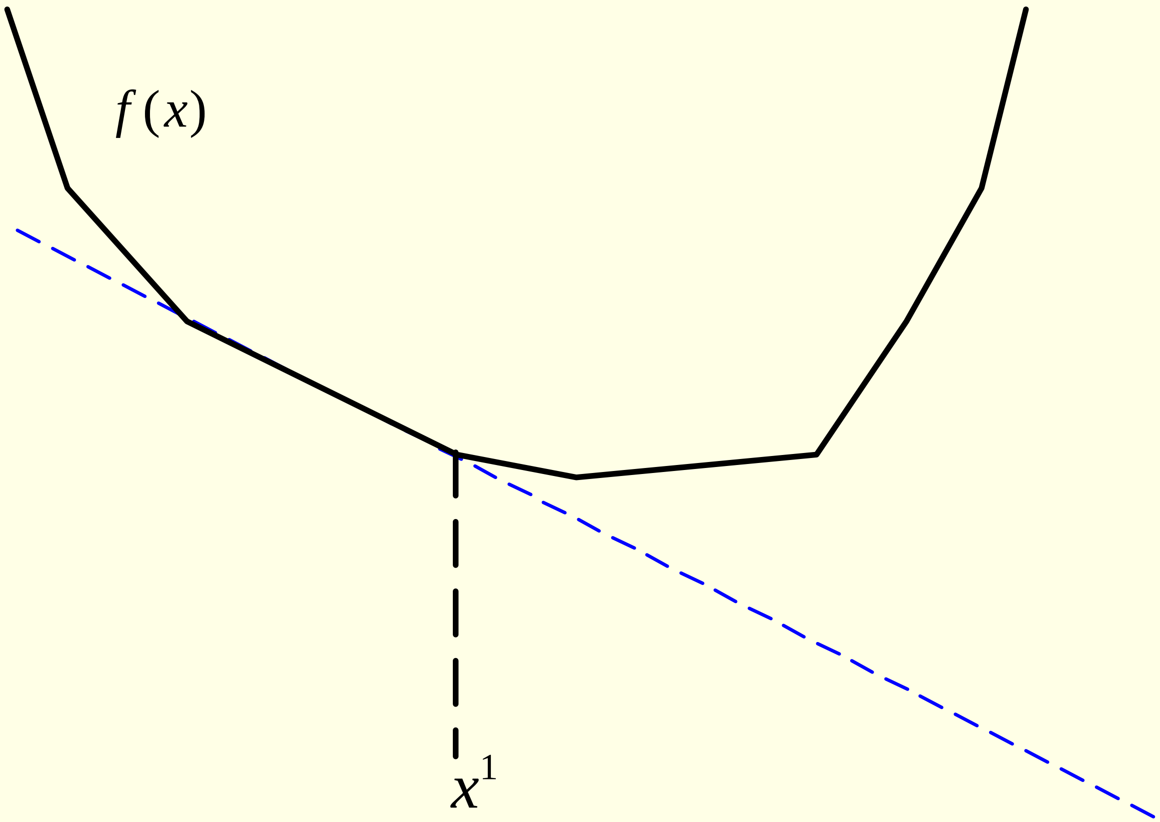
Monotonicity defeats instability and oscillations: the sequence of function values at green-spot iterates converges

- Bundle Methods select green-spot iterates using a descent rule

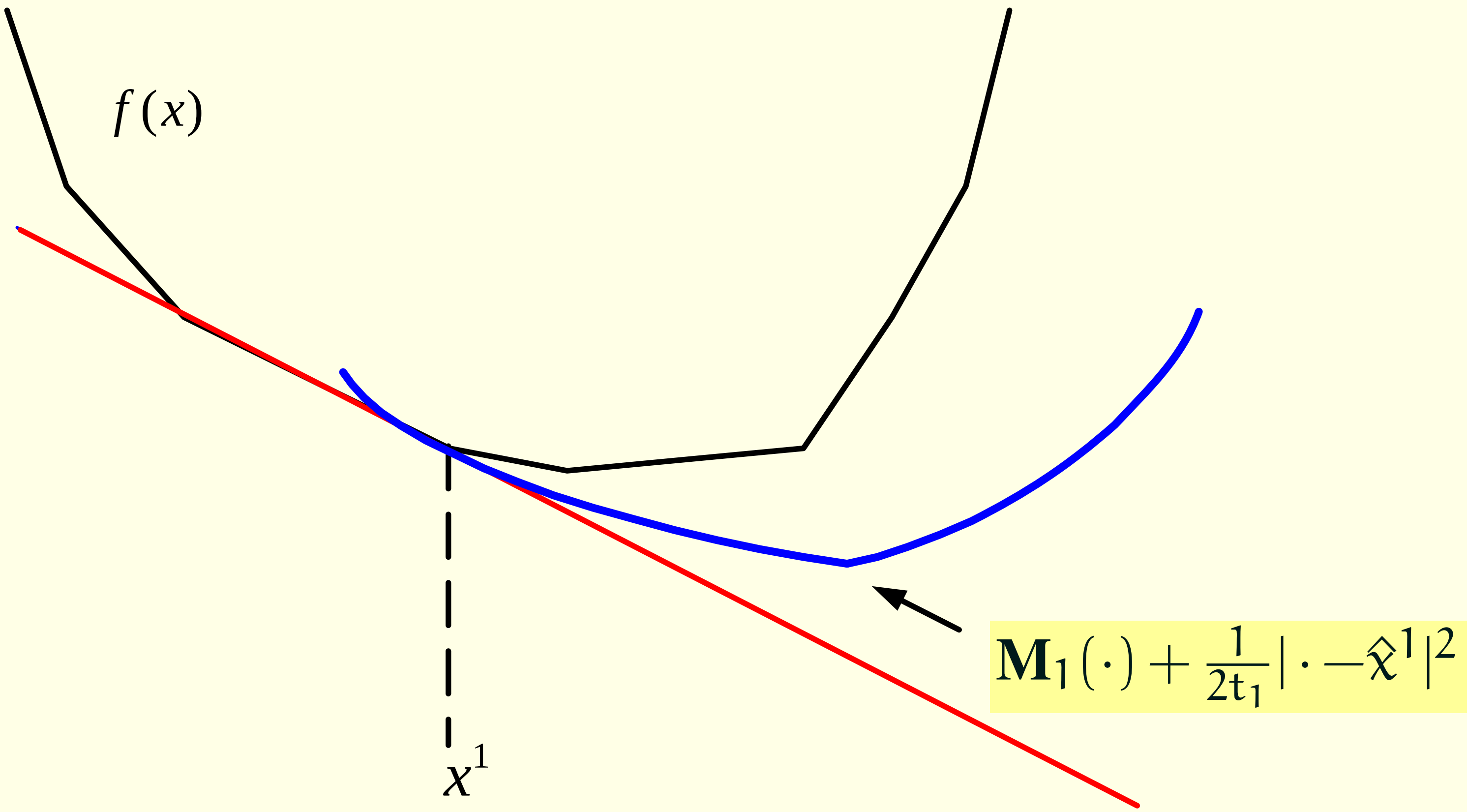
  $f(\hat{x}^{k+1}) \leq f(\hat{x}^k) - m\delta_k$ where δ_k **is a positive quantity** $< f(\hat{x}^k)$

  limit points of the **serious-step** subsequence $\{\hat{\mathbf{x}}^k\}$ minimize f

Bundle Methods



Bundle Methods



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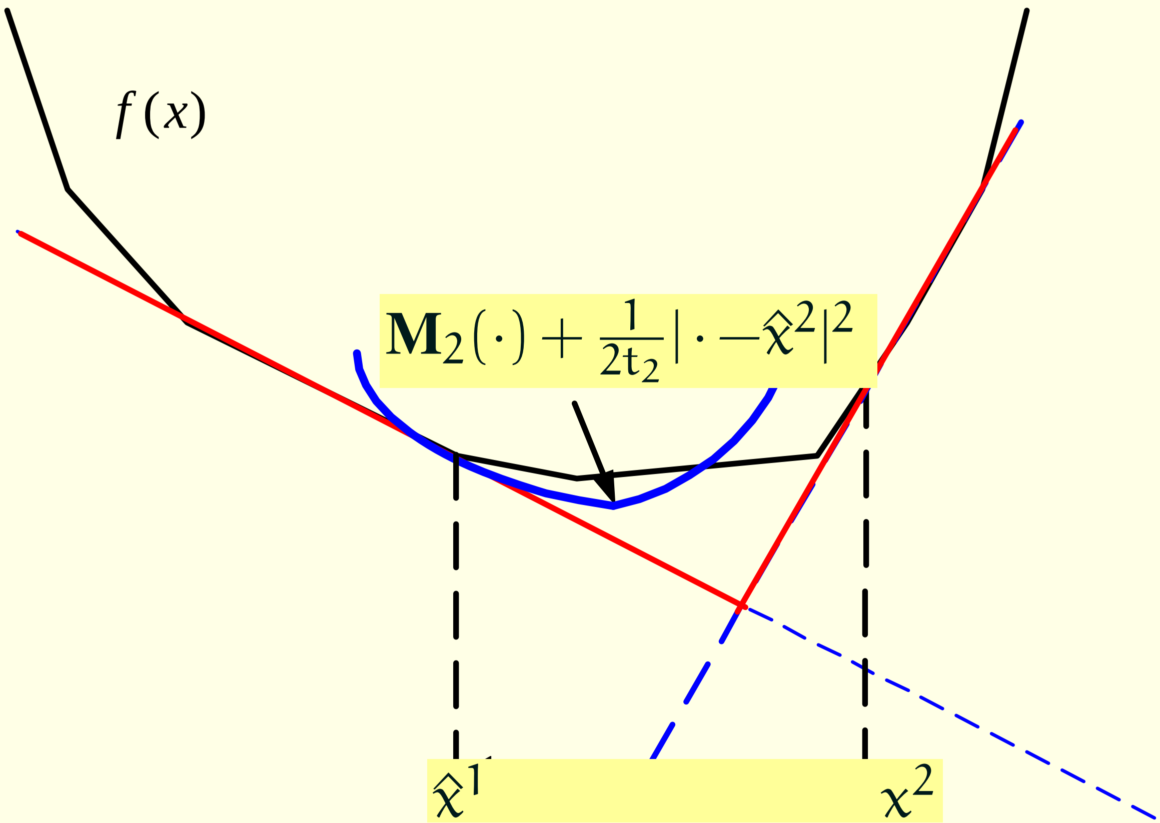
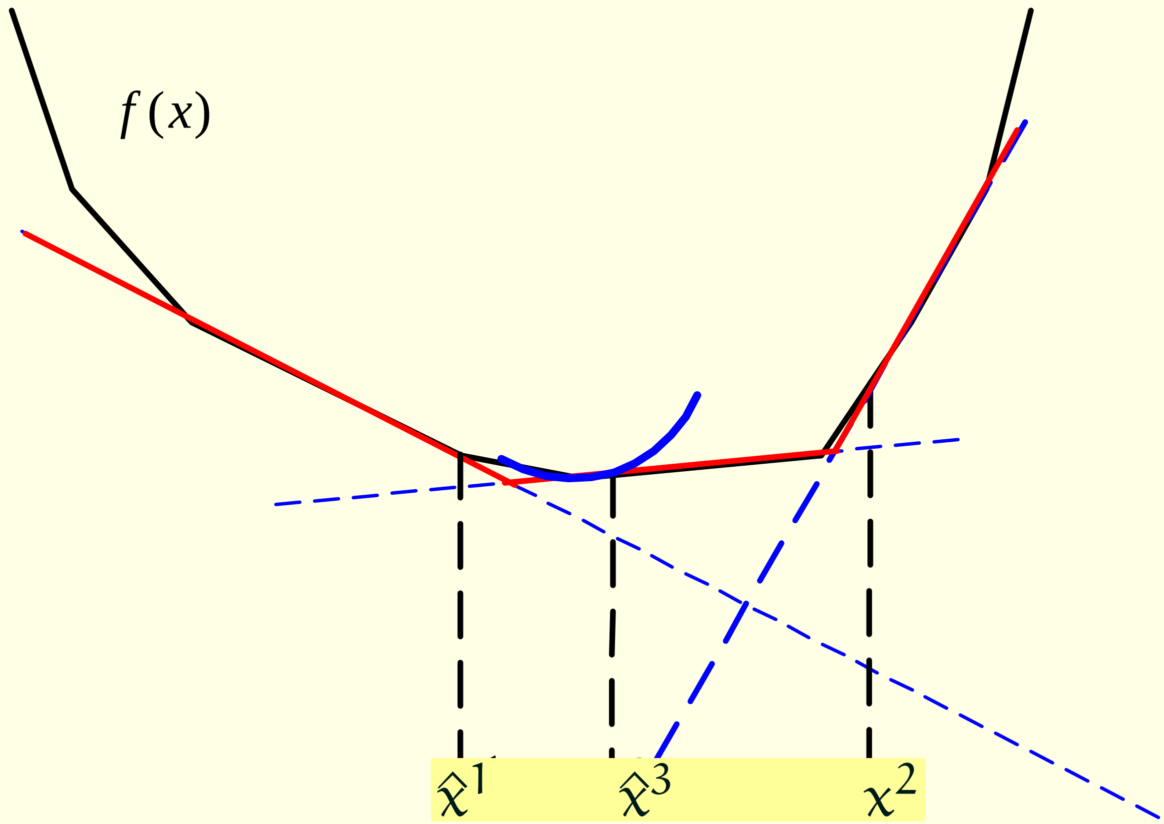
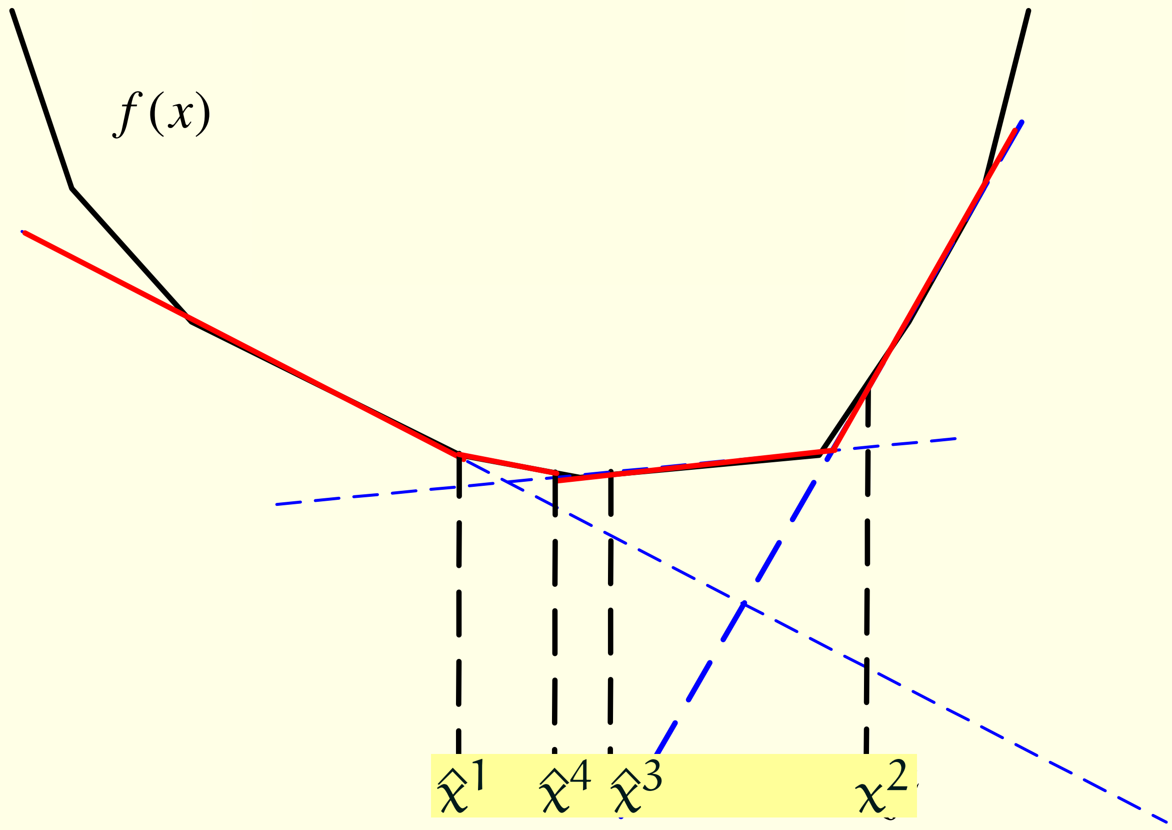


Bundle Methods



Bundle Methods



Bundle Methods

0 Choose x^1, set $k = 1$, and let $\hat{x}^1 = x^1$.

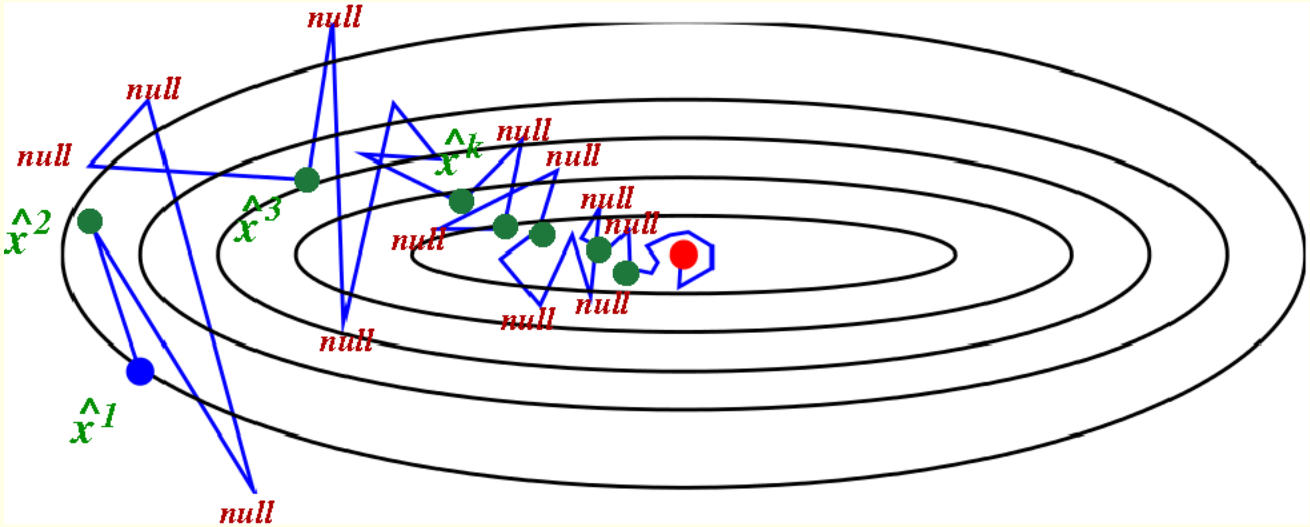
1 Compute $x^{k+1} \in \arg\min \mathbf{M}_k(x) + \frac{1}{2t_k}|x - \hat{x}^k|^2$

2 If $\delta_{\mathbf{k}} := f(\hat{x}^k) - \mathbf{M}_k(x^{k+1}) \leq \mathrm{tol}$ STOP

3 Call the oracle at x^{k+1}.

If $f(x^{k+1}) \leq f(\hat{x}^k) - m\delta_k$, set $\hat{x}^{k+1} = x^{k+1}$ • (Serious Step)
Otherwise, maintain $\hat{x}^{k+1} = \hat{x}^k$ (Null Step)

4 Define $\mathbf{M}_{k+1}$, t_{k+1}, make $k = k + 1$, and loop to 1.



Bundle Methods

Unlike **CP** $\mathbf{M}_{k+1}(\cdot) = \max\left(\mathbf{M}_k(\cdot), f^k + g^{k\top}(\cdot - x^k)\right)$,

now the choice of the new model is more flexible:

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$$\begin{cases} \min_{r \in \mathbb{R}, x \in \mathbb{R}^n} & r + \frac{1}{2t_k}|x - \hat{x}^k|^2 \\ \text{s.t.} & r \ge f^i + g^{i\top}(x - x^i) \text{ for } i \le k \end{cases}$$

A posteriori, the solution remains the same if ...

