Discretization and Dualization of Linear-Quadratic Control Problems with Bang-Bang Solutions

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Where is Jena?

Is it a beautiful place?

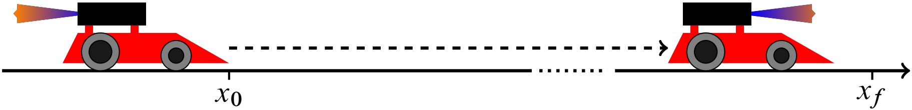
Outline

Part 1

Motivation: Optimal Control with ODEs

The Rocket Car

A car powered by a rocket engine has to reach its aim as precise as possible in a given time $t_f > 0$. Therefor let $x_1(t)$ be the position, $x_2(t)$ the velocity and $u(t)$ the acceleration (**control**) of the car at time $t \in [0, t_f]$.



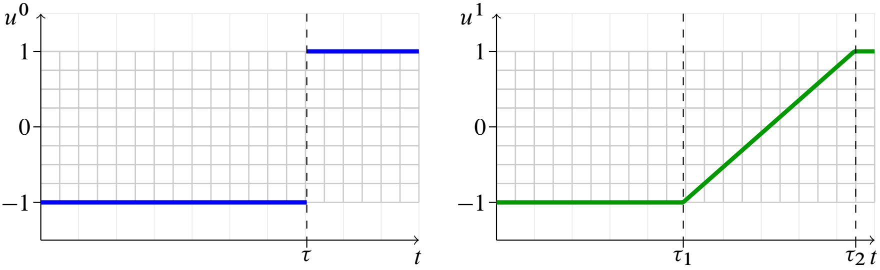
$$
\begin{aligned}
\min \quad & \frac{1}{2}\left(x_1(t_f)^2 + x_2(t_f)^2\right) + \frac{\alpha}{2}\,\|u\|_2^2 \\
\text{s. t.} \quad & \dot{x}_1(t) = x_2(t)\,,\ \ \dot{x}_2(t) = u(t) \quad \text{a.e. on } [0, t_f]\,, \\
& x_1(0) = 6\,, \qquad x_2(0) = 1\,, \\
& u(t) \in [-1, 1] \qquad \text{a.e. on } [0, t_f]\,.
\end{aligned}
$$

x_1 ... position of the car

x_2 ... velocity of the car

u ... acceleration of the car (**control**)

Optimal control for $\alpha = 0$ and $\alpha = 1$ ($t_f = 5$):



$$u^0(t) = \begin{cases} -1\,, & \text{for } 0 \le t \le \tau\,, \\ +1\,, & \text{for } \tau < t \le t_f \end{cases}$$

$$u^\alpha(t) = \mathrm{Pr}_{[-1,1]}\left[-\frac{1}{\alpha}\lambda_2^\alpha(t)\right]$$

Parts 3 and 4 are joint work with **Walter Alt** and **Martin Seydenschwanz**.

Part 5 is joint work with **Walter Alt** and **Yalcin Kaya**.

Part 2

Linear-Quadratic Control Problems

Basic Results

Problem (PQ)

$$\begin{aligned} \min \quad & f(x,u) \\ \text{s.t.} \quad & \dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) \qquad \text{a.e. on } [t_0, t_f]\,, \\ & x(0) = x_0\,, \\ & u(t) \in U := \{u \in \mathbb{R}^m \mid b_\ell \le u \le b_u\} \text{ a.e. on } [t_0, t_f]\,, \end{aligned}$$

where f is a linear-quadratic cost functional defined by

$$\begin{aligned} f(x,u) = {} & \frac{1}{2} x(t_f)^\mathsf{T} Q x(t_f) + q^\mathsf{T} x(t_f) \\ & + \int_{t_0}^{t_f} \frac{1}{2} x(t)^\mathsf{T} W(t) x(t) + w(t)^\mathsf{T} x(t) + x(t)^\mathsf{T} S(t) u(t) + r(t)^\mathsf{T} u(t) \,\mathrm{d}t\,. \end{aligned}$$

$u(t) \in L^{\infty}(t_0, t_f; \mathbb{R}^m)$ is the control and $x(t) \in W^1_{\infty}(t_0, t_f; \mathbb{R}^n)$ is the state of the system at time t. The functions W, S, w, r, A, B, and b are Lipschitz continuous.

