Stochastic Optimization with Ambiguity in Distribution

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1 How to Deal with a Single Stochastic Constraint

Consider a single constraint under uncertainty

$$v(x, \tilde{z}) \leq w, \quad (x, w) \in \mathbb{R}^n \times \mathbb{R}$$

Robust Optimization

$$\sup_{z \in C} v(x, z) \leq w \tag{1}$$

Stochastic Optimization

$$\mathbb{E}_{\mathbb{P}_0} v(x, \tilde{z}) \leq w, \ \mathbb{P}_0 \text{ is the joint distribution of } \tilde{z} \tag{2}$$

Special case of (2) - Chance constraint

$$\mathbb{P}(a(\tilde{z})^T x \geq b(\tilde{z})) \geq 1 - \epsilon \iff \mathbb{E}\left(\mathbf{1}_{a(\tilde{z})^T x \leq b(\tilde{z})}\right) \leq \epsilon$$

Sto. Opt . with Ambiguity in Distribution

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} v(x,\tilde{z}) \le w, \ \mathcal{P} \text{ is the ambiguity set of distributions.} \tag{3}$$

(1) and (2) $\subset$ (3).

Notations.

$\mathcal{P}_0(\mathbb{R}^m)$ - The set of distributions on $\mathbb{R}^m$

$|\tilde{z}|, |\tilde{u}|$ - cardinality of $\tilde{z}$ and $\tilde{u}$, resp.

$\mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{|\tilde{z}|} \times \mathbb{R}^{|\tilde{u}|})$ is a joint probability distribution of $(\tilde{z}, \tilde{u})$.

$\prod_{\tilde{z}} \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{|\tilde{z}|})$ - the marginal distribution of $\tilde{z}$ under $\mathbb{P}$

Assumption. $v(x,\tilde{z})$ is convex in x and measurable in $\tilde{z}$ $\forall x$

Let $\mathcal{K}$ be a proper cone, we use $x \succeq_{\mathcal{K}} 0$ to represent $x \in \mathcal{K}$.

1. $\mathcal{K} = \mathbb{R}^n_+ \iff x_1, ..., x_n \geq 0$.
2. $\mathcal{K} = \mathbb{L}^{n+1} \iff x_0 \geq (x_1^2 + \cdots + x_n^2)^{1/2}$.
3. $\mathcal{K} = \mathbb{S}^n_+ \iff X$ is a positive semidefinite matrix.

More Notations.

$\mathcal{K}^* := \{y : \langle y, x \rangle \geq 0 \ \forall x \in \mathcal{K}\}$.

$B \preceq_{\mathcal{K}} A$ or $A \succeq_{\mathcal{K}} B \iff A - B \in \mathcal{K}$.

We assume that the *ambiguity set* $\mathcal{P}$ in (3) is representable in the "standard form"

$$\mathcal{P} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^P \times \mathbb{R}^Q) \ : \ \begin{array}{ll} \mathbb{E}_{\mathbb{P}}\left[A\tilde{z} + B\tilde{u}\right] = b, & \\ \mathbb{P}\left[(\tilde{z}, \tilde{u}) \in \mathcal{C}_i\right] \in \left[\underline{p}_i, \overline{p}_i\right] & \forall i \in \mathcal{I} \end{array} \right\}, \tag{4}$$

where $\mathbb{P}$ represents a joint probability distribution of the random vector $\tilde{z} \in \mathbb{R}^P$ appearing in the constraint function v in (3) and some auxiliary random vector $\tilde{u} \in \mathbb{R}^Q$. We assume that $A \in \mathbb{R}^{K \times P}$, $B \in \mathbb{R}^{K \times Q}$, $b \in \mathbb{R}^K$ and $\mathcal{I} = \{1, \ldots, I\}$, while the confidence sets $\mathcal{C}_i$ are defined as

$$\mathcal{C}_i = \left\{(z, u) \in \mathbb{R}^P \times \mathbb{R}^Q \ : \ C_i z + D_i u \preccurlyeq_{\mathcal{K}_i} c_i\right\} \tag{5}$$

with $C_i \in \mathbb{R}^{L_i \times P}$, $D_i \in \mathbb{R}^{L_i \times Q}$, $c_i \in \mathbb{R}^{L_i}$ and $\mathcal{K}_i$ being proper cones.

More requirements on $\mathcal{P}$

We require that the ambiguity set $\mathcal{P}$ satisfies the following two regularity conditions (**support-interior assumption**).