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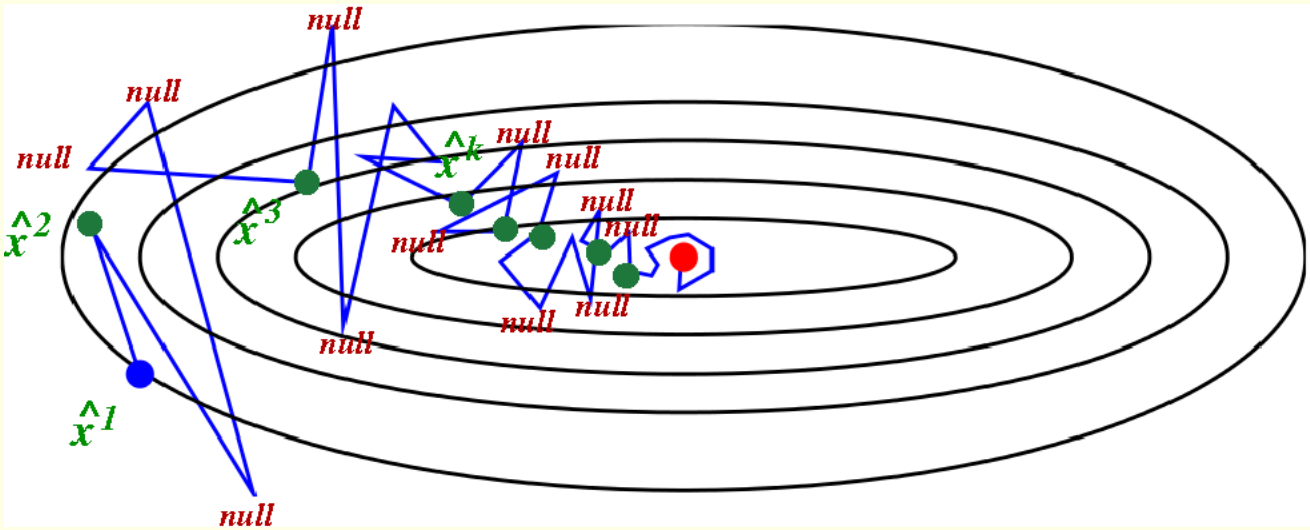
Bundle Compression: QP with 2 constraints

Bundle Methods

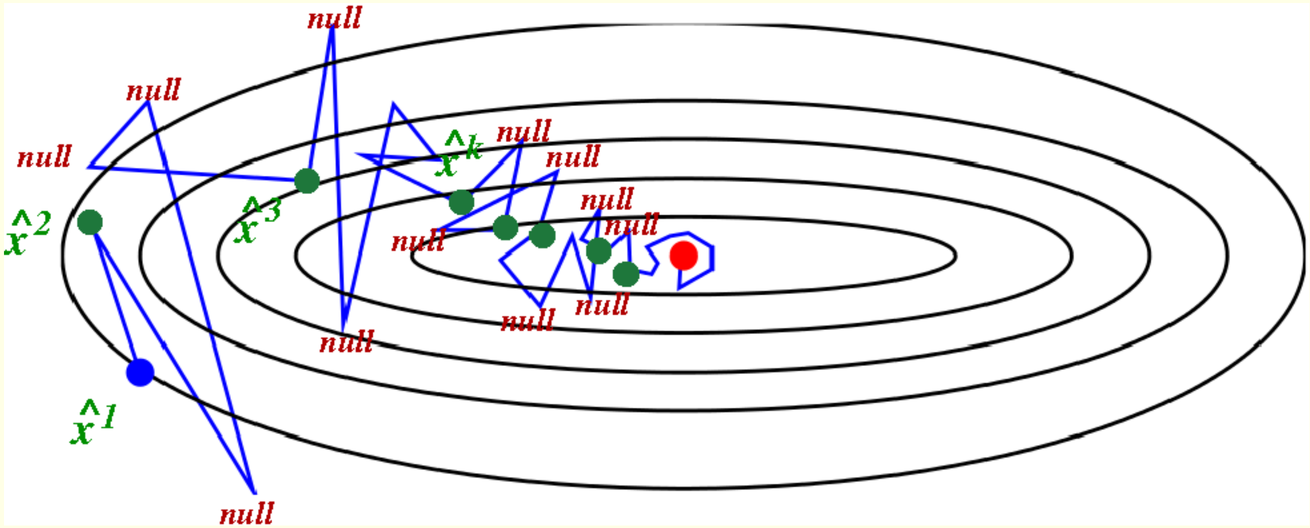


When k $\rightarrow \infty$, the algorithm generates two subsequences.

Convergence analysis addresses the mutually exclusive situations

- either the SS subsequence is infinite
- or there is a last SS, followed by infinitely many null steps

Bundle Methods



When $k \to \infty$, the algorithm generates two subsequences.

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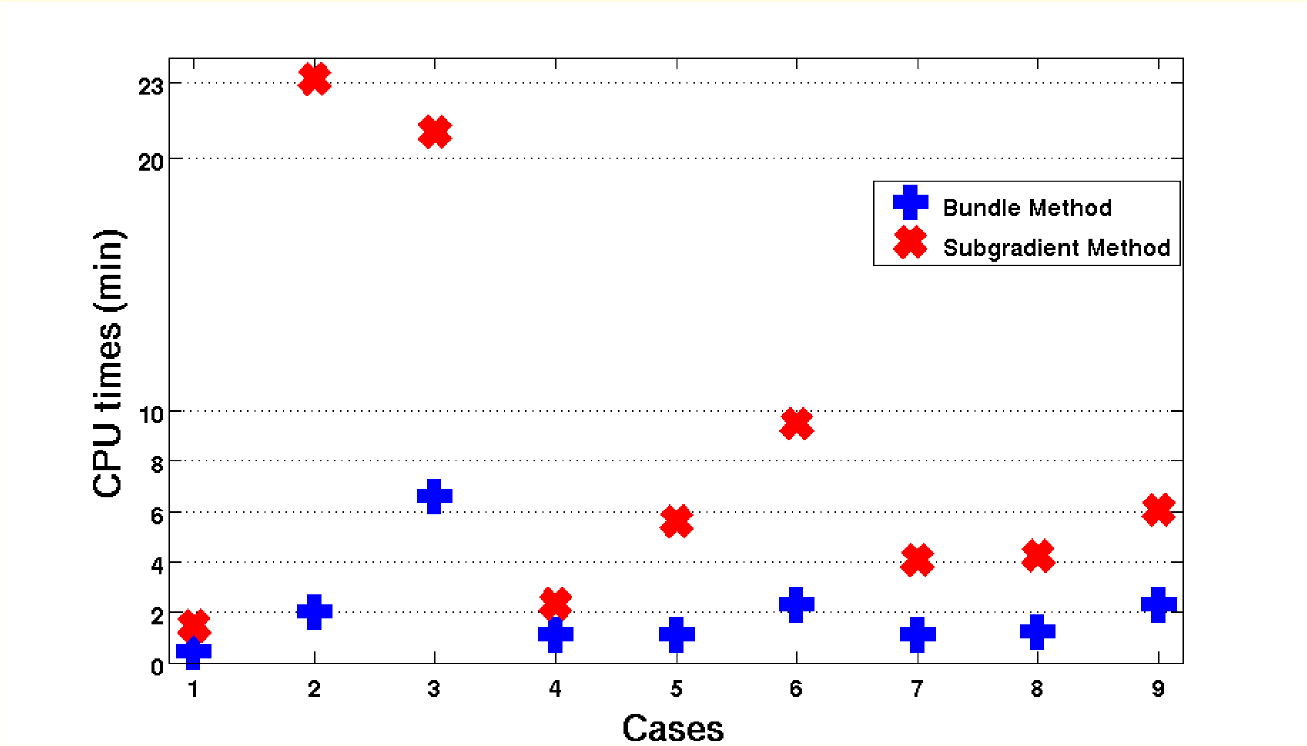
- either the SS subsequence is infinite **(limit point minimizes f)**
- or there is a last SS, followed by infinitely many null steps **(last SS minimizes f and null$\to$ last SS)**

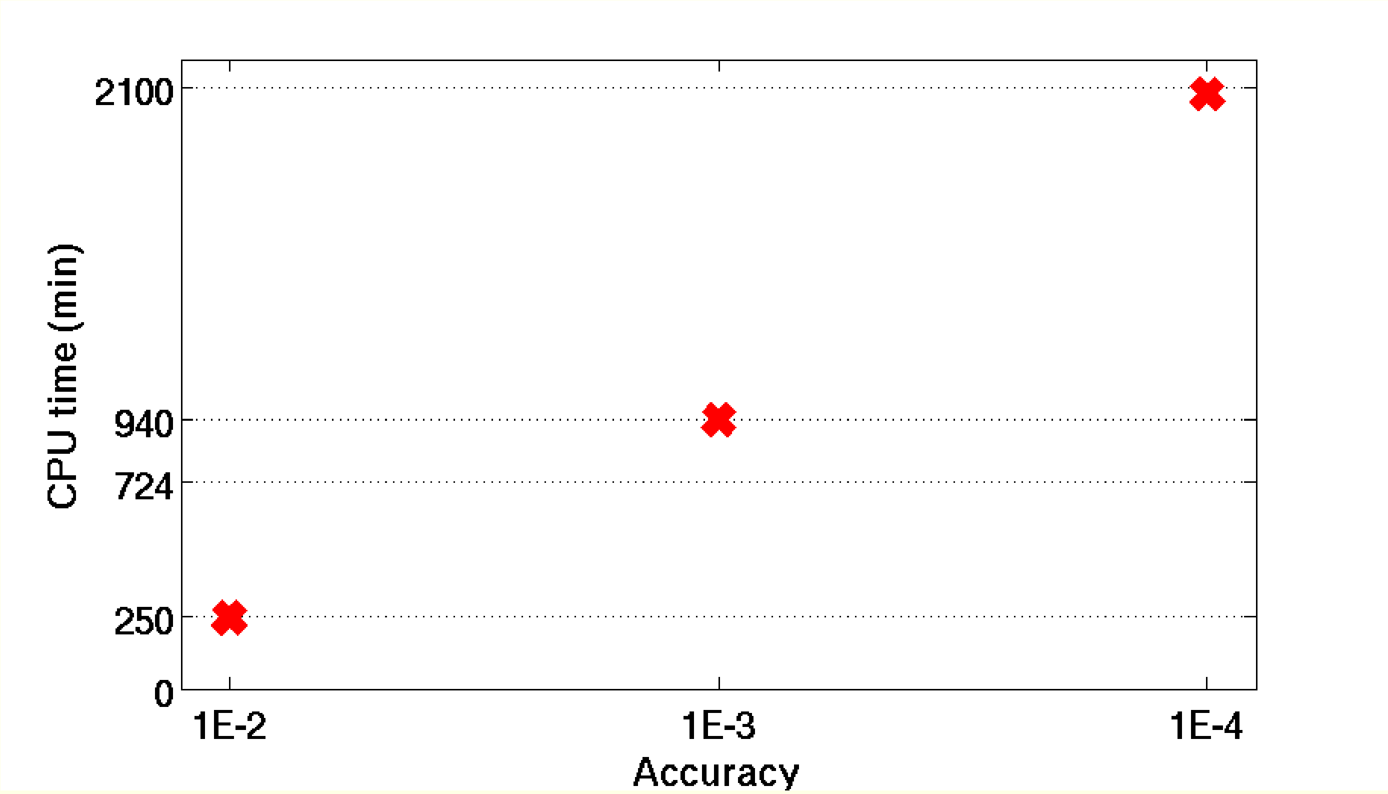
Comparing the methods: bundle and SG

Typical performance on a battery of Unit Commitment problems



Comparing the methods: bundle and CP

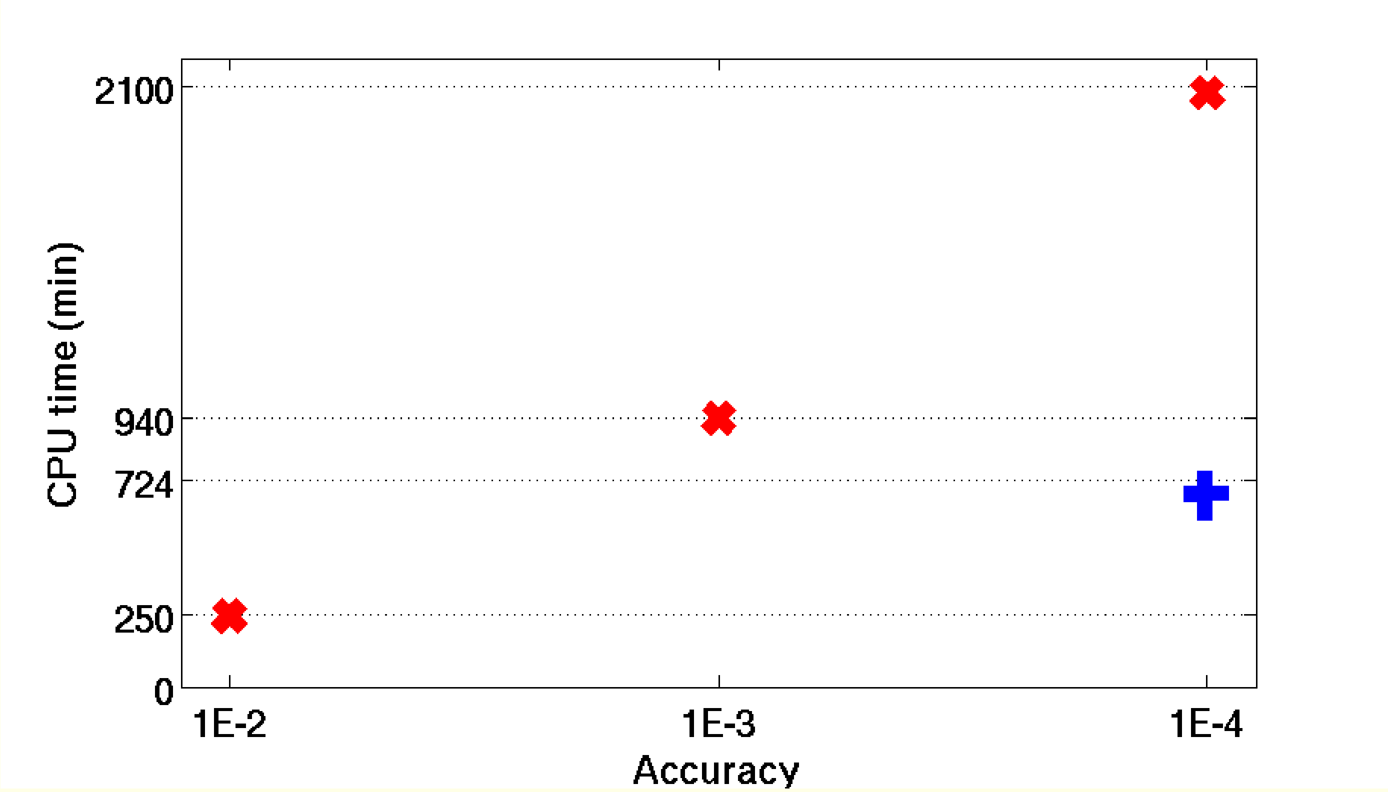
On a battery of probabilistically constrained problems



X CP is fast to reach a few digits of accuracy, then stalls

Comparing the methods: bundle and CP

On a battery of probabilistically constrained problems



x CP is fast to reach a few digits of accuracy, then stalls

+ Bundle is consistently 3 times faster

Comparing the methods

SG ok if low precision -for instance in combinatorial optimization

CP ok if not many iterations -usually not the case

Bundle ok if f complex and high precision is required

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a good recipe

Can we do any better??

Can we do any better??