(AC) The matrices Q and $W(t)$, $t \in [t_0, t_f]$, are symmetric and

$$x(t_f)^{\mathsf{T}} Q x(t_f) + \int_{t_0}^{t_f} x(t)^{\mathsf{T}} W(t) x(t) + 2x(t)^{\mathsf{T}} S(t) u(t) \,\mathrm{d}t \geq 0$$

for all $(x, u) \in X$ with

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) && \text{a.e. on } [t_0, t_f]\,, \\ x(t_0) &= 0\,, \\ u(t) &\in U - U && \text{a.e. on } [t_0, t_f]\,. \end{aligned}$$

Then (PQ) is a **convex optimization problem** and a **solution exists**.

Optimality Conditions

(x^*, u^*) is a **solution** for Problem (PQ) iff there exists a function λ^* such that the **adjoint equation**

$$-\dot{\lambda}^*(t) = A(t)^\mathsf{T}\lambda^*(t) + W(t)x^*(t) + S(t)u^*(t) + w(t) \quad \text{a.e. on } [t_0, t_f]\,,$$
$$\lambda^*(t_f) = Qx^*(t_f) + q\,,$$

holds, and the **minimum principle**

$$\Big[B(t)^\mathsf{T}\lambda^*(t) + S(t)^\mathsf{T}x^*(t) + r(t)\Big]^\mathsf{T}\big(u - u^*(t)\big) \geq 0 \quad \text{for all } u \in U$$

is satisfied for a.e. $t \in [t_0, t_f]$.

Bang-Bang Structure

We denote the **switching function** by

$$\sigma(t) := B(t)^\top \lambda^*(t) + S(t)^\top x^*(t) + r(t)\,.$$

Then the minimum principle implies for $i \in \{1, \ldots, m\}$

$$u_i^*(t) = \begin{cases} b_{\ell,i}\,, & \text{if } \sigma_i(t) > 0\,, \\ b_{u,i}\,, & \text{if } \sigma_i(t) < 0\,, \\ \text{undetermined}\,, & \text{if } \sigma_i(t) = 0\,. \end{cases}$$

If the switching function σ has only finitely many isolated zeros, the optimal control u^* is of **bang-bang type**.

(B1) The set Σ of zeros of the components σ_i, $i = 1\ldots,m$, of the switching function σ is finite and $t_0, t_f \notin \Sigma$, i.e., $\Sigma = \{s_1,\ldots,s_l\}$ with $t_0 < s_1 < \ldots < s_l < t_f$.

Let $\mathcal{I}(s_j) := \{1 \le i \le m \mid \sigma_i(s_j) = 0\}$ be the set of active indices for the components of the function σ. In order to obtain stability of the bang-bang structure under perturbations, we need an additional assumption:

(B2) There exist $\bar{\sigma} > 0$, $\bar{\tau} > 0$ such that

$$|\sigma_i(\tau)| \ge \bar{\sigma}|\tau - s_j|$$

for all $j \in \{1,\ldots,l\}$, $i \in \mathcal{I}(s_j)$, and all $\tau \in [s_j - \bar{\tau}, s_j + \bar{\tau}]$.

$(\mathrm{B1}) + (\mathrm{B2}) \Rightarrow$ Problem (PQ) has a **unique solution of bang-bang type**.

Part 3

Stability of Solutions

w.r.t. Perturbations and L^2-Regularization

We introduce **standard perturbations $p = (\phi, \xi, \zeta, \eta)$**, where

$$\phi \in \mathbb{R}^n\,,\ \xi \in L^\infty(t_0, t_f; \mathbb{R}^n)\,,\ \zeta \in L^\infty(t_0, t_f; \mathbb{R}^m)\,,\ \eta \in L^\infty(t_0, t_f; \mathbb{R}^n)\,,$$

and a **regularization parameter $\alpha \geq 0$** and consider the L^2-regularized parametric LQP