(C1) The confidence set $\mathcal{C}_I$ is bounded and has probability one, that is, $\underline{p}_I = \overline{p}_I = 1$.

(C2) There is a distribution $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}\left[(\tilde{z}, \tilde{u}) \in \mathcal{C}_i\right] \in \left(\underline{p}_i, \overline{p}_i\right)$ whenever $\underline{p}_i < \overline{p}_i$, $i \in \mathcal{I}$.

We require that the constraint function v in (3) satisfies the following condition (**piecewise bi-linear assumption**).

(C3) The constraint function $v(x, z)$ can be written as

$$\boldsymbol{v(x, z) = \max_{l \in \mathcal{L}} v_l(x, z)} \tag{6}$$

where $\mathcal{L} = \{1, \ldots, L\}$ and the auxiliary functions $v_l : \mathbb{R}^N \times \mathbb{R}^P \to \mathbb{R}$ are of the form

$$\boldsymbol{v_l(x, z) = s_l(z)^\top x + t_l(z)} \tag{7}$$

with $s_l(z) = S_l z + s_l$, $S_l \in \mathbb{R}^{N \times P}$ and $s_l \in \mathbb{R}^N$, $t_l(z) = t_l^\top z + t_l$, $t_l \in \mathbb{R}^P$ and $t_l \in \mathbb{R}$.

The tractability of optimization problems with constraints of the type (3) critically depends on the following **nesting condition** for the confidence sets in the definition of $\mathcal{P}$:

(N) For all $i, i' \in \mathcal{I}$, $i \neq i'$, we have either $\mathcal{C}_i \Subset \mathcal{C}_{i'}$, $\mathcal{C}_{i'} \Subset \mathcal{C}_i$ or $\mathcal{C}_i \cap \mathcal{C}_{i'} = \emptyset$.

we denote by $\mathcal{A}(i) = \{i\} \cup \{i' \in \mathcal{I} \, : \, \mathcal{C}_i \Subset \mathcal{C}_{i'}\}$ and $\mathcal{D}(i) = \{i' \in \mathcal{I} \, : \, \mathcal{C}_{i'} \Subset \mathcal{C}_i\}$ the index sets of all supersets (antecedents) and all strict subsets (descendants) of $\mathcal{C}_i$, respectively.

Theorem 1.1 (Wiesemann, Kuhn, Sim, 2014) *Assume that the conditions (**C1**)-(**C3**) and (**N**) hold. Then, the constraint (3) is satisfied for the ambiguity set (4) if and only if there is $\beta \in \mathbb{R}^K$, κ, $\lambda \in \mathbb{R}^I_+$ and $\phi_{il} \in \mathcal{K}_i^\star$, $i \in \mathcal{I}$ and $l \in \mathcal{L}$, that satisfy the constraint system*

$$\boldsymbol{b^\top \beta + \sum_{i \in \mathcal{I}} \left[\overline{p}_i \kappa_i - \underline{p}_i \lambda_i \right] \leq w,}$$

$$\left.\begin{array}{l} \boldsymbol{c_i^\top \phi_{il} + s_l^\top x + t_l \leq \sum_{i' \in \mathcal{A}(i)} [\kappa_{i'} - \lambda_{i'}]} \\ C_i^\top \phi_{il} + A^\top \beta = S_l^\top x + t_l \\ D_i^\top \phi_{il} + B^\top \beta = 0 \end{array}\right\} \quad \forall i \in \mathcal{I},\ \forall l \in \mathcal{L}. \tag{8}$$

Theorem 1.2 (Lifting Theorem of WKS) *Let $f \in \mathbb{R}^M$ and $g : \mathbb{R}^P \to \mathbb{R}^M$ be a function with a conic representable $\mathcal{K}$-epigraph, and consider the ambiguity set*

$$\mathcal{P}' = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^P) \ : \ \begin{array}{l} \mathbb{E}_{\mathbb{P}}\left[g(\tilde{z})\right] \preccurlyeq_{\mathcal{K}} f, \\ \mathbb{P}\left[\tilde{z} \in \mathcal{C}_i\right] \in \left[\underline{p}_i, \overline{p}_i\right] \qquad \forall i \in \mathcal{I} \end{array} \right\}, \tag{9}$$

as well as the lifted ambiguity set

$$\mathcal{P} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^P \times \mathbb{R}^M) \ : \ \begin{array}{l} \mathbb{E}_{\mathbb{P}}\left[\tilde{u}\right] = f, \\ \mathbb{P}\left[g(\tilde{z}) \preccurlyeq_{\mathcal{K}} \tilde{u}\right] = 1, \\ \mathbb{P}\left[\tilde{z} \in \mathcal{C}_i\right] \in \left[\underline{p}_i, \overline{p}_i\right] \qquad \forall i \in \mathcal{I} \end{array} \right\}, \tag{10}$$

which involves the auxiliary random vector $\tilde{u} \in \mathbb{R}^M$. We then have that (i) *$\mathcal{P}' = \Pi_{\tilde{z}}\mathcal{P}$ and* (ii) *$\mathcal{P}$ can be reformulated as an instance of the standardized ambiguity set (4).*