YES, WE CAN

Bundle Methods with on-demand accuracy

the new generation

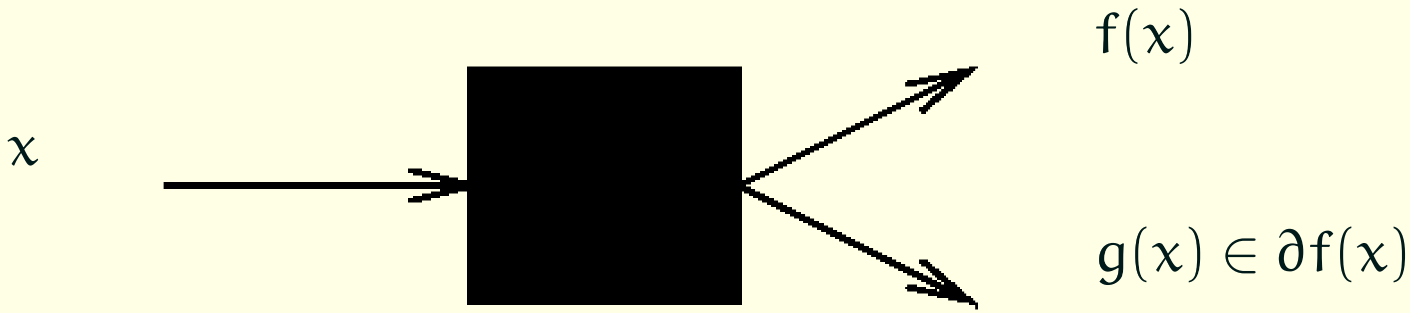
(or the perfect caipirinha)

First, the bad news

For a convex nonsmooth function, solving

$$\min f(x)$$

with a black box method



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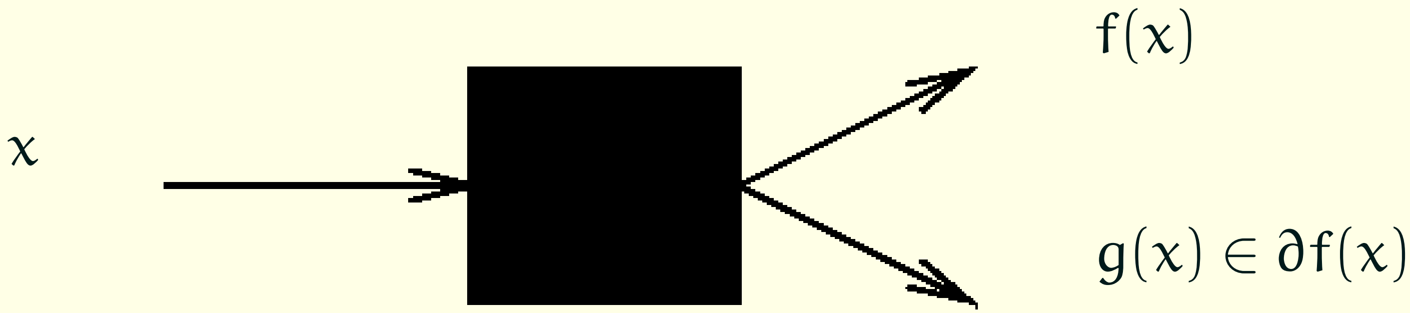

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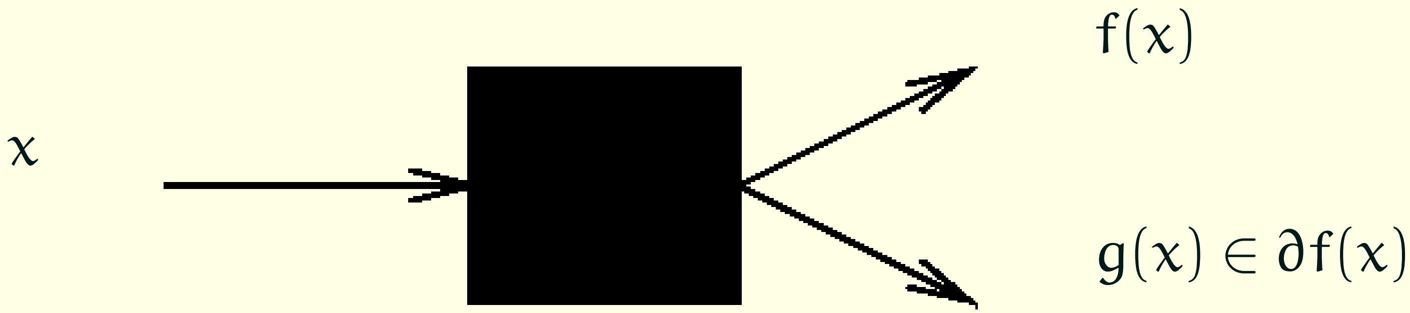

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For instance, for strongly convex f complexity drops to $O(\frac{1}{k})$

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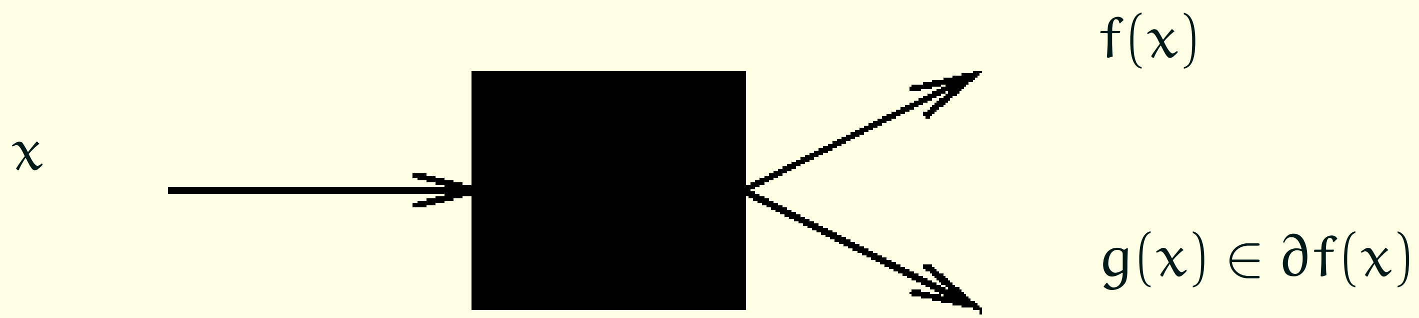

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Note: complexity results assume black box always called as above

How does structure appear?

– Explicitly

as a sum

as a composition

– Implicitly

U-Lagrangian

VU-decomposition

partly smooth functions

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digging tools

Explicit Structure:

Opening the Black Box

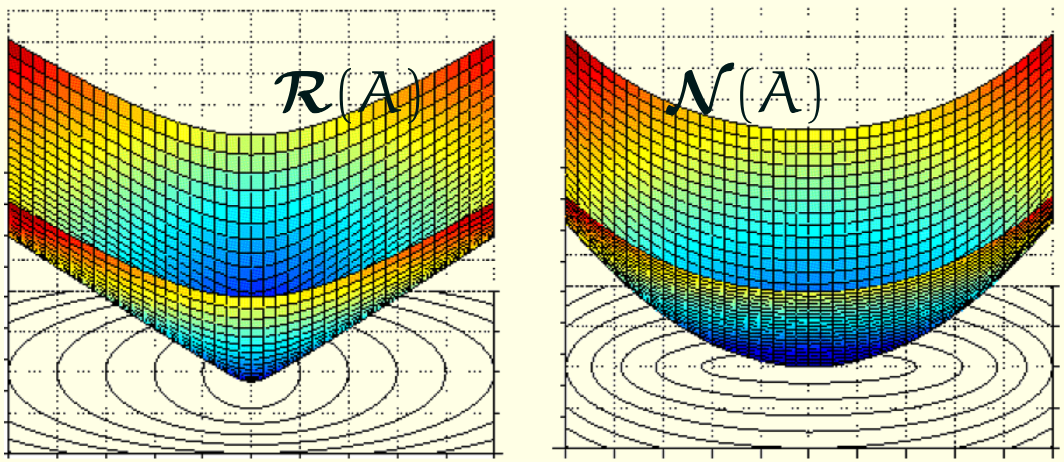
A convex partly nonsmooth function

For $x \in \mathbb{R}^n$, given matrices $A \succeq 0$, $B \succ 0$,

$$f(x) = \sqrt{x^\top A x} + x^\top B x$$

has a unique minimizer at 0.

On $\mathcal{N}(A)$ the function is not differentiable, and the first term vanishes: $f|_{\mathcal{N}(A)}$ looks smooth.



This function has several interesting structures

If no structure at all

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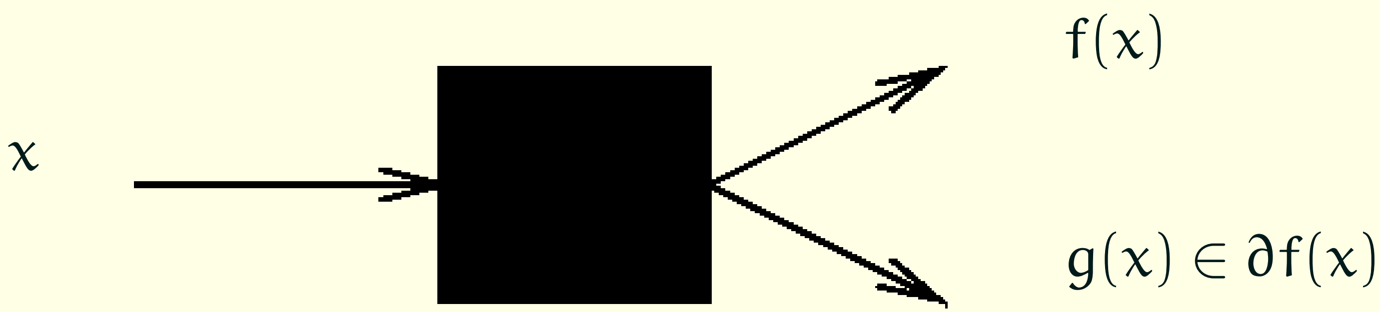

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Sum structure

$$f(x) = f_1(x) + f_2(x) \text{ with } \begin{cases} f_1(x) = \sqrt{x^\top A x} \\ f_2(x) = x^\top B x \end{cases}$$

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This defines a **sum black box**:

x

$f_1(x), f_2(x)$

$g_j(x) \in \partial f_j(x)_{j=1,2}$

This function has several interesting structures

Composite structure

$$f(x) = (h \circ c)(x) \text{ with } \begin{cases} c(x) = (x, x^\top B x) \in \mathbb{R}^{n+1} \\ h(C) = \sqrt{C_{1:n}^\top A C_{1:n}} + C_{n+1} \end{cases}$$

for C smooth and h positively homogeneous

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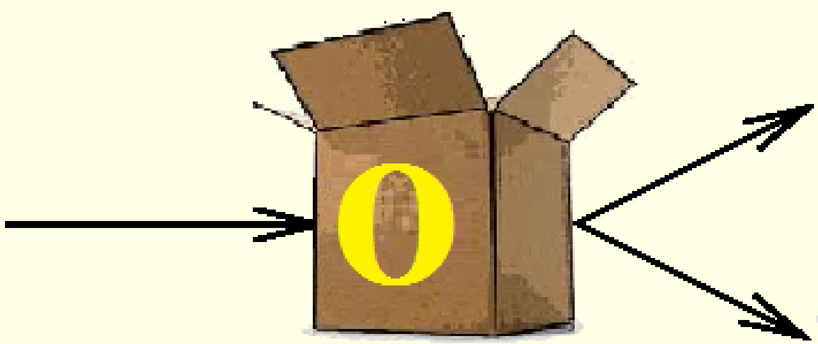

$C := c(x)$ and $h(C)$

Jacobian $Dc(x)$ and

$G(C) \in \partial h(C)$

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Inexact information

Suppose not all of A/B is known/accessible,

so that only **estimates** are available for f

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Inexact information

Suppose not all of A/B is known/accessible,

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This defines a **noisy black box**:

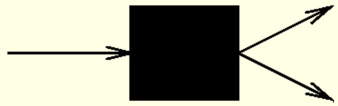
x

$f_x \approx f(x)$

$g_x \approx g(x) \in \partial f(x)$

Structured models for f

No structure



$$\mathbf{M}(x) = \max_i \left\{ f^i + g^{i\top}(x - x^i) \right\}$$
$$= \max_i \left\{ (f_1^i + f_2^i) + (g_1^i + g_2^i)^\top (x - x^i) \right\}$$

Sum structure

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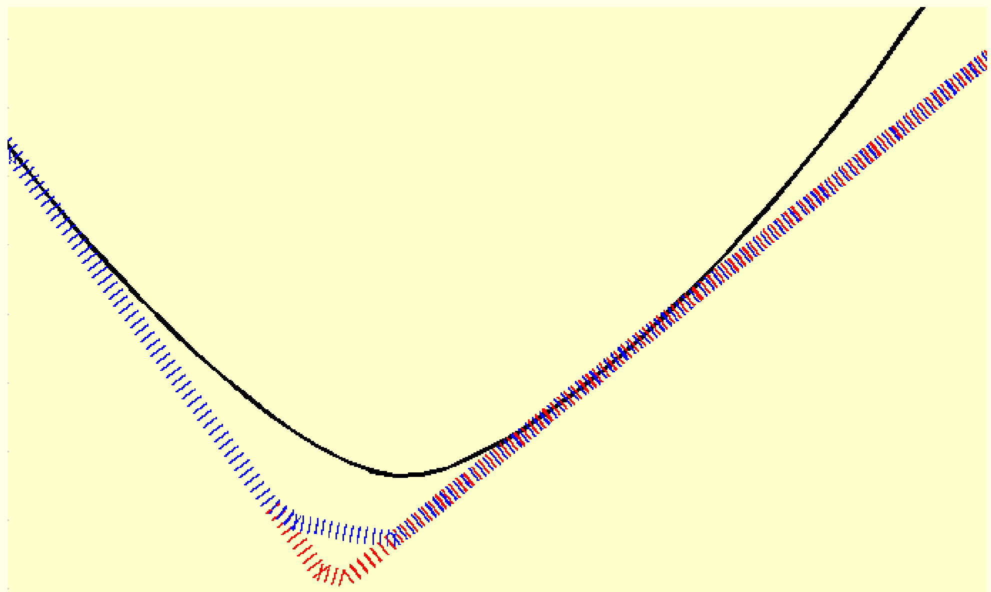

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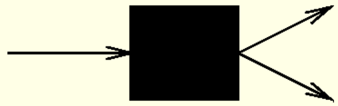

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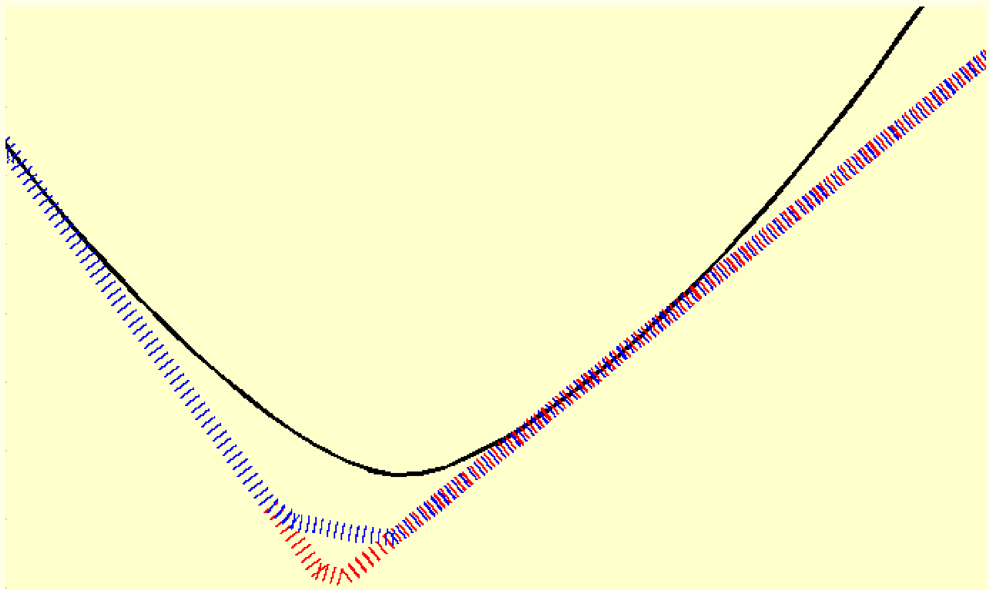


Larger

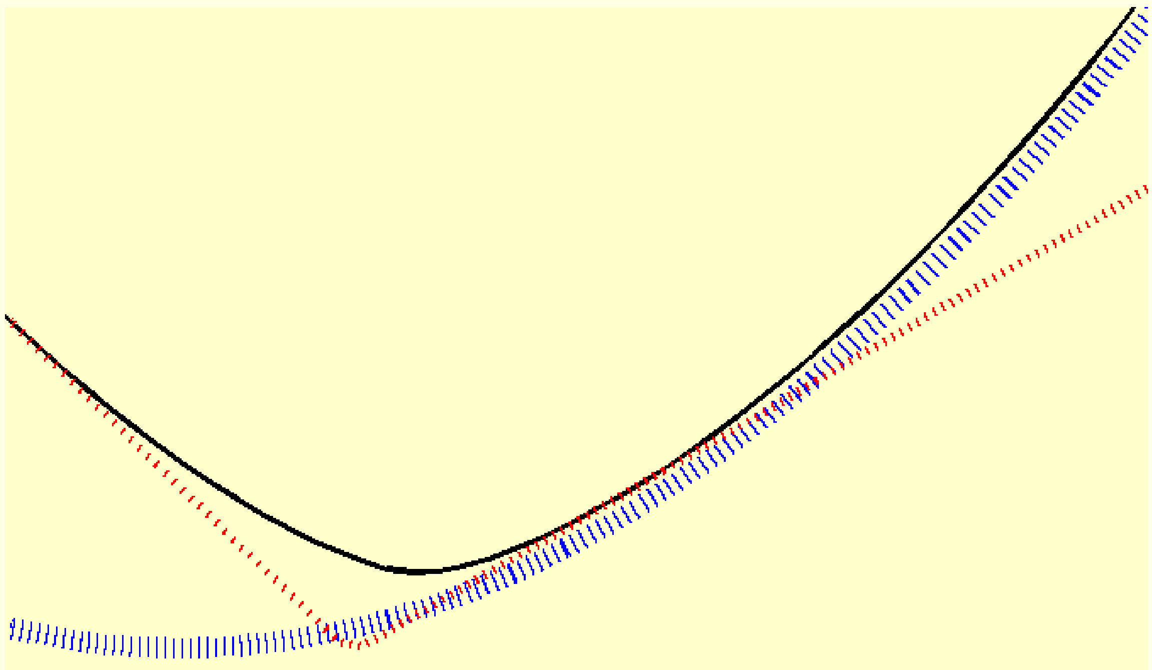
QP

Structured models for f

Composite structure

$$\mathbf{M}(x) = \max_i \left\{ G^{i\top}\Big(c(\hat{x}) + Dc(\hat{x})(x - \hat{x})\Big)\right\}$$

$$\approx h(c(\hat{x}) + Dc(\hat{x})(x - \hat{x}))$$



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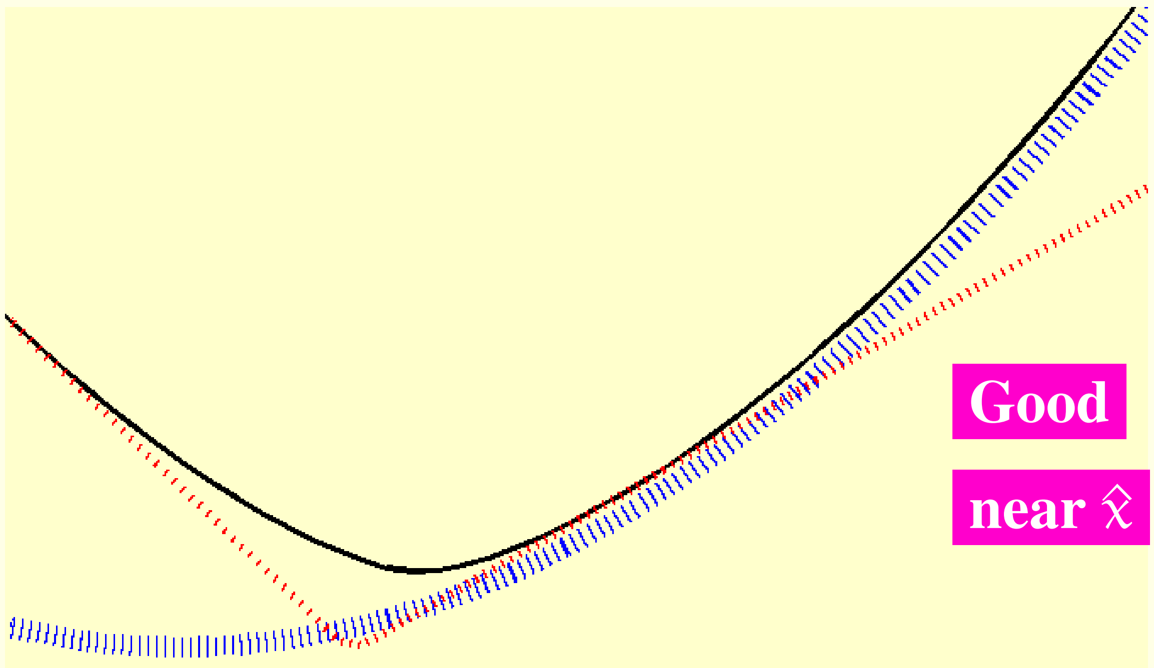