Problem $(\mathrm{PQ})_p^\alpha$

$$\begin{aligned}
\min \quad & f_p(x,u) + \frac{\alpha}{2}\,\|u\|_2^2 \\
\text{s.t.} \quad & \dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) + \eta(t) && \text{a.e. on } [t_0, t_f]\,, \\
& x(0) = x_0\,, \\
& u(t) \in U := \{u \in \mathbb{R}^m \mid b_\ell \leq u \leq b_u\} && \text{a.e. on } [t_0, t_f]\,.
\end{aligned}$$

Stability of Solutions

Here, $\boldsymbol{f_p}$ is a linear-quadratic cost functional defined by

$$f_p(x,u) = \frac{1}{2}x(t_f)^\top Q x(t_f) + [q + \boldsymbol{\phi}]^\top x(t_f) + \int_{t_0}^{t_f} \frac{1}{2}x(t)^\top W(t)x(t) + [w(t) + \boldsymbol{\xi(t)}]^\top x(t)\,\mathrm{d}t + \int_{t_0}^{t_f} x(t)^\top S(t)u(t) + [r(t) + \boldsymbol{\zeta(t)}]^\top u(t)\,\mathrm{d}t\,.$$

The parameters $p = 0$, $\alpha = 0$ are the **reference parameters**, and Problem (PQ) is the **reference problem**. We are interested in the behavior of solutions (x_p^α, u_p^α) of Problem $(\mathrm{PQ})_p^\alpha$ in dependence of the parameters p and α.

Optimality Conditions

(x_p^α, u_p^α) is a **solution** for Problem $(\mathrm{PQ})_h^\alpha$ iff there exists a function λ_p^α such that the **adjoint equation**

$$-\dot{\lambda}_p^\alpha(t) = A(t)^\top \lambda_p^\alpha(t) + W(t)x_p^\alpha(t) + S(t)u_p^\alpha(t) + w(t) + \boldsymbol{\xi}(\boldsymbol{t}) \quad \text{a.e.},$$
$$\lambda_p^\alpha(t_f) = Q x_p^\alpha(t_f) + q + \boldsymbol{\phi}\,,$$

holds, and the **minimum principle**

$$\left[\alpha u_p^\alpha(t) + B(t)^\top \lambda_p^\alpha(t) + S(t)^\top x_p^\alpha(t) + r(t) + \boldsymbol{\zeta}(\boldsymbol{t})\right]^\top \left(u - u_p^\alpha(t)\right) \geq 0$$

is satisfied for all $u \in U$ and a.e. $t \in [t_0, t_f]$.

Theorem (Calmness of Solutions)
Let Assumptions (AC), (B1) and (B2) be satisfied. Then there exist constants c_u, c_x and c_λ independent of p and α such that for the optimal solutions of Problems $(\mathrm{PQ})_p^\alpha$ the estimates

$$\left\| u_p^\alpha - u^* \right\|_1 \leq c_u \left(\|p\| + \alpha \right), \quad \left\| x_p^\alpha - x^* \right\|_{1,1} \leq c_x \left(\|p\| + \alpha \right)$$

and

$$\left\| \lambda_p^\alpha - \lambda^* \right\|_{1,1} \leq c_\lambda \left(\|p\| + \alpha \right)$$

hold for all p and α.

L^2-Regularization

We consider Problem $(\mathrm{PQ})_0^\alpha$ with the cost functional

$$f_0(x,u) + \frac{\alpha}{2}\,\|u\|_2^2\,.$$

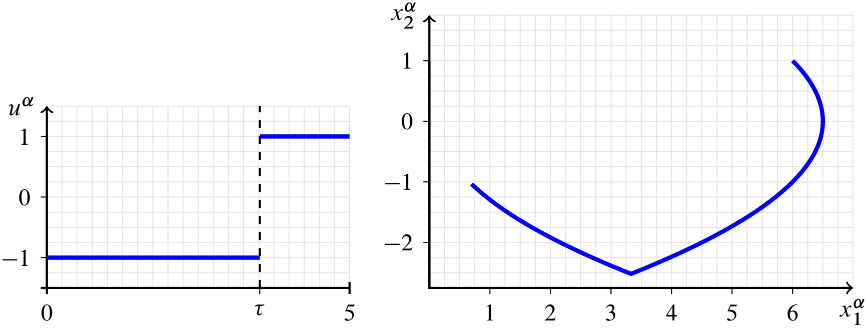
This problem has a **unique Lipschitz continuous** optimal control:

$$u^\alpha(t) = \mathrm{Pr}_{[b_\ell,b_u]}\left(-\frac{1}{\alpha}\left[B(t)^\mathsf{T}\lambda_0^\alpha(t) + S(t)^\mathsf{T}x_0^\alpha(t) + r(t)\right]\right).$$