Example 1.1 (Mean) *Assume that* $\boldsymbol{G}\,\mathbb{E}_{\mathbb{Q}^0}[\tilde{\boldsymbol{z}}] \preccurlyeq_{\boldsymbol{\mathcal{K}}} \boldsymbol{f}$ *for a proper cone* $\mathcal{K}$ *and* $G \in \mathbb{R}^{M \times P}$, $f \in \mathbb{R}^M$, *and consider the following instance of the ambiguity set (4), which involves the auxiliary random vector* $\tilde{u} \in \mathbb{R}^M$.

$$\mathcal{P} = \left\{ \mathbb{P} \in \mathcal{P}_0\left(\mathbb{R}^P \times \mathbb{R}^M\right) \,:\, \mathbb{E}_{\mathbb{P}}[\tilde{u}] = f, \;\; \mathbb{P}\left[G\tilde{z} \preccurlyeq_{\mathcal{K}} \tilde{u}\right] = 1 \right\} \qquad (11)$$

We then have $\mathbb{Q}^0 \in \Pi_{\tilde{z}}\mathcal{P} = \left\{ \mathbb{Q} \in \mathcal{P}_0\left(\mathbb{R}^P\right) \,:\, G\,\mathbb{E}_{\mathbb{Q}}[\tilde{z}] \preccurlyeq_{\mathcal{K}} f \right\}$.

Example 1.2 (Variance) *Assume that* $\boldsymbol{\mu} = \mathbb{E}_{\mathbb{Q}^0}\left[\tilde{\boldsymbol{z}}\right]$ *and* $\mathbb{E}_{\mathbb{Q}^0}\left[(\tilde{z}-\mu)(\tilde{z}-\mu)^\top\right] \preccurlyeq \Sigma$ *for a given* $\Sigma \in \mathbb{S}^P_+$. *Consider the following instance of (4), which involves the auxiliary random matrix* $\tilde{U} \in \mathbb{R}^{P\times P}$.

$$\boldsymbol{\mathcal{P}} = \{\mathbb{P} \in (\mathbb{R}^{\boldsymbol{P}} \times \mathbb{R}^{\boldsymbol{P}\times\boldsymbol{P}}) \ : \ \mathbb{E}_{\mathbb{P}}\left[\tilde{\boldsymbol{z}}\right] = \boldsymbol{\mu}, \ \ \mathbb{E}_{\mathbb{P}}\left[\tilde{\boldsymbol{U}}\right] = \boldsymbol{\Sigma},$$

$$\mathbb{P}\left(\begin{bmatrix} 1 & (\tilde{z}-\mu)^\top \\ (\tilde{z}-\mu) & \tilde{U} \end{bmatrix} \succcurlyeq 0\right) = 1\} \tag{12}$$

We then have $\mathbb{Q}^0 \in \Pi_{\tilde{z}}\mathcal{P} = \left\{\mathbb{Q} \in \mathcal{P}_0(\mathbb{R}^P) \ : \ \mathbb{E}_{\mathbb{Q}}\left[\tilde{z}\right] = \mu, \ \ \mathbb{E}_{\mathbb{Q}}\left[(\tilde{z}-\mu)(\tilde{z}-\mu)^\top\right] \preccurlyeq \Sigma\right\}$.

Example 1.3 (The plus-function, high-order moment, ...)