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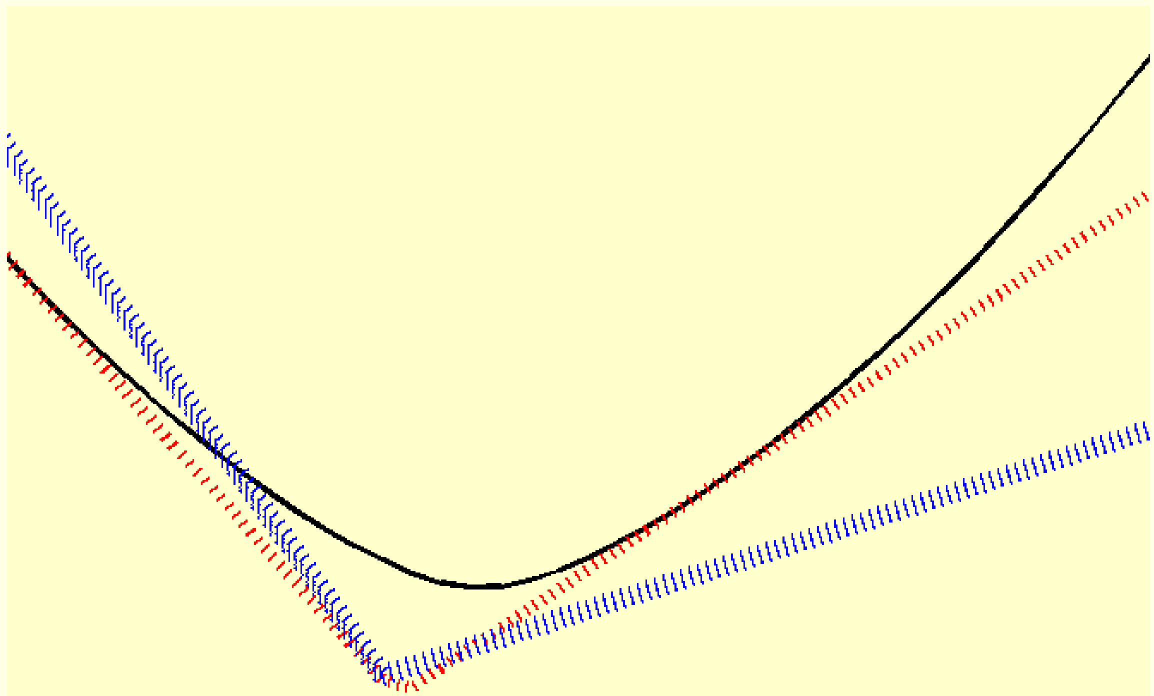

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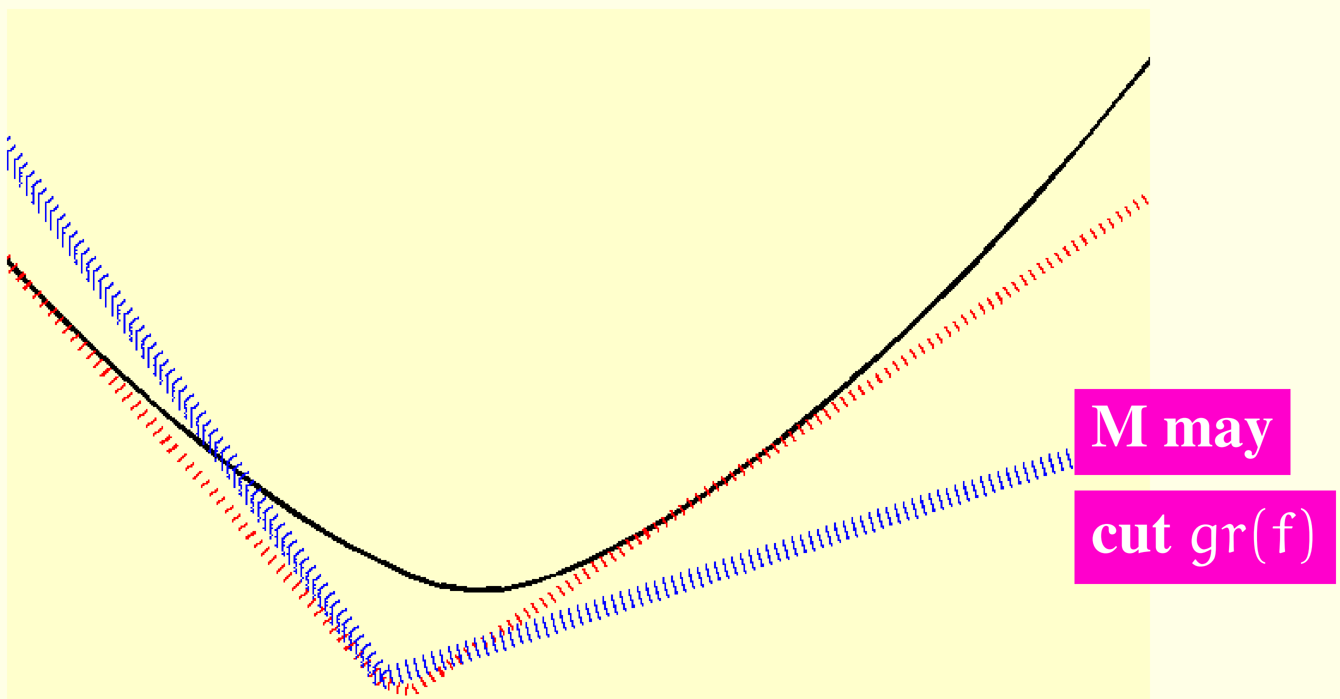
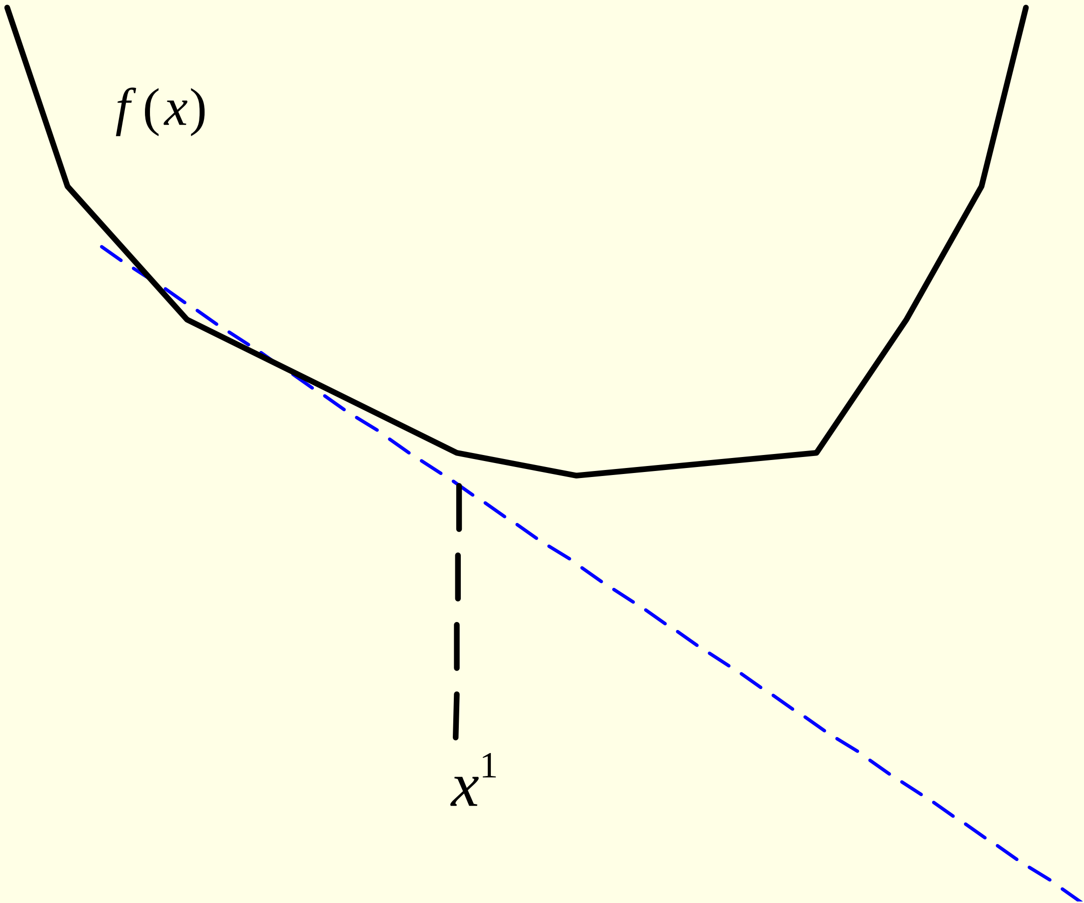


excessive noise is attenuated via stepsize t_k

Bundle Methods with Inexact Information



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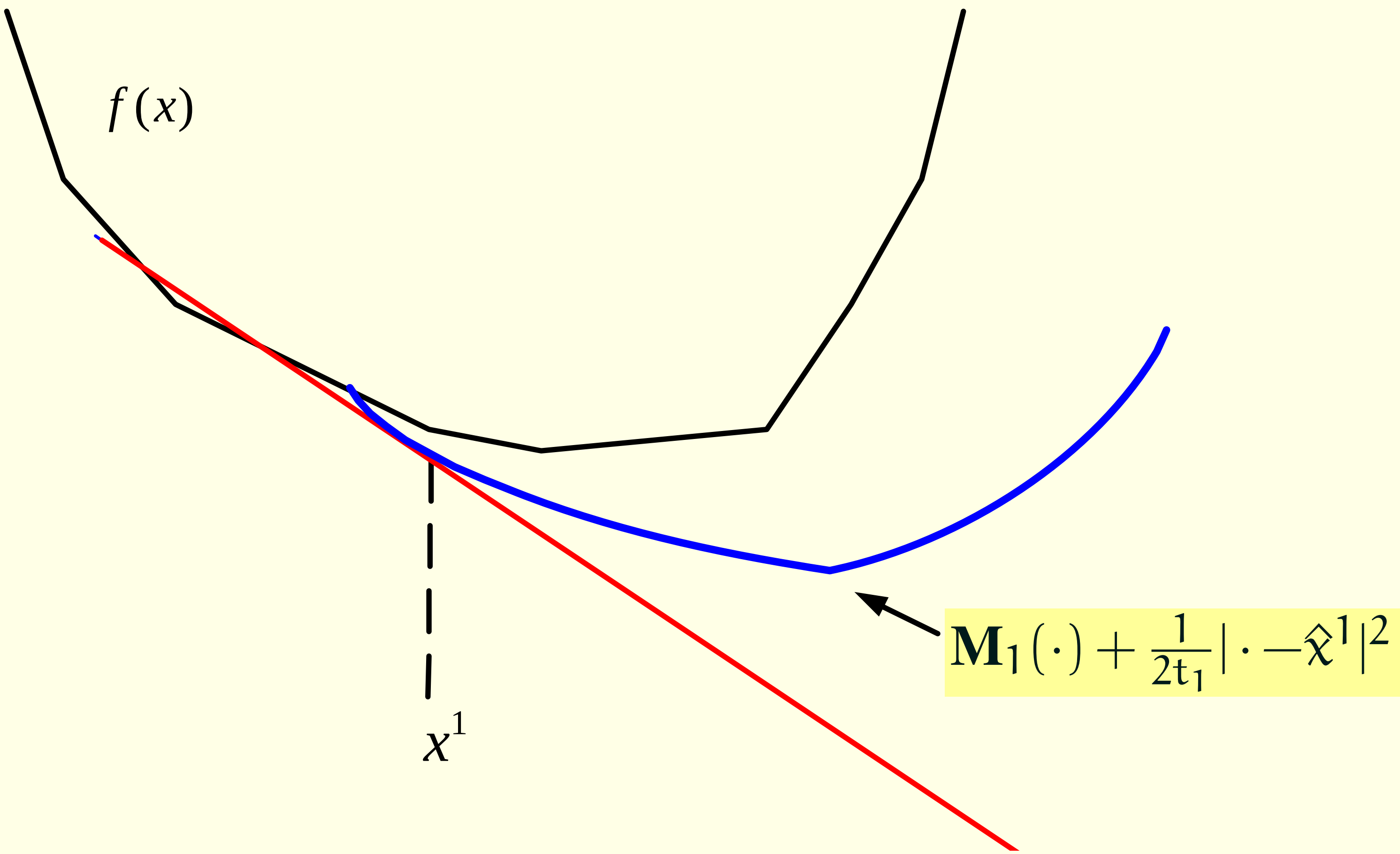

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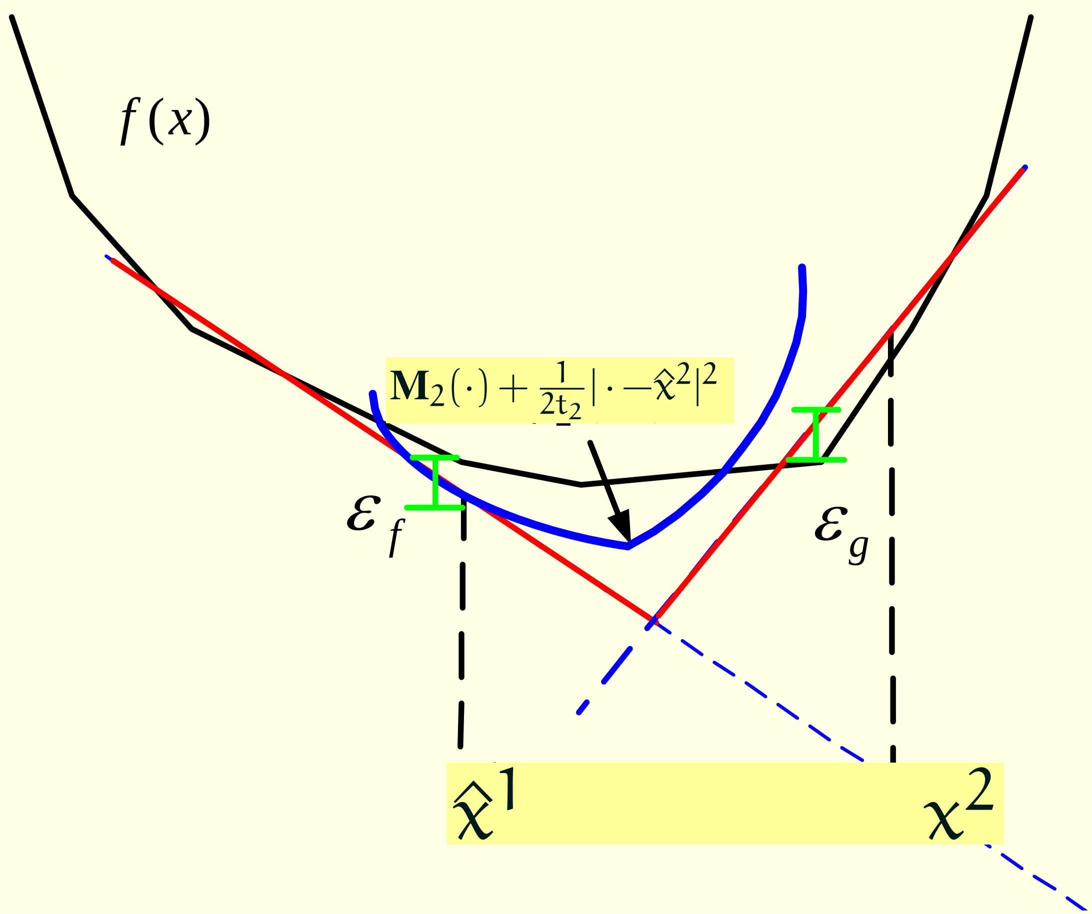