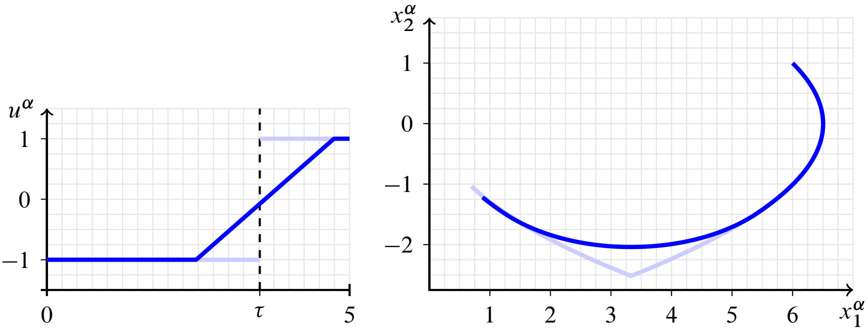
Theorem

For $\alpha \to 0$ we obtain $u_0^\alpha \to u^*$ and $x_0^\alpha \to x^*$.

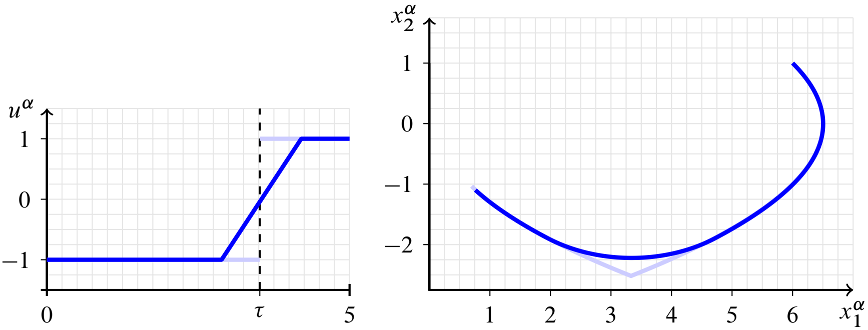
Optimal solution for $\alpha = 0$:



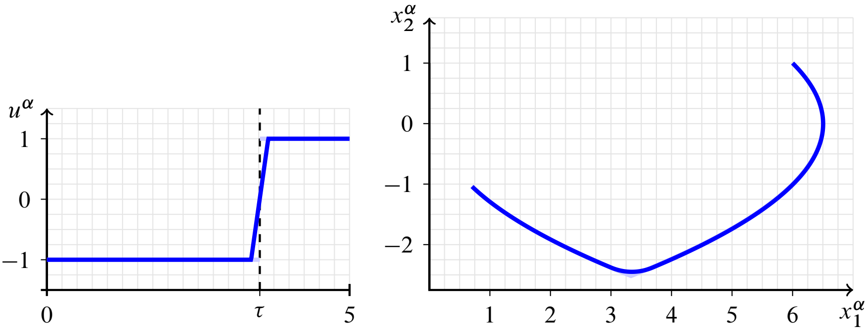
Optimal solution for $\alpha = 1$:



Optimal solution for $\alpha = 0.5$:



Optimal solution for $\alpha = 0.1$:



For two functions $u_1, u_2 \in L^\infty(t_0, t_f; \mathbb{R}^m)$ we define

$$d^\#(u_1, u_2) := \operatorname{meas}\left\{t \in [t_0, t_f] \mid u_1(t) \neq u_2(t)\right\}.$$

Theorem

Let Assumptions (AC), (B1) and (B2) be satisfied. Then there exists a constant κ independent of (p, α) such that for any solution (x_p^α, u_p^α) of Problems $(\mathrm{PQ})_p^\alpha$ the estimate

$$d^\#(u_p^\alpha, u^*) \leq \kappa\,(\|p\| + \alpha)$$

holds, if $\|p\| + \alpha$ is sufficiently small.