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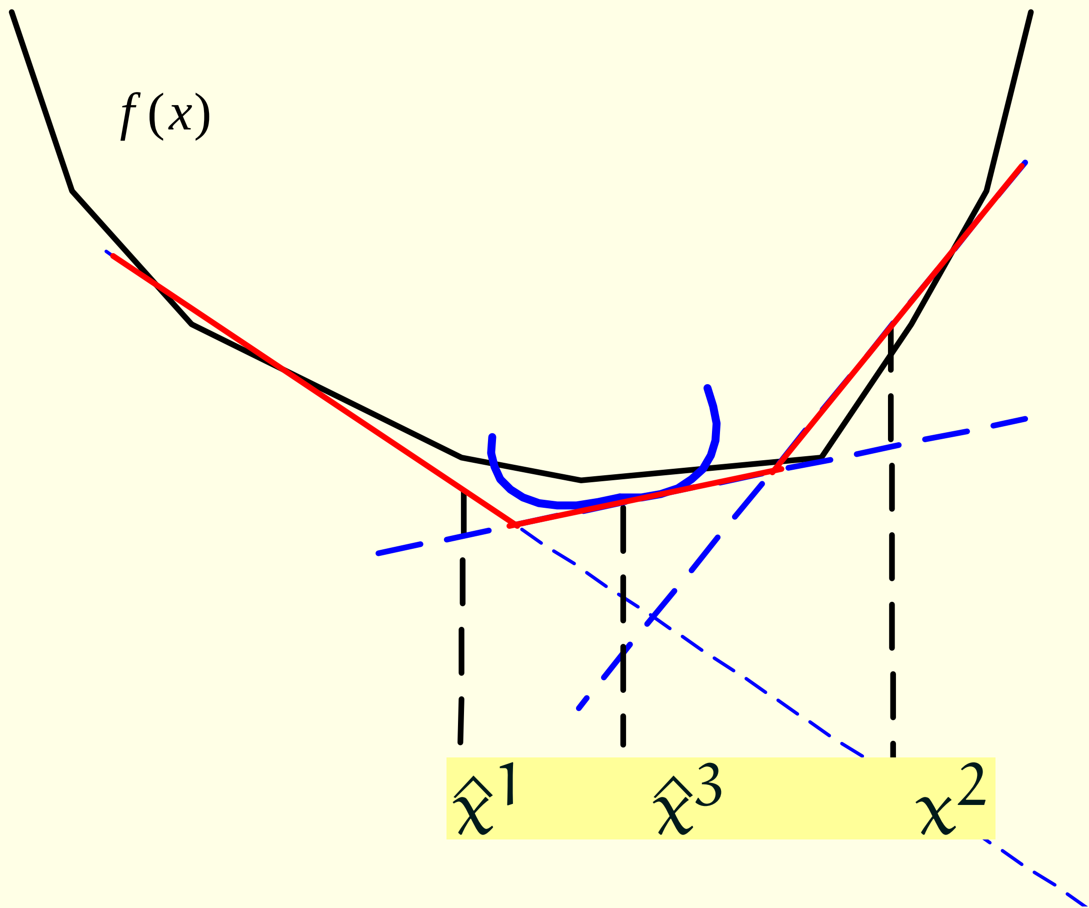
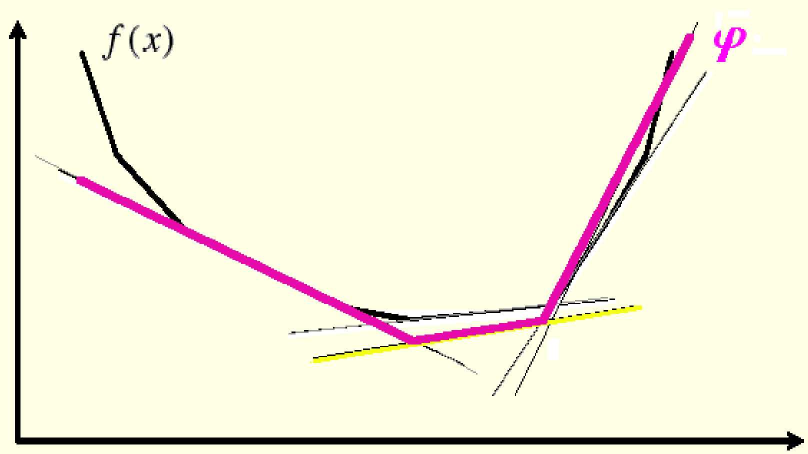


Controlling the impact of noise

$$x^{k+1} = \arg\min_x \mathbf{M}(x) + \frac{1}{2t_k}|x - \hat{x}|^2$$

now linearizations may be inexact:

x^j →  $f^j = f_{x^j}$, $g^j = g_{x^j}$ $\Longrightarrow \mathbf{M}(x) = \max_{j \le i}\left\{f^j + {g^j}^\top(x - x^j)\right\}$



and the model may be "wrong"

If too wrong: noise needs to be attenuated

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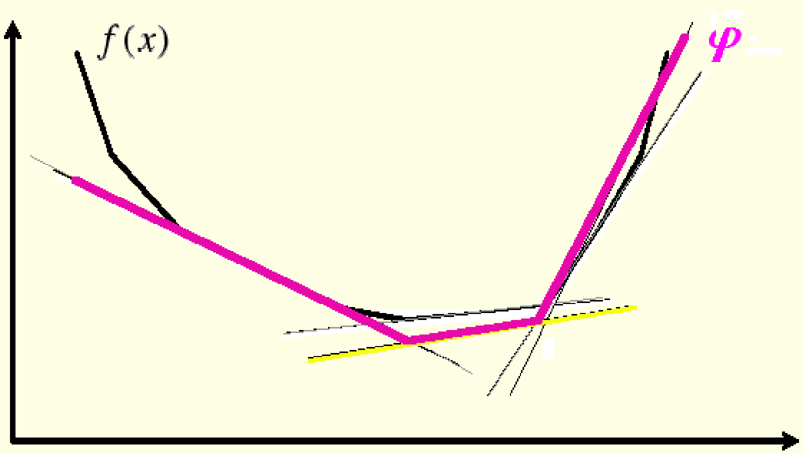
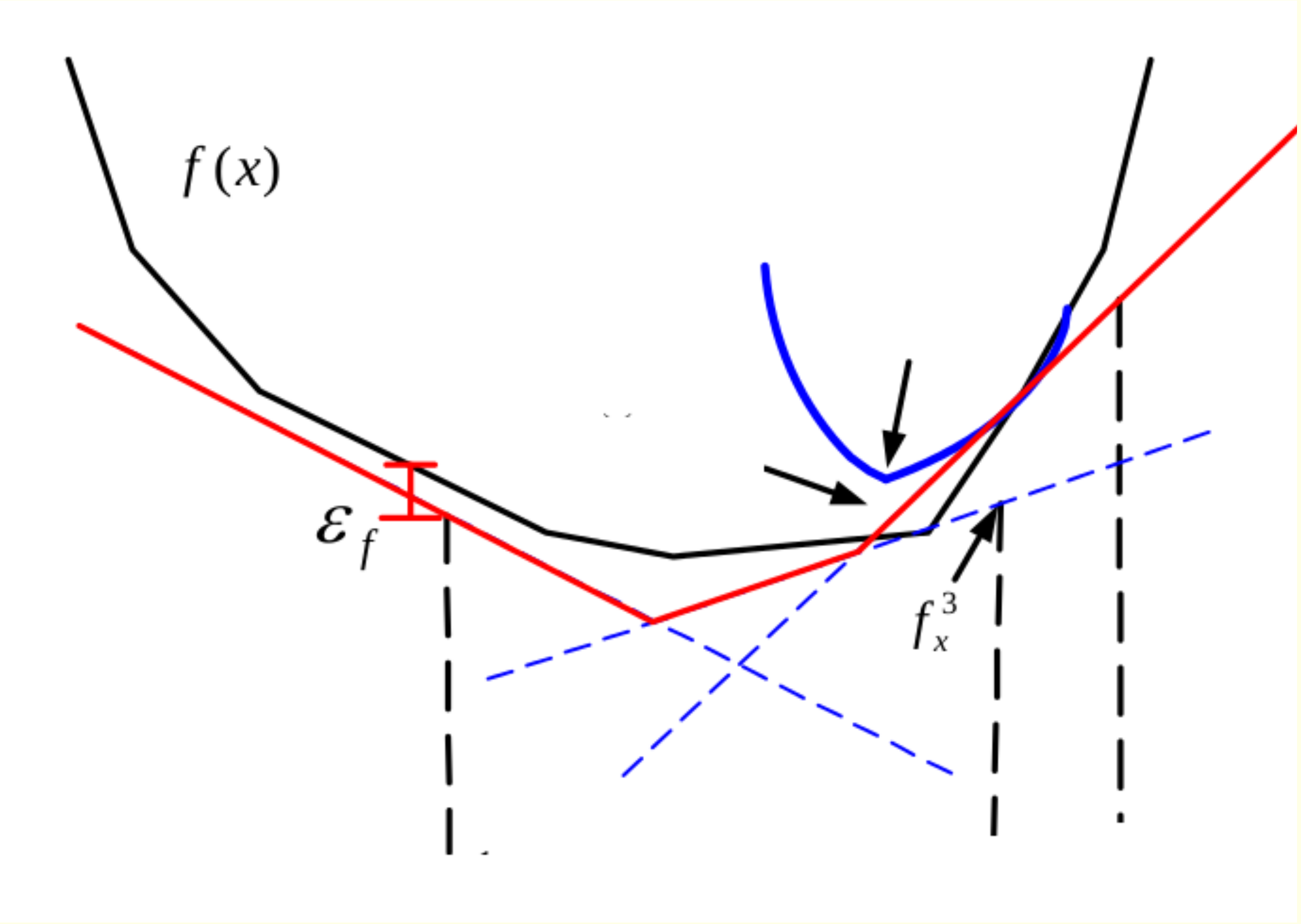


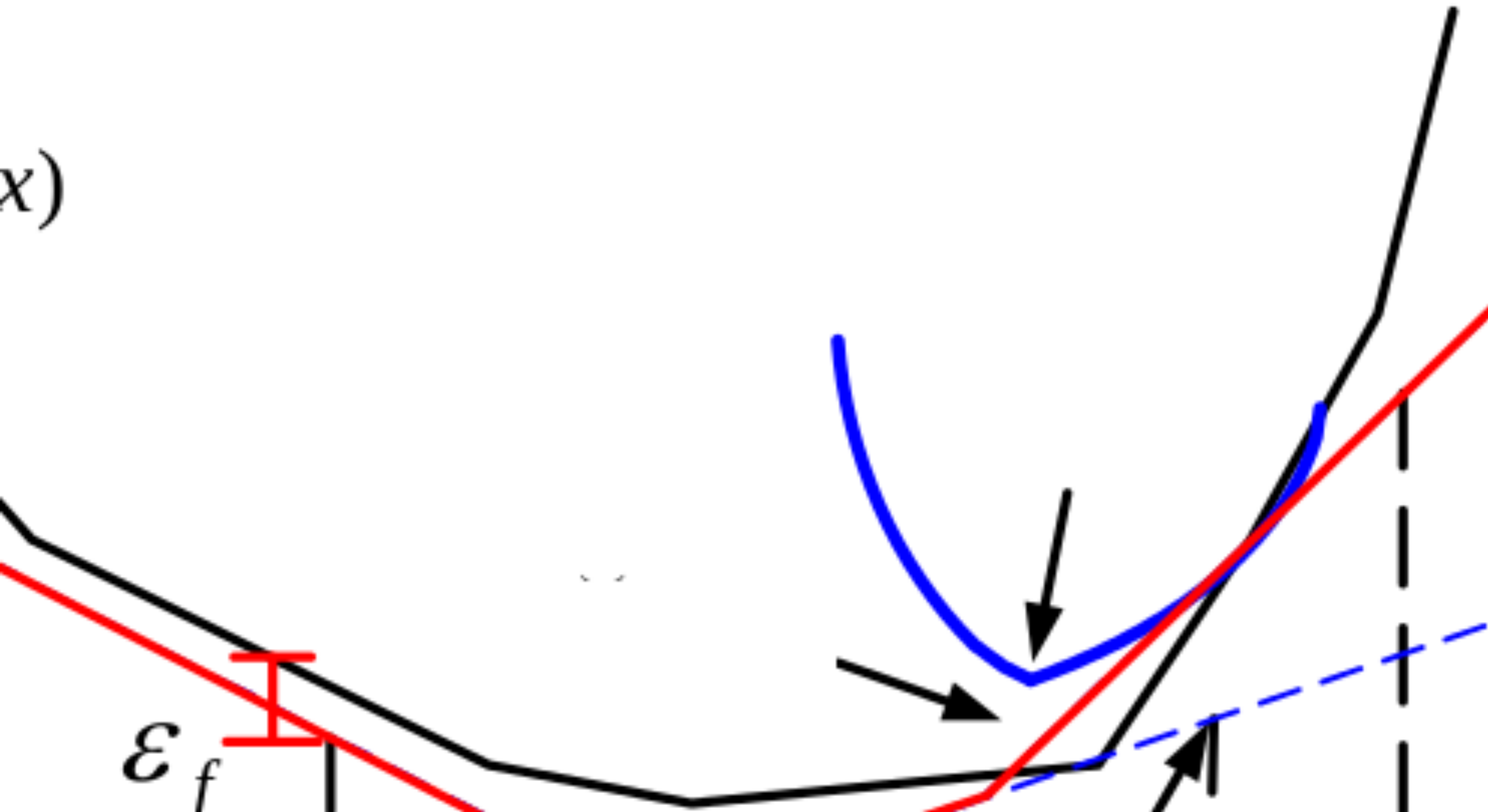
and the model may be "wrong"

Noise attenuated by increasing t, hence lowering QP value

Detecting excessive noise by checking δ_k



Detecting excessive noise: $\delta_k < 0$



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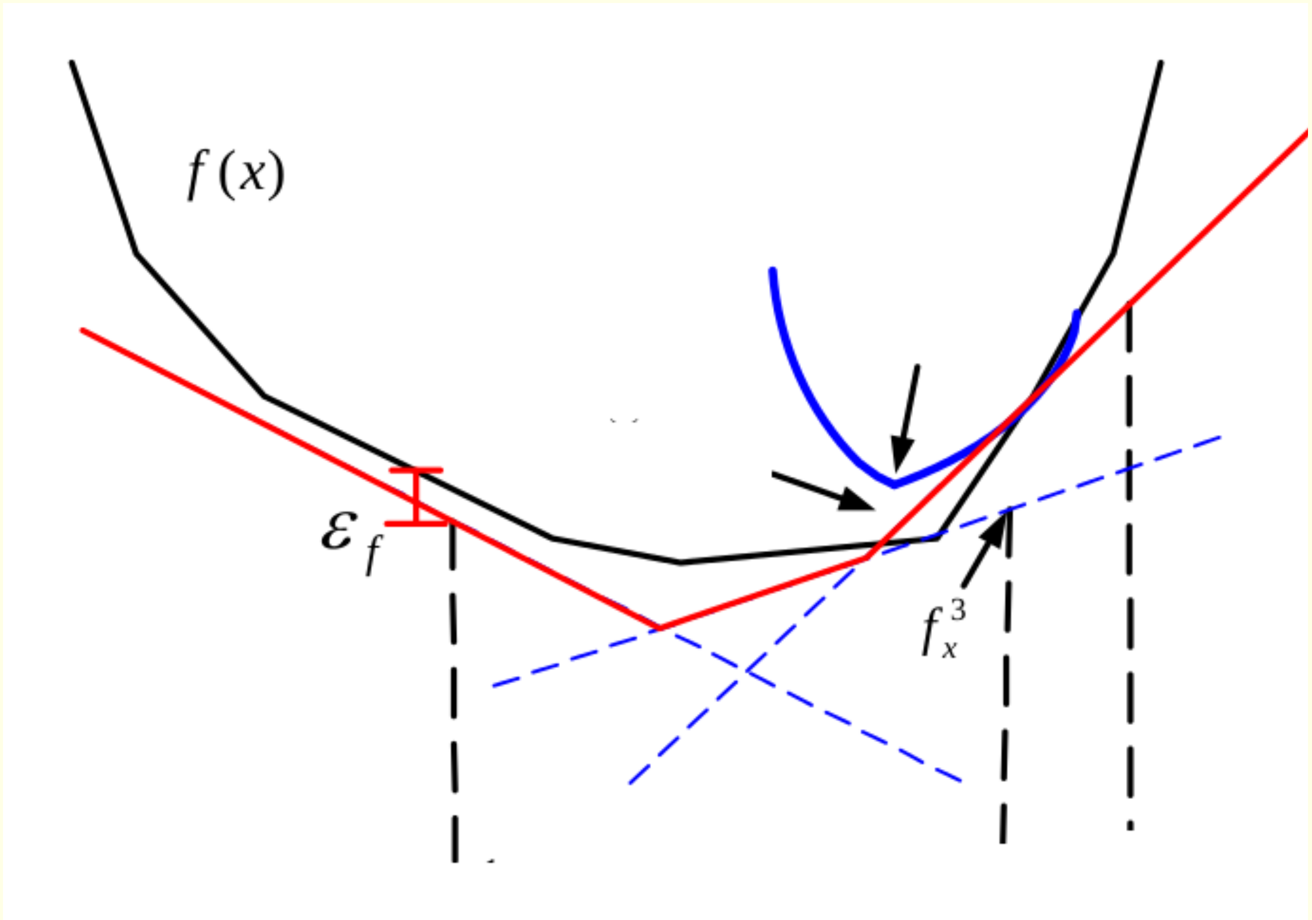

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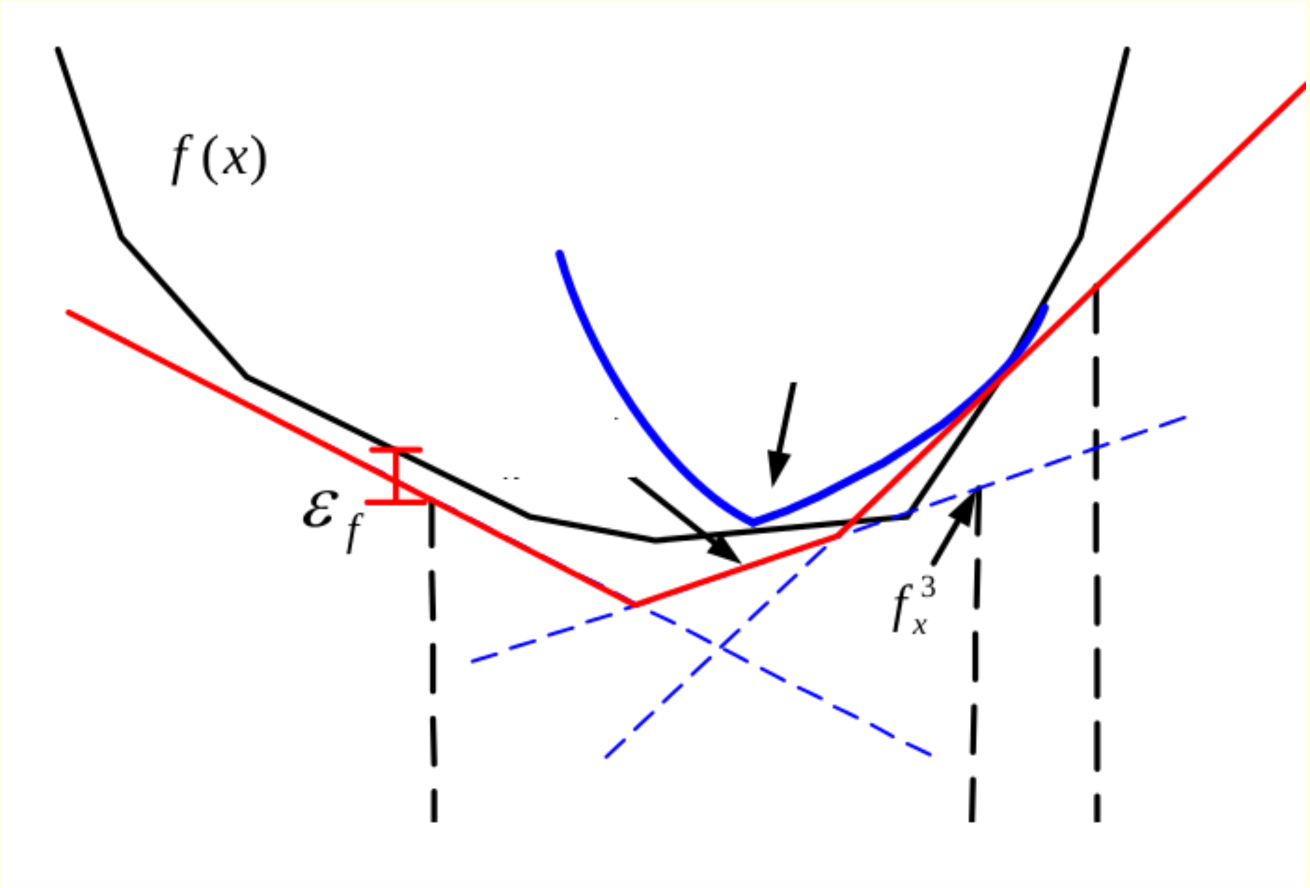