Part 4

Discretization

Explicit and Implicit Euler Method

The discretization of an optimal control problem depends on the choice of the discretization scheme for the system equation. This section is devoted to the **Euler discretization** of Problem $(\mathrm{PQ})_0^\alpha$.

This results in the following finite dimensional optimization problem:

Problem $(\mathrm{PQ})_h^\alpha$

$$\min \quad f_h^\alpha(x_h, u_h)$$

$$\begin{aligned} \text{s.t.} \quad x_{h,j+1} &= x_{h,j} + h\left[A(t_j)x_{h,j} + B(t_j)u_{h,j} + b(t_j)\right], && j \in \mathcal{J}_0^{N-1} \\ x_{h,0} &= x_0\,, \\ u_{h,j} &\in U\,, && j \in \mathcal{J}_0^{N-1} \end{aligned}$$

$$
\begin{aligned}
f_h^\alpha(x_h, u_h) = {} & \frac{1}{2} x_{h,N}^\top Q x_{h,N} + q^\top x_{h,N} \\
& + h \sum_{j=0}^{N-1} \left[\frac{1}{2} x_{h,j}^\top W(t_j) x_{h,j} + w(t_j)^\top x_{h,j} + x_{h,j}^\top S(t_j) u_{h,j} \right] \\
& + h \sum_{j=0}^{N-1} \left[r(t_j)^\top u_{h,j} + \frac{\alpha}{2} u_{h,j}^\top u_{h,j} \right]
\end{aligned}
$$

A solution (x_h^α, u_h^α) of Problem $(\mathrm{PQ})_h^\alpha$ **exists**. Since it may happen that one of the zeros of the discrete switching function is a discretization point, **the optimal control has not to be unique in the case $\alpha = 0$**. For $\alpha > 0$, the optimal control is uniquely determined.

We are able to show that (x_h^α, u_h^α) **also solves** some Problem $(\mathrm{PQ})^\alpha_{p_h^\alpha}$ for a parameter $p_h^\alpha = (\phi_h^\alpha, \xi_h^\alpha, \zeta_h^\alpha, \eta_h^\alpha)$ with

$$\|p_h^\alpha\| \leq ch\,,$$

where the constant c is independent of $N \in \mathbb{N}$ and $\alpha \geq 0$.

Therefore, we apply the **calmness result** to prove convergence of the discretization.

In the same way, convergence of the **implicit Euler discretization** can be proved.

Theorem (Convergence)

Let Assumptions (AC), (B1) and (B2) be satisfied. For any $N \in \mathbb{N}$, the corresponding mesh size $h = (t_f - t_0)/N$ and any $\alpha \geq 0$, and for any solution (x_h^α, u_h^α) with associated multiplier λ_h^α the error estimate

$$\left\| u_h^\alpha - u^* \right\|_1 + \left\| x_h^\alpha - x^* \right\|_{1,1} + \left\| \lambda_h^\alpha - \lambda^* \right\|_{1,1} \leq c\,(h + \alpha)$$

holds with some constant c independent of N.

With some constant γ we **choose $\boldsymbol{\alpha = \gamma h}$** and obtain

$$\left\| u_h^\alpha - u^* \right\|_1 + \left\| x_h^\alpha - x^* \right\|_{1,1} + \left\| \lambda_h^\alpha - \lambda^* \right\|_{1,1} \leq \bar{c} h\,.$$

Theorem

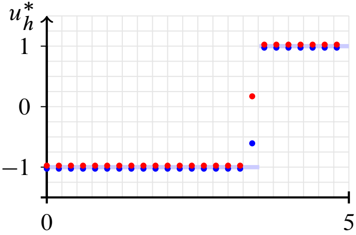
Let Assumptions (AC), (B1) and (B2) be satisfied. There exists a constant κ independent of $N \in \mathbb{N}$, the corresponding mesh size h and $\alpha \geq 0$ such that

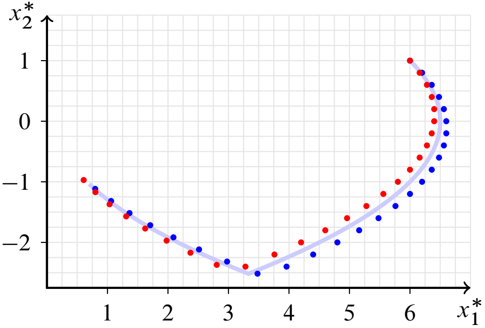
$$d^{\#}(u_h^{\alpha}, u^*) \leq \kappa\,(h + \alpha)$$

holds for all $N \in \mathbb{N}$ and $\alpha \geq 0$.