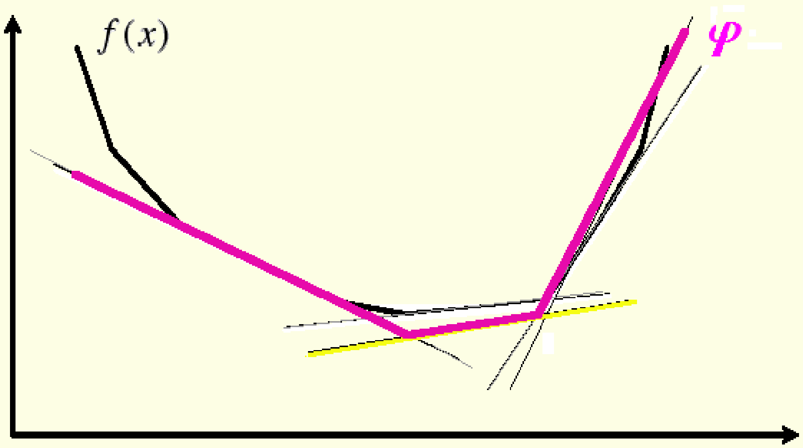
Controlling the impact of noise:

oracles with on-demand accuracy

$$x^{k+1} = \arg\min_x \mathbf{M}(x) + \frac{1}{2t_k}|x - \hat{x}|^2$$

now linearizations may be inexact:

x^j  $f^j = f_{x^j}$, $g^j = g_{x^j}$ $\Longrightarrow \mathbf{M}(x) = \max_{j \le i}\left\{f^j + g^{j\top}(x - x^j)\right\}$



If we have the ability of computing f_x/g_x

with more or less accuracy

compute (asympt.) exactly SS

and do not waste time in Null

On-demand accuracy scheme

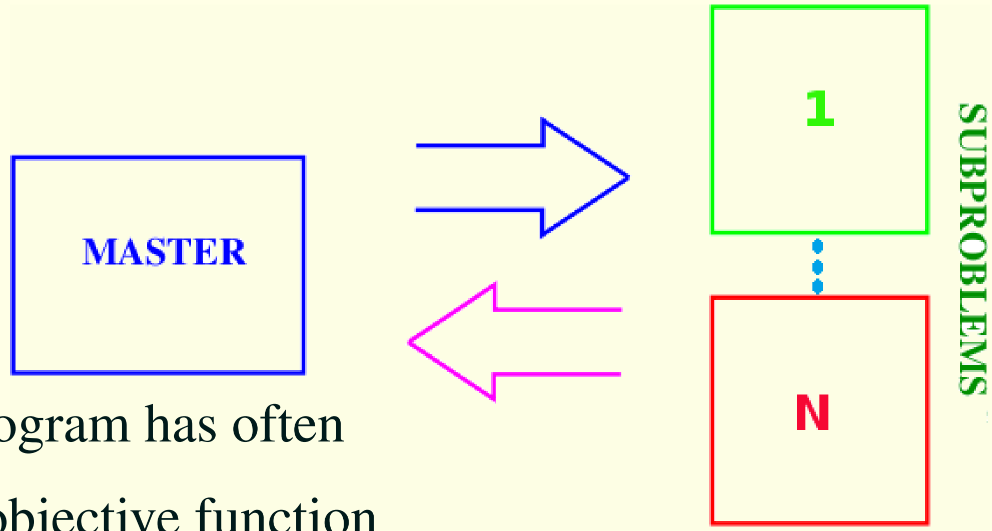
Explicit structure, induced by some **decomposition** method

by Lagrangian relaxation

by Benders decomposition

Principle: if a problem is difficult to solve directly,

solve instead a sequence of easier subproblems.



The master program has often

a nonsmooth objective function

On-demand accuracy scheme

Explicit structure, induced by some **decomposition** method

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Principle: if a problem is difficult to solve directly,
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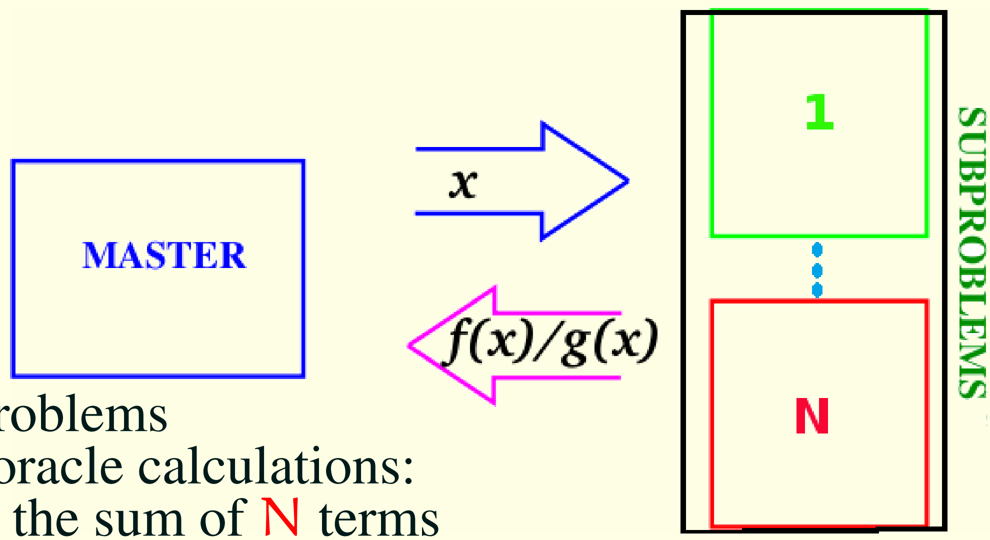

Separate subproblems
allow for fast oracle calculations:
f/g defined as the sum of N terms

Lagrangian Relaxation Example

Real-life optimization problems

$$
(\texttt{primal}) \quad \begin{cases} \max & \displaystyle\sum_{j \in J} -\mathcal{C}^j(p^j) \\ & p^j \in \mathcal{P}^j, j \in J \\ & \displaystyle\sum_{j \in J} g^j(p^j) = 0 \end{cases}
$$

Lagrangian Relaxation Example

Real-life optimization problems

$$
(\texttt{primal}) \quad \begin{cases} \min \quad \displaystyle\sum_{j\in J} \mathcal{C}^j(p^j) \\ \qquad\quad p^j \in \mathcal{P}^j, j \in J \\ \qquad\quad \displaystyle\sum_{j\in J} g^j(p^j) = 0 \end{cases} \qquad \longleftarrow x
$$

Lagrangian Relaxation Example

Real-life optimization problems

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(\texttt{primal}) \quad \begin{cases} \max \quad \displaystyle\sum_{j \in J} -\mathcal{C}^j(p^j) \\ \qquad\quad p^j \in \mathcal{P}^j, j \in J \\ \qquad\quad \boxed{\displaystyle\sum_{j \in J} g^j(p^j) = 0} \end{cases} \quad \boxed{\leftarrow x}
$$

often exhibit separable structure after dualization

Lagrangian Relaxation Example

Real-life optimization problems

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often exhibit separable structure after dualization

$$(\texttt{dual}) \quad \min_x \sum_{j\in J} \begin{cases} \max & -\mathcal{C}^j(p^j) + \left\langle x, g^j(p^j) \right\rangle \\ & p^j \in \mathcal{P}^j \end{cases}$$

Lagrangian Relaxation Example

Real-life optimization problems

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(\texttt{primal}) \quad \begin{cases} \max & \sum_{j\in J} -\mathcal{C}^j(p^j) \\ & p^j \in \mathcal{P}^j, j \in J \\ & \boxed{\sum_{j\in J} g^j(p^j) = 0} \quad \boxed{\leftarrow x} \end{cases}
$$

often exhibit separable structure after dualization

$$
(\texttt{dual}) \quad \min_x \sum_{j\in J} f^j(x)
$$

$$
f^j(x) := \begin{cases} \max & -\mathcal{C}^j(p^j) + \left\langle x, g^j(p^j) \right\rangle \\ & p^j \in \mathcal{P}^j \end{cases}
$$

Energy management problems

Typically, evaluating $f^j(x) := \begin{cases} \max & -\mathcal{C}^j(p^j) + \left\langle x, g^j(p^j) \right\rangle \\ & p^j \in \mathcal{P}^j \end{cases}$

corresponds to local subproblems, related to one power plant, requiring sometimes heavy calculations

Energy management problems

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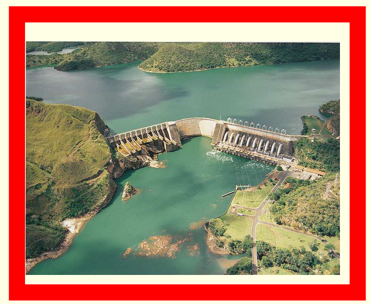
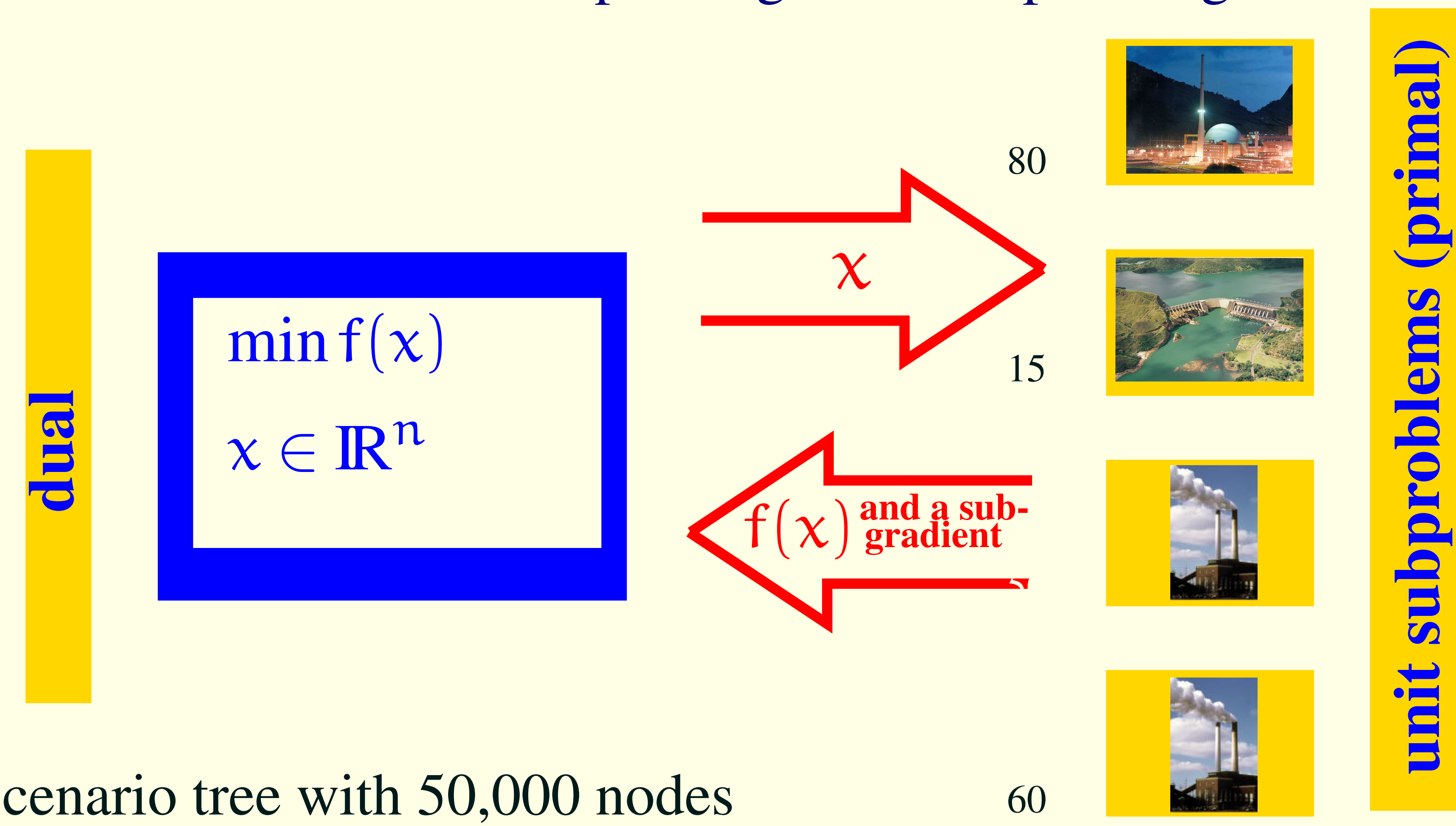


One subgradient for free: $g^j(p^j(x))$ once a solution $p^j(x)$ is available

Often, most of the CPU time is spent in the oracle calculations. For mid-term power generation planning:



Scenario tree with 50,000 nodes

Nuclear subproblems are LPs with 100,000 variables

and 300,000 constraints, consuming **99%** total running time

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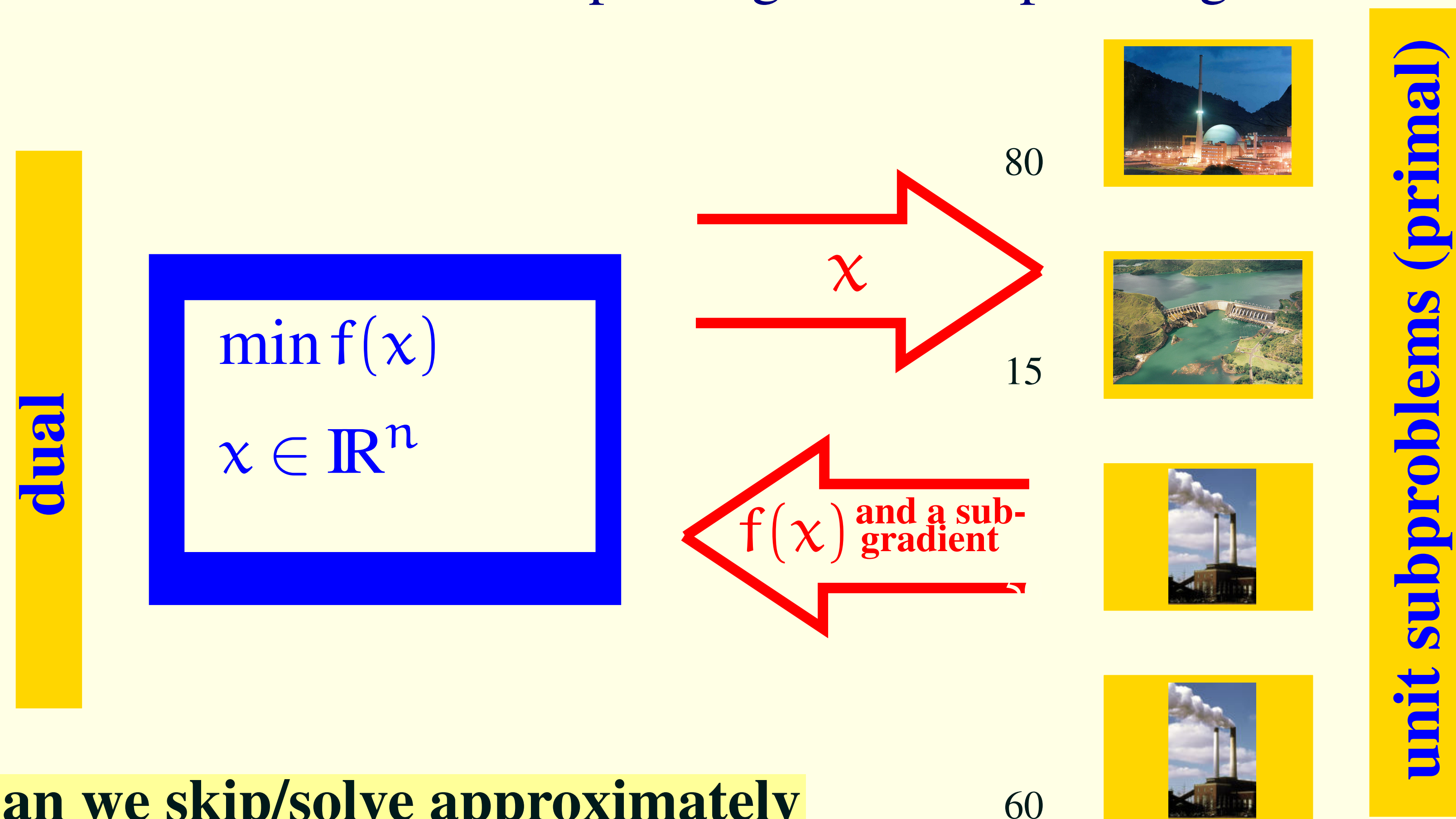
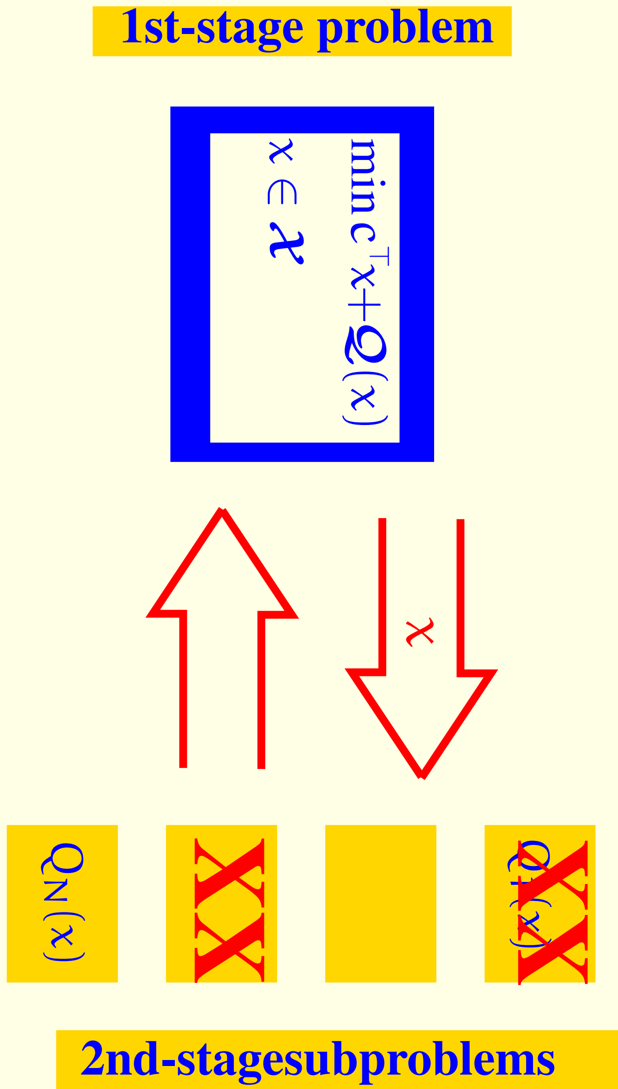


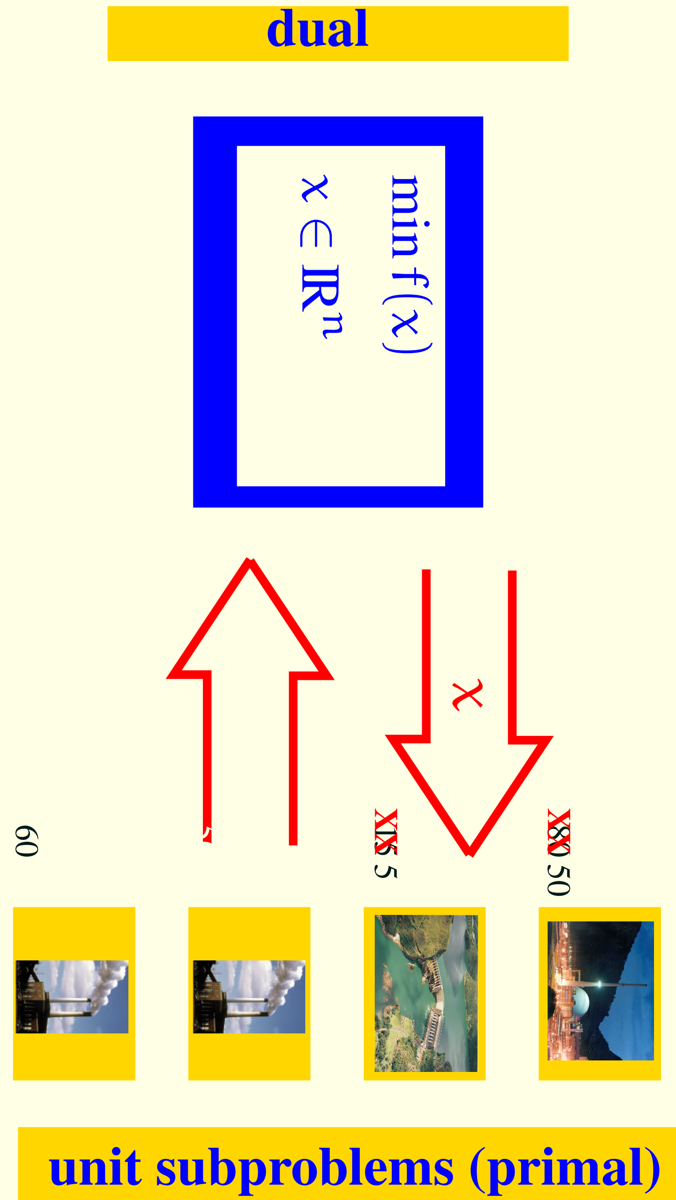
Can we skip/solve approximately

nuclear subproblems,

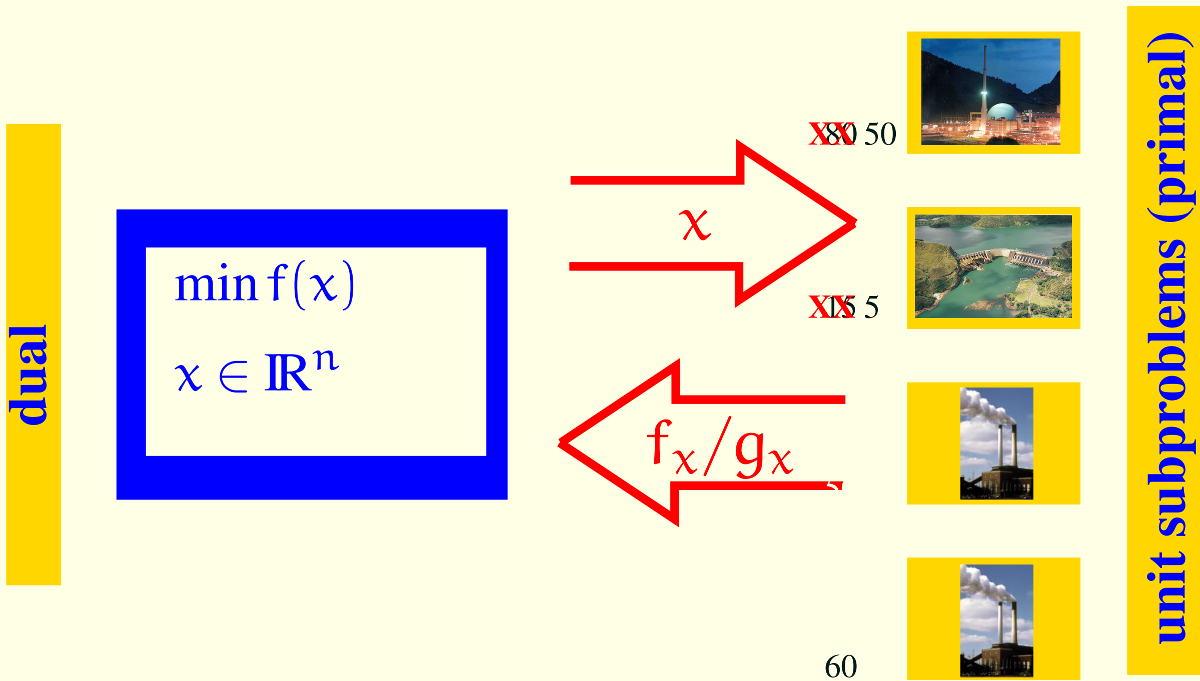
consuming LESS running time without losing accuracy?

Can we adapt the oracle response to the solver needs?

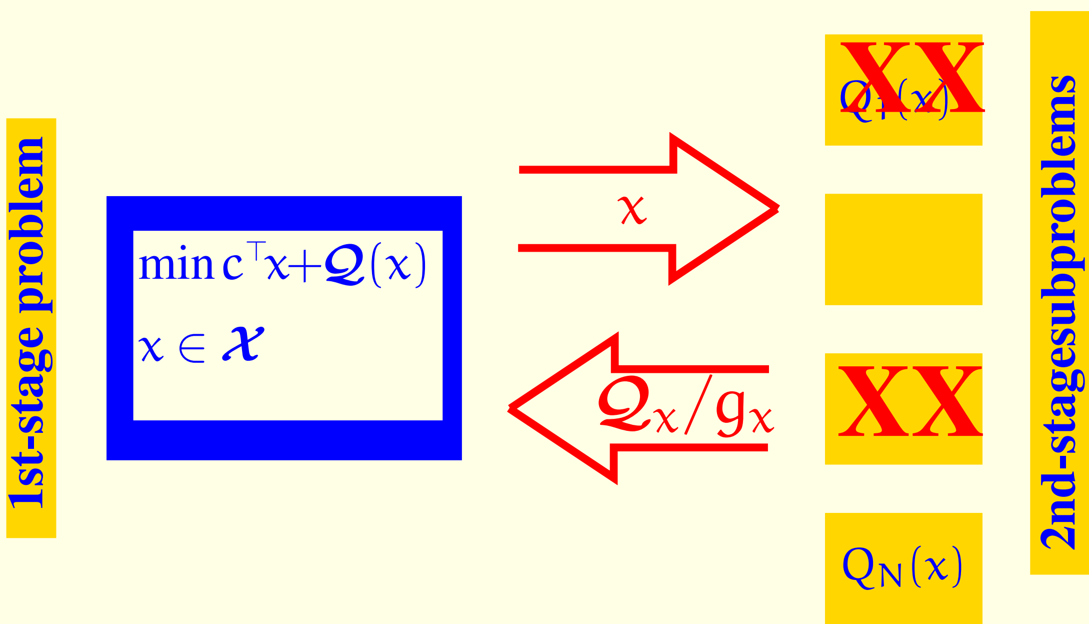




Can we adapt the oracle response to the solver needs?



now the oracle returns INEXACT values



Can we adapt the oracle response to the solver needs? **YES!**

with a NSO method capable of handling

oracles with On-demand Accuracy

Can we adapt the oracle response to the solver needs? **YES!**

with a NSO method capable of handling **oracles with On-demand Accuracy** created over noisy black-boxes

x

$$f_x \approx f(x)$$

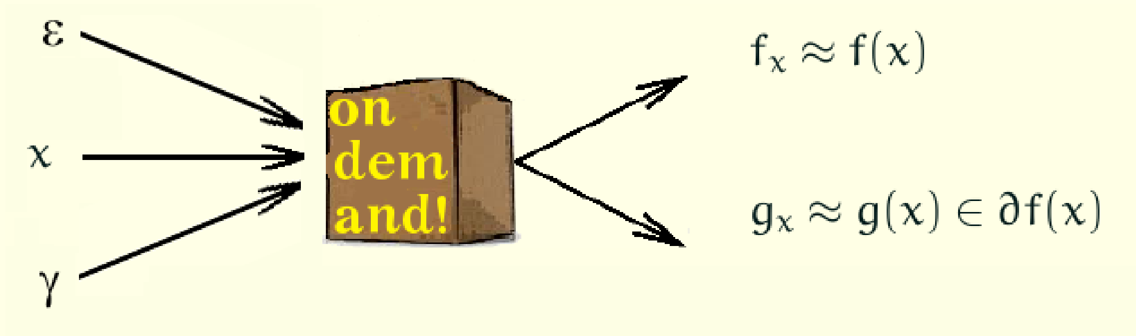
$$g_x \approx g(x) \in \partial f(x)$$

when we have the ability of computing f_x/g_x with more or less accuracy

Oracle with on-demand accuracy

For $f^j(x) := \begin{cases} \max & -\mathcal{C}^j(p^j) + \left\langle x, g^j(p^j) \right\rangle \\ & p^j \in \mathcal{P}^j \end{cases}$

we design a noisy black box that gets additional input:



an **error bound ε and a descent target γ** such that

$$\left.\begin{array}{l} f_x = f(x) - \eta(x) \\ g_x \in \partial_{\eta(x)} f(x) \end{array}\right\} \quad \text{for all } x, \text{ with } \eta(x) \geq 0$$

$$\eta(x) \leq \varepsilon \qquad \text{if } x \text{ gave enough descent: } f_x \leq \gamma$$

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For $f^j(x) := \begin{cases} \max & -\mathcal{C}^j(p^j) + \left\langle x, g^j(p^j) \right\rangle \\ & p^j \in \mathcal{P}^j \end{cases}$

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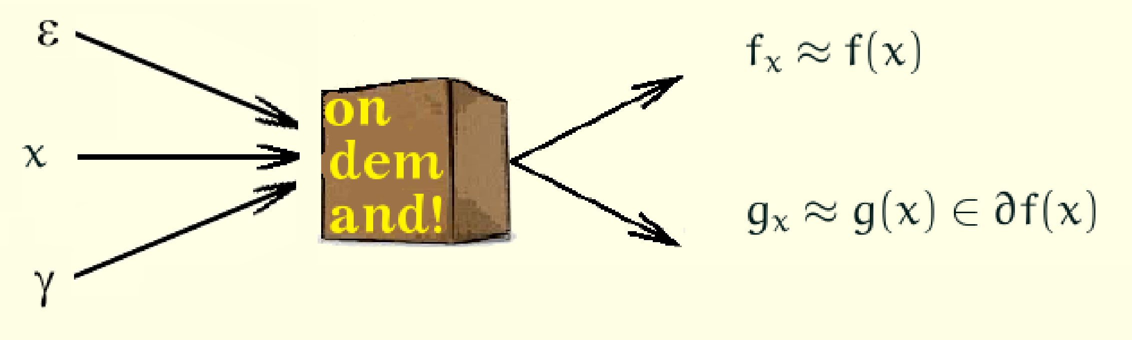

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$$\left.\begin{array}{l} f_x = f(x) - \eta(x) \\ g_x \in \partial_{\eta(x)} f(x) \end{array}\right\} \quad \text{for all } x\text{, with } \eta(x) \geq 0 \ \textbf{unknown}$$

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Classical Bundle Method

0 Choose x^1, set $k = 1$ $\hat{x}^1 = x^1$.

1 Compute $x^{k+1} \in \arg\min \mathbf{M}_k(x) + \frac{1}{2t_k}|x - \hat{x}^k|^2$

2 If $\delta_k = f(\hat{x}^k) - \mathbf{M}_k(x^{k+1}) \leq \mathtt{tol}$ STOP

3 Call the oracle at x^{k+1}.

If $f(x^{k+1}) \leq f(\hat{x}^k) - m\delta_k$, set $\hat{x}^{k+1} = x^{k+1}$ • (Serious Step)
Otherwise, maintain $\hat{x}^{k+1} = \hat{x}^k$ (Null Step)

4 Define $\mathbf{M}_{k+1}$, t_{k+1}, make $k = k+1$, and loop to 1.

Partly Exact Bundle Method

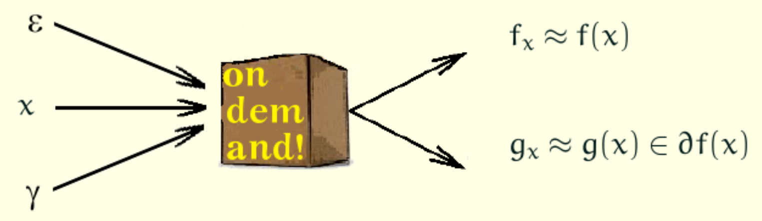
0 Choose x^1, ε_1, set $k=1$ $\hat{x}^1 = x^1$.

1 Compute $x^{k+1} \in \arg\min \mathbf{M}_k(x) + \frac{1}{2t_k}|x-\hat{x}^k|^2$

2 If $\delta_k = f_{\hat{x}^k} - \mathbf{M}_k(x^{k+1})$ **"is too negative"** $t_{k+1} = 10t_k$, go to **1**

Otherwise, if $\delta_k \le \mathtt{tol}$ STOP

3 Call the oracle at x^{k+1} **with $\gamma = f_{\hat{x}^k} - m\delta_k$, decreasing ε_k**



If $f_{x^{k+1}} \le f_{\hat{x}^k} - m\delta_k$, set $\hat{x}^{k+1} = x^{k+1}$ • (Serious Step)
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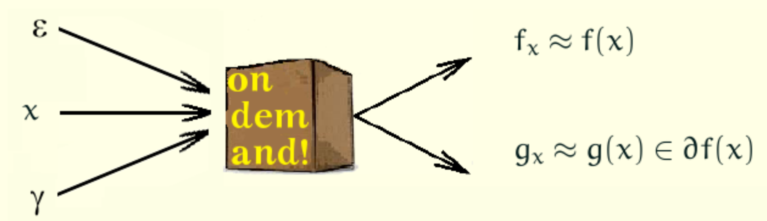

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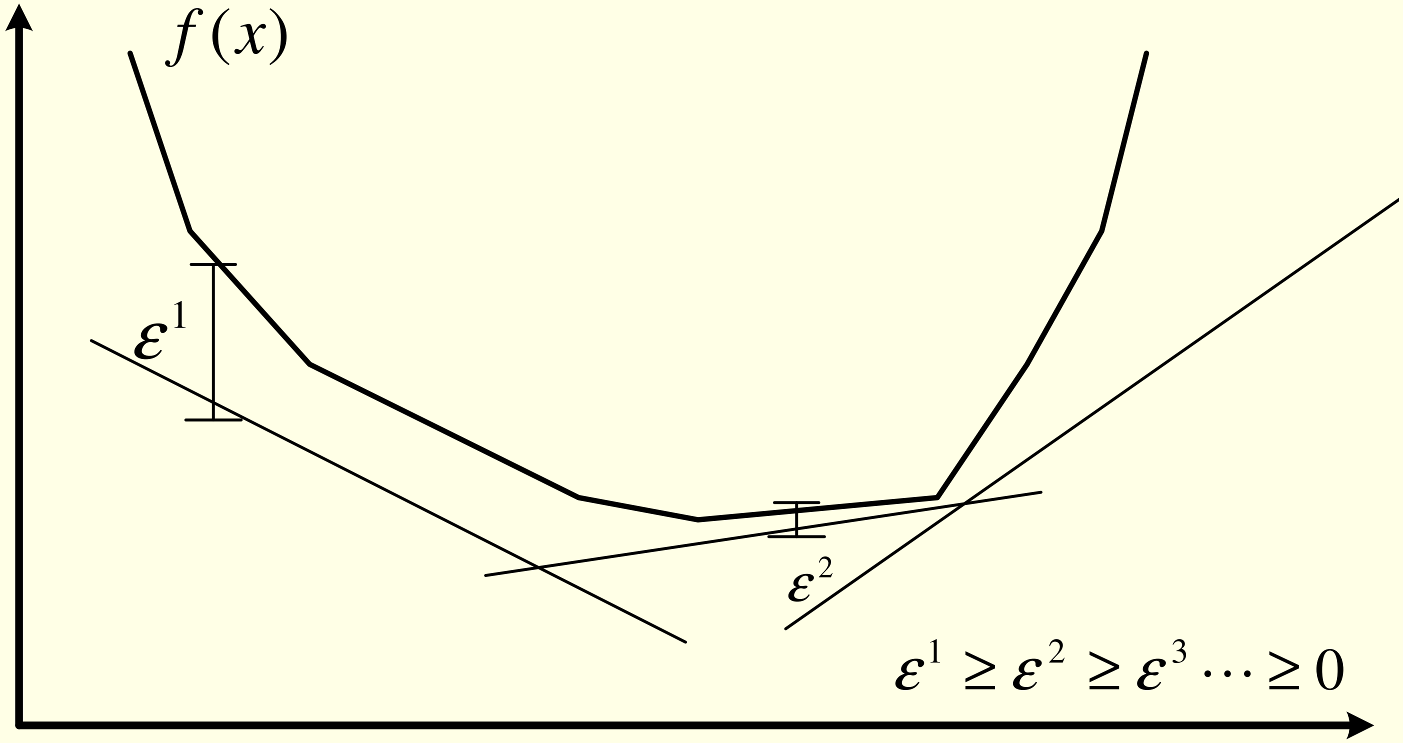
Otherwise, maintain $\hat{x}^{k+1} = \hat{x}^k$ (Null Step)

4 Define $\mathbf{M}_{k+1}$, t_{k+1}, make $k = k + 1$, and loop to 1.

as $\varepsilon_k \to 0$, $f_{\hat{x}^k} \to f(\hat{x}^k)$, the method finds exact solutions!