Numerical Results for the Rocket Car Example

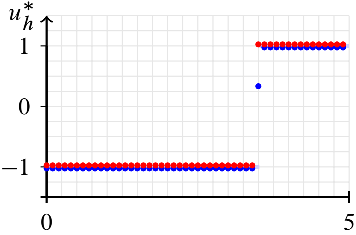
Explicit Euler vs. **Implicit Euler**, $N = 25$:

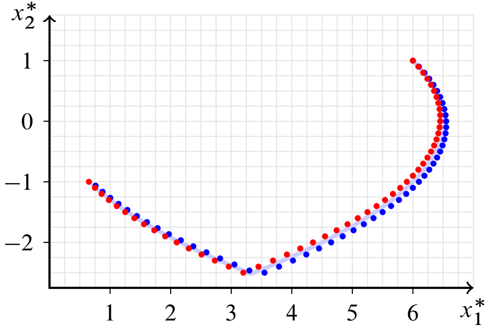




Numerical Results for the Rocket Car Example

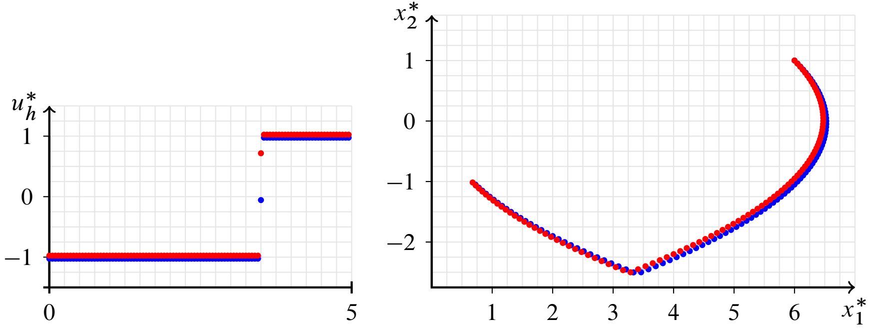
Explicit Euler vs. **Implicit Euler**, $N = 50$:





Numerical Results for the Rocket Car Example

Explicit Euler vs. **Implicit Euler**, $N = 100$:



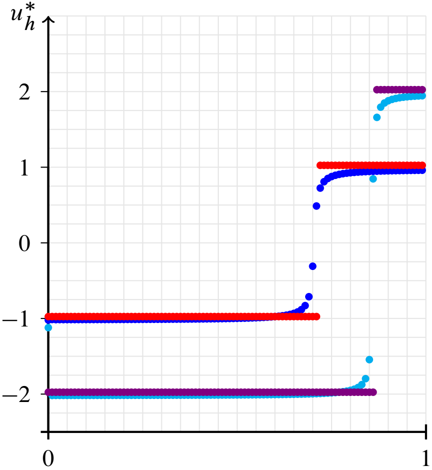
Numerical experiments **confirm the theoretical findings**. Here, u_{Exp} is the solution of the (explicit) Euler discretized problem and u_{Imp} is the solution of the implicit Euler discretized problem.