$f(x)$
ε^1
ε^2
$\varepsilon^1 \geq \varepsilon^2 \geq \varepsilon^3 \cdots \geq 0$

Oracle with on-demand accuracy: versatility

$$\left.\begin{aligned} f_x &= f(x) - \eta(x) \\ g_x &\in \partial_{\eta(x)} f(x) \end{aligned}\right\} \quad \text{for all } x, \text{ with } \eta(x) \geq 0$$

$$\eta(x) \leq \varepsilon \qquad \text{if } x \text{ gave enough descent: } f_x \leq \gamma$$

We control both ε and γ, which can vary with x:

newline – $\varepsilon_x = 0$ and $\gamma_x = +\infty$ is an exact oracle.

newline – $\varepsilon_x \to 0$ along the iterative process and $\gamma_x = +\infty$ is an asymptotically exact oracle

newline – $\varepsilon_x = 0$ with finite γ_x gives a partly inexact oracle

newline – $\varepsilon_x > 0$ unknown, but bounded, with $\gamma_x = +\infty$ is an inexact oracle

Theoretical Results

Convex proximal bundle methods in depth: a unified analysis for inexact oracles

W. de Oliveira, C. Sagastizábal, C. Lemaréchal

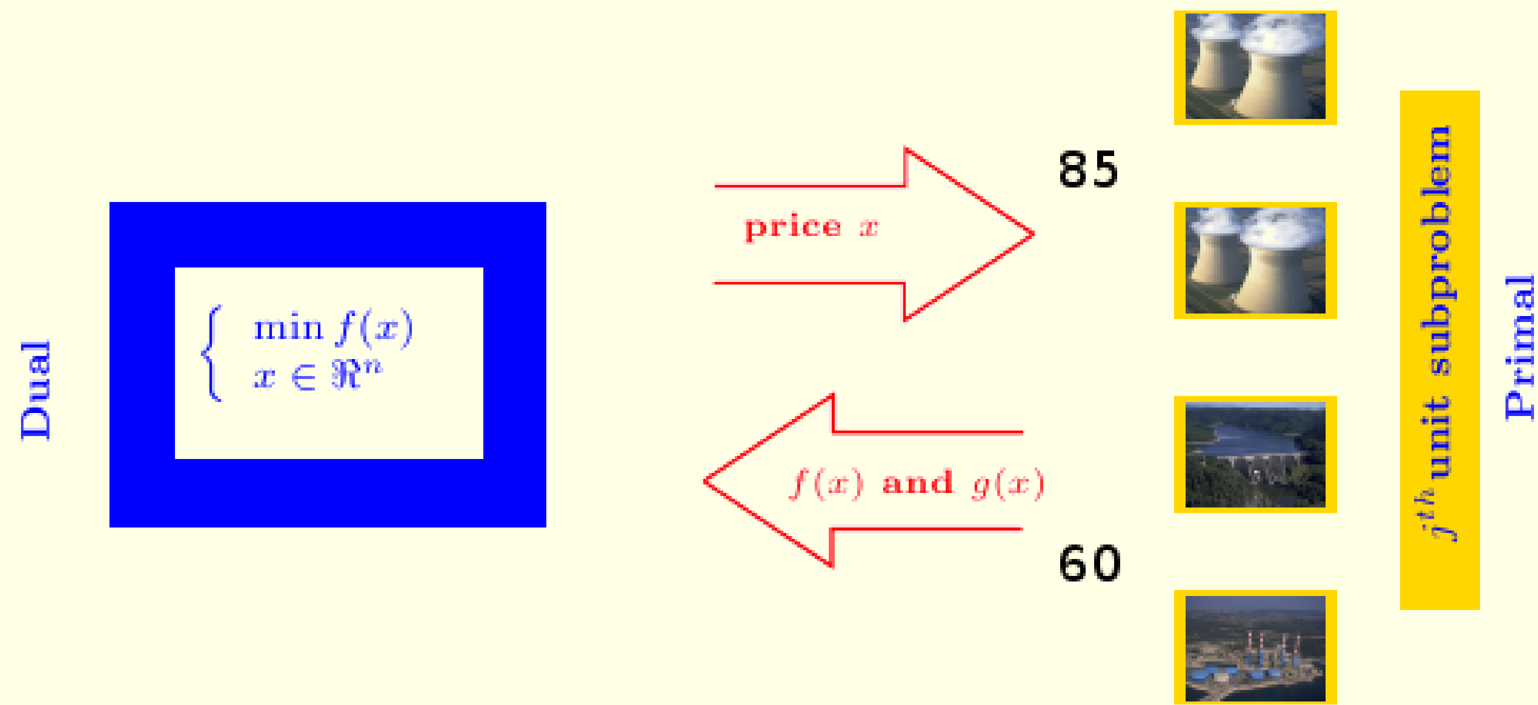
MathProg 148, pp 241-277, 2014

General and versatile convergence theory for inexact oracles, including

- asymptotically exact ones (driving ε to 0).
- inexact oracles (convergence within accuracy bound)
- lower an dupper oracles
- previous exact bundle variants
- new ones

Application in Energy I

Mid-term planning for power generation

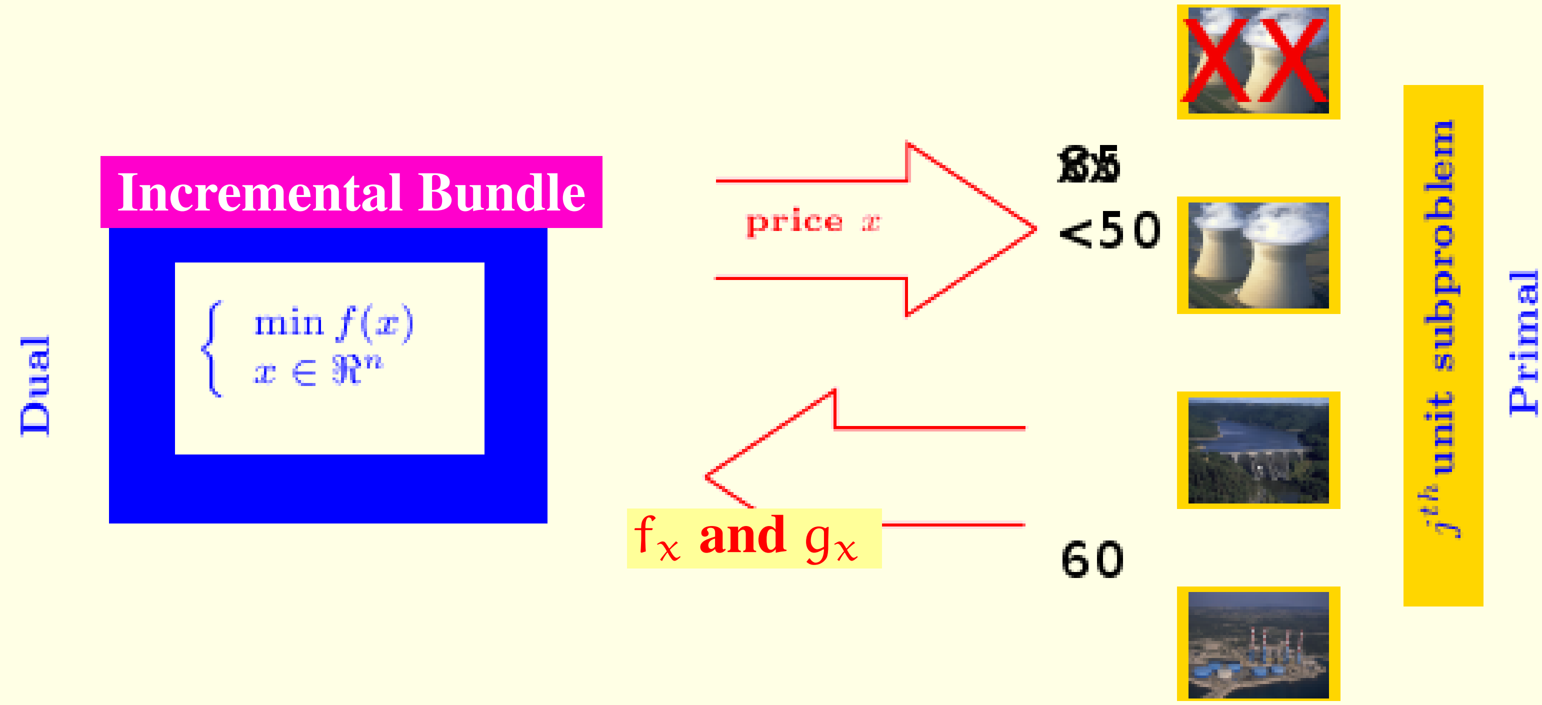


Scenario tree with 50,000 nodes

Nuclear LPs with 100,000 variables and 300,000 constraints

Application in Energy I

Mid-term planning for power generation

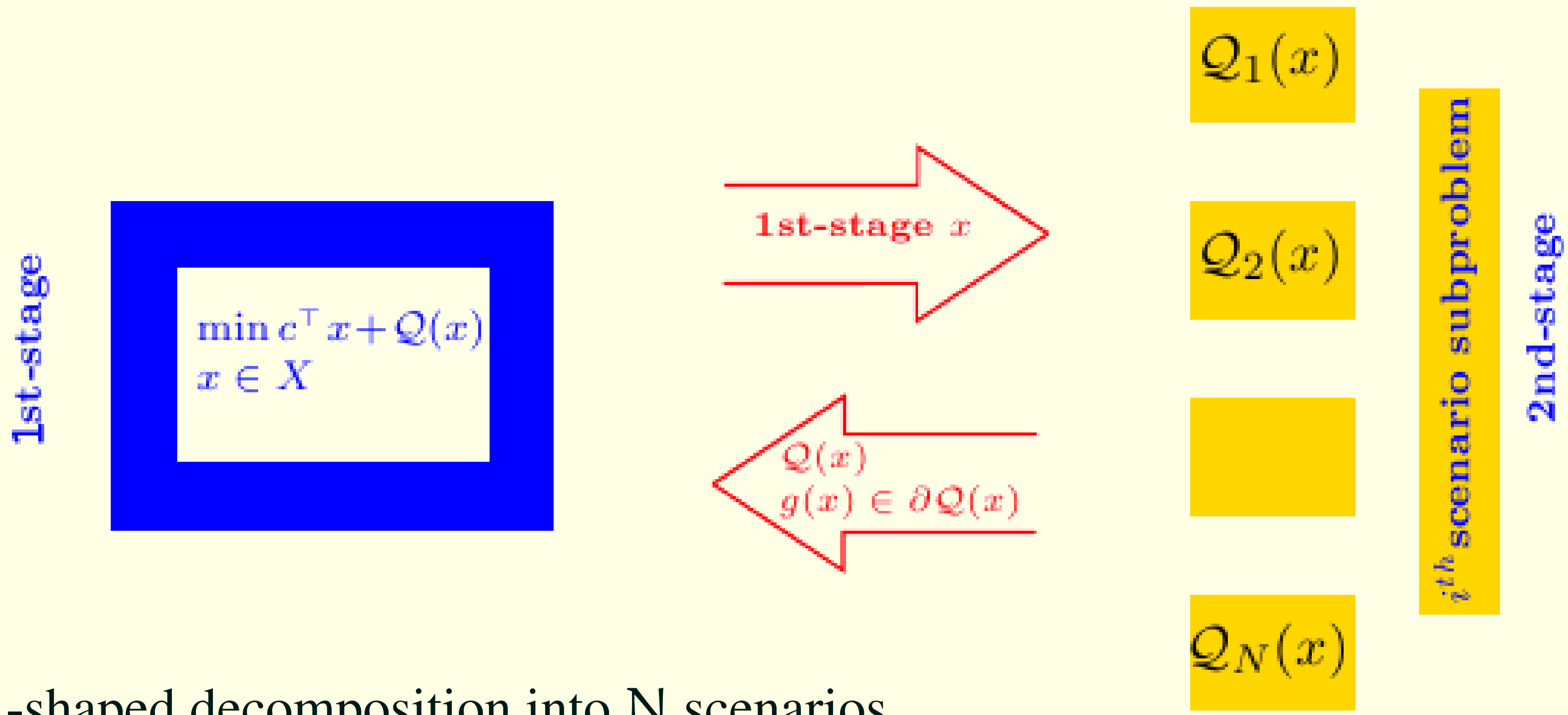


Skips Nuclear LPs (alternating) $\equiv$ noisy black box

25% less CPU time than exact bundle, same accuracy

Application in Energy II

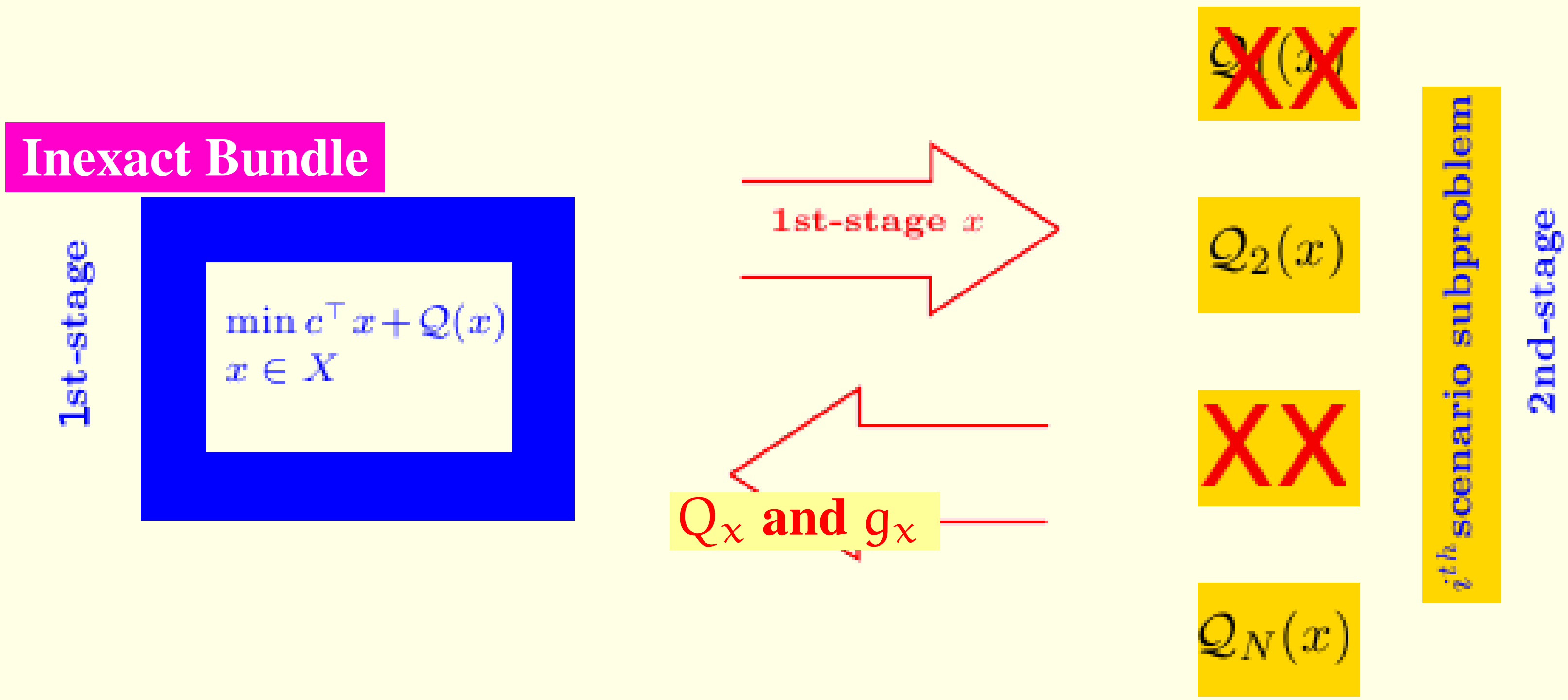
2-stage stochastic linear programs



L-shaped decomposition into N scenarios

Application in Energy II

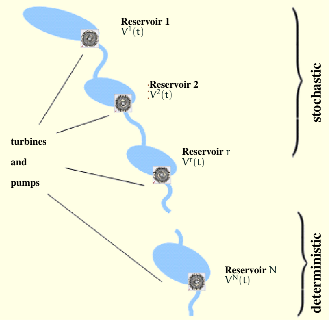
2-stage stochastic linear programs



Skips 80% LPs solution ≡ noisy black box

4 times faster than L-shaped, same accuracy

Applications in Energy III



Maximize revenue of hydro producers keeping reservoir levels between min-zones with 90% confidence (numerical integration in dimension 192!)

Comparison with previous values obtained by Wim van Ackooij, from R&D at EDF on several instances from Val d'Isère (Alpes), using a method by A. Prékopa.

Huge reduction in CPU times: **drops from almost 3h to 3 minutes**

Closing remarks

- Thanks to Welington de Oliveira and Marc Schmidt for some of the images.
- Credits to some co-authors: Welington de Oliveira, Claude Lemaréchal, Wim van Ackooij
- **Warning:** This tutorial does not intend to encourage drinking caipirinha.

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- Credits to some co-authors: Welington de Oliveira, Claude Lemaréchal, Wim van Ackooij
- **Warning:** This tutorial does not intend to encourage drinking caipirinha.

  It is rather meant to facilitate the use of modern (on-demand accuracy) bundle methods.

Any doubts or questions, just e-mail me

To learn more

(exact) Bundle books

J.F. Bonnans, J.C. Gilbert, C. Lemaréchal, and C. Sagastizábal, Numerical Optimization: Theoretical and Practical Aspects, Springer, 2nd ed., 2006.

J.B. Hiriart-Urruty and C. Lemaréchal, Convex Analysis and Minimization Algorithms II, no. 306 in Grund. der math. Wissenschaften, Springer, 2nd ed., 1996.

Inexact Bundle theory

M. Hintermüller, A proximal bundle method based on approximate subgradients, COAp, 20 (2001), pp. 245–266.

K.C. Kiwiel, A proximal bundle method with approximate subgradient linearizations, SiOpt, 16 (2006), pp. 1007–1023.

W. de Oliveira, C. Sagastizábal, and C. Lemaréchal, Convex proximal bundle methods in depth: a unified analysis for inexact oracles, MathProg, 148 (2014), pp. 241–277.

Inexact Bundle variants with applications

G. Emiel and C. Sagastizábal, Incremental-like bundle methods with application to energy planning, COAp, 46 (2010), pp. 305–332.

W. de Oliveira, C. Sagastizábal, and S. Scheimberg, Inexact bundle methods for two-stage stochastic programming, SiOpt, 21 (2011), pp. 517–544.

W. van Ackooij and C. Sagastizábal, Constrained bundle methods for upper inexact oracles with application to joint chance constrained energy problems, SiOpt, 24 (2014), pp. 733–765.

W. de Oliveira and C. Sagastizábal, Level bundle methods for oracles with on-demand accuracy, OMS 29 (2014), pp. 1180 –1209

W. de Oliveira and C. Sagastizábal, Bundle methods in the xxi century: A birds'-eye view, Pesquisa Operacional, 34 (2014), pp. 647 – 670.

W. de Oliveira and M. Solodov, A doubly stabilized bundle method for nonsmooth convex optimization, accepted in MathProg, 2015.

and my web-page: `http://www.impa.br/~sagastiz`

Save the date!