N	100	200	400	800	1600	3200
$\|u_{\text{Exp}} - u^*\|_1$	0.1167	0.0602	0.0293	0.0151	0.0074	0.0036
$\frac{\|u_{\text{Exp}} - u^*\|_1}{h}$	2.3343	2.4085	2.3408	2.4227	2.3798	2.2873
$\|u_{\text{Imp}} - u^*\|_1$	0.0805	0.0421	0.0202	0.0106	0.0052	0.0026
$\frac{\|u_{\text{Imp}} - u^*\|_1}{h}$	1.6105	1.6826	1.6160	1.7007	1.6547	1.6906

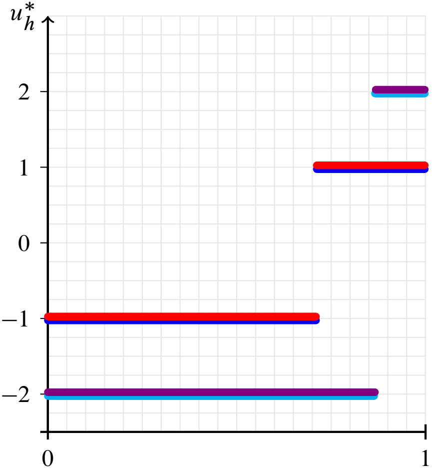
A Stiff Optimal Control Problem

$$
\begin{aligned}
\min \quad & \int_0^1 2x_1(t) + 6x_2(t) - u_1(t) - 0.5u_2(t)\,\mathrm{d}t \\
\text{s.t.} \quad & \dot{x}_1(t) = 0.5\,(c_1 + c_2)\,x_1(t) + 0.5\,(c_1 - c_2)\,x_2(t) + u_1(t) && \text{a.e.,} \\
& \dot{x}_2(t) = 0.5\,(c_1 - c_2)\,x_1(t) + 0.5\,(c_1 + c_2)\,x_2(t) + u_2(t) && \text{a.e.,} \\
& x_1(0) = 0\,, \quad x_2(0) = 0\,, \\
& u_1(t) \in [-1, 1]\,, \quad u_2(t) \in [-2, 2] && \text{a.e.}
\end{aligned}
$$

We choose $c_1 = -1$ and $c_2 = -1000$, so the problem becomes **stiff**.



$N = 100$



$N = 500$

Part 5

The Dual Problem

Continuous-Time Problem and Discretization

Burachik/Kaya/Majeed (SICON, 2014)

Computation of the dual problem for linear-quadratic control problems with continuous solutions. Strong duality holds.

Numerical experiments illustrate that by solving the dual problem computational savings can be achieved.

The Dual Problem

$$\begin{aligned}\min \quad & \frac{1}{2}\left(p(t_f)+q\right)^\top Q^{-1}\left(p(t_f)+q\right)+p(0)^\top a \\ & +\int_{t_0}^{t_f}\frac{1}{2}\left(x^*(t)-w(t)\right)^\top W(t)^{-1}\left(x^*(t)-w(t)\right)+\psi^\alpha(p(t),t)\,\mathrm{d}t \\ \text{s. t.} \quad & \dot{p}(t)=-A(t)^\top p(t)+x^*(t) \quad \text{a.e. on } [t_0,t_f]\,,\end{aligned}$$

where for the **$\alpha = 0$ (bang-bang case)** $\psi^0(p,t)$ is defined by

$$\psi_i^0(p,t)=\begin{cases} b_{\ell,i}\left(B(t)^\top p\right)_i - b_{\ell,i} r_i(t) & \text{if } r_i(t)-\left(B(t)^\top p\right)_i \geq 0\,, \\ b_{u,i}\left(B(t)^\top p\right)_i - b_{u,i} r_i(t)\,, & \text{if } r_i(t)-\left(B(t)^\top p\right)_i < 0\,. \end{cases}$$

The Regularized Case

Remember (**$\alpha = 0$**)

$$\psi_i^0(p,t) = \begin{cases} b_{\ell,i}\left(B(t)^\top p\right)_i - b_{\ell,i} r_i(t)\,, & \text{if } r_i(t) - \left(B(t)^\top p\right)_i \geq 0\,, \\ b_{u,i}\left(B(t)^\top p\right)_i - b_{u,i} r_i(t)\,, & \text{if } r_i(t) - \left(B(t)^\top p\right)_i < 0\,. \end{cases}$$

For the **$\alpha > 0$ (regularized case)** $\psi^\alpha(p,t)$ is defined by

$$\psi_i^\alpha(p,t) = \begin{cases} \frac{1}{2\alpha}\left(B(t)^\top p\right)_i^2 - \frac{1}{\alpha} r_i(t)\left(B(t)^\top p\right)_i + \frac{1}{2\alpha} r_i(t)^2\,, & \dots \\ b_{\ell,i}\left(B(t)^\top p\right)_i - b_{\ell,i} r_i(t) - \frac{\alpha}{2} b_{\ell,i}^2\,, & \dots \\ b_{u,i}\left(B(t)^\top p\right)_i - b_{u,i} r_i(t) - \frac{\alpha}{2} b_{u,i}^2\,, & \dots \end{cases}$$

Theorem (Strong Duality)

Let (x, u) be an optimal solution of the primal problem with adjoint variable λ. We set

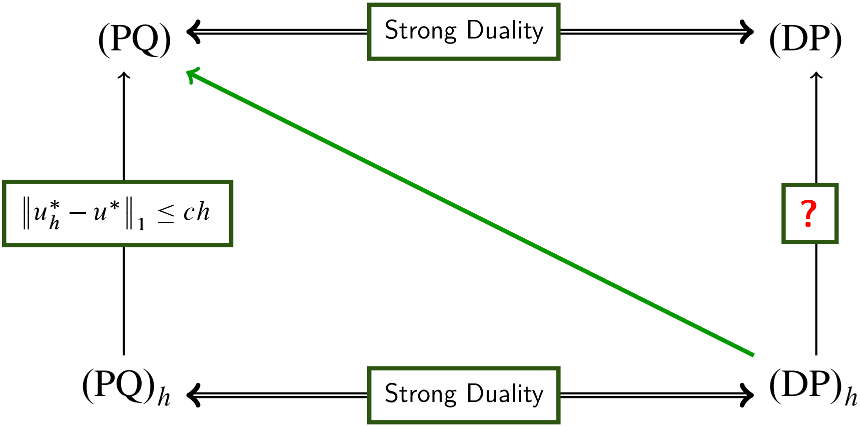
$$p(t) = -\lambda(t) \quad \text{a.e. on } [t_0, t_f], \quad p(t_f) = -Qx(t_f) - q,$$

and

$$x^*(t) = W(t)x(t) + w(t) \quad \text{a.e. on } [t_0, t_f].$$

Then the optimal values of the primal and dual problem are equal and (p, x^*) is a solution of the dual problem.

Discretization of the Dual Problem



Revisiting the Rocket Car

$$\begin{aligned} \min \quad & \frac{1}{2}\left(x_1(t_f)^2 + x_2(t_f)^2\right) + \frac{\alpha}{2}\|u\|_2^2 \\ \text{s.t.} \quad & \dot{x}_1(t) = x_2(t)\,, \ \dot{x}_2(t) = u(t) \quad \text{a.e. on } [0, t_f]\,, \\ & x_1(0) = 6\,, \quad x_2(0) = 1\,, \\ & u(t) \in [-1, 1] \quad \text{a.e. on } [0, t_f]\,. \end{aligned}$$

The Dual Problem

Problem	Regularization	CPU time [s]	Ratio
Primal	$\alpha = 0$	0.264	100%
Primal	$\alpha = h$	0.283	107%
Dual	$\alpha = 0$	0.192	73%
Dual	$\alpha = h$	0.150	57%

$N = 5000$, average over 1000 runs. Solver: IPOPT.

Diabetes Mellitus

$$\begin{aligned}
\min \quad & \int_0^1 \frac{1}{2} x_1(t)^2 \, \mathrm{d}t + \frac{\alpha}{2} \|u\|_2^2 \\
\text{s. t.} \quad & \dot{x}_1(t) = -0.1 x_1(t) - x_2(t) && \text{a.e. on } [0,1]\,, \\
& \dot{x}_2(t) = 0.2 x_1(t) + 0.1 x_2(t) + u(t) && \text{a.e. on } [0,1]\,, \\
& x_1(0) = 1\,, \quad x_2(0) = 0\,, \\
& u(t) \in [0,4] && \text{a.e. on } [0,1]
\end{aligned}$$

Problem	Regularization	CPU time [s]	Ratio
Primal	$\alpha = 0$	0.576	100%
Primal	$\alpha = h$	0.359	61%
Dual	$\alpha = 0$	0.211	37%
Dual	$\alpha = h$	0.182	32%

$N = 5000$, average over 1000 runs. Solver: IPOPT.

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Questions?

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