$\mathbb{E}_{\mathbb{P}}\left[|a_1^T z|\right] \le \mu, \mathbb{E}_{\mathbb{P}}\left(a_2^T z\right)^2 \le \delta, \mathbb{E}_{\mathbb{P}}\left[((a_3^T z)_+\right]^3 \le \sigma, \ ...$

$$
\begin{array}{ll}
(1) & a_1^T z \le \nu_1, -a_1^T z \le \nu_1 \\
(2) & \sqrt{(a_2^T z)^2 + (\frac{\nu_2 - 1}{2})^2} \le \frac{\nu_2 + 1}{2} \\
(3) & \mu_1 \ge 0, \mu_1 \ge a_3^T z, \\
 & [\mu_1^2 + (\frac{\mu_1 - 1}{2})^2]^{1/2} \le \frac{\mu_2 - 1}{2}, \\
 & [\mu_2^2 + (\frac{\mu_1 - \nu_3}{2})^2]^{1/2} \le \frac{\nu_3 + \mu_1}{2}
\end{array}
$$

Application in two-stage stochastic linear programming

$$\min_{x \in X} \left\{ c'x + \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[Q(x, \tilde{z})] \right\}. \tag{13}$$

$$\begin{aligned} Q(x, \tilde{z}) = \min_{y(\tilde{z})} \quad & d'y(\tilde{z}) \\ \text{s. t.} \quad & A(\tilde{z})x + Dy(\tilde{z}) = b(\tilde{z}), \\ & y(\tilde{z}) \geq 0, \end{aligned}$$

$$\Rightarrow \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[Q(x, \tilde{z})] \leq t - c'x = w$$

A production planning problem. A company manager is considering the amount of steel to purchase (at \$58/lb) for producing wrenches and pliers in next month. The manufacturing process involves molding the tools on a molding machine and then assembling the tools on an assembly machine. Here is the technical data.

	Wrench	Plier	Total
Steel (lbs.)	1.5	1	x
Molding Machine (hours)	1	1	z_1
Assembly Machine (hours)	.3	.5	z_2
Contribution to Earnings (\$/1000 units)	130	100	

If z_1 and z_2 are fixed, then the problem is

$$\begin{array}{ll} \min & 57x - 130w - 100p \\ \text{s.t.} & w + p \leq z_1 \text{(Mold constraint)} \\ & .3w + .5p \leq z_2 \text{(Assembly constraint)} \\ & 1.5w + p \leq x \text{(Steel constraint)} \\ & x, w, p \geq 0 \end{array}$$

The Two-stage Stochastic Programming Formulation

Decision Variables: x: the steel to purchase now; w_i, p_i: production plan under scenario $i = 1, 2, 3, 4$.

Scenario	Assembly Hours	Molding Hours	Probability
1	8000	25000	.25
2	8000	21000	.25
3	10000	25000	.25
4	10000	21000	.25

We minimize the expected cost subject to "scenario constraints".

$$\begin{array}{lll} \min & 58x - \sum_{i=1}^{4} .25(130w_i + 100p_i) & \\ \text{s.t.} & w_1 + p_1 \le 25000 & \text{(Mold constraint for scenario 1)} \\ & .3w_1 + .5p_1 \le 8000 & \text{(Assembly constraint for scenario 1)} \\ & -x + 1.5w_1 + p_1 \le 0 & \text{(Steel constraint for scenario 1)} \\ & w_2 + p_2 \le 21000 & \text{(Mold constraint for scenario 2)} \\ & .3w_2 + .5p_2 \le 8000 & \text{(Assembly constraint for scenario 2)} \end{array}$$

$-x + 1.5w_2 + p_2 \leq 0$	(Steel constraint for scenario 2)
$w_3 + p_3 \leq 25000$	(Mold constraint for scenario 3)
$.3w_3 + .5p_3 \leq 10000$	(Assembly constraint for scenario 3)
$-x + 1.5w_3 + p_3 \leq 0$	(Steel constraint for scenario 3)
$w_1 + p_1 \leq 21000$	(Mold constraint for scenario 4)
$.3w_4 + .5p_4 \leq 10000$	(Assembly constraint for scenario 4)
$-x + 1.5w_4 + p_4 \leq 0$	(Steel constraint for scenario 4)
$x, w_i, p_i \geq 0,\ i = 1, ..., 4.$	

The solutions are as follows. $x = 27,250$,
minimal expected cost = -802,833, and the production plans under various scenarios are as follows.

Scenario	w_i	p_i
1	12,500	8,500
2	12,500	8,500
3	8,056	15,167
4	12,500	8,500

The SOAD formulation. We have $X = \{x : x \geq 0\}$ and $\mathbb{P} \in \mathcal{P}$

$$\mathcal{P} := \left\{ \mathbb{P} : \begin{array}{c} \mathbb{E}_{\mathbb{P}}(\tilde{z}_j) = \mu_j, \;\; j = 1, \ldots, m, \\ \mathbb{E}_{\mathbb{P}}(\tilde{z}_j^2) \leq \eta_j, \;\; j = 1, \ldots, m, \\ \mathbb{P}\{\tilde{z} \in \Omega\} = 1. \end{array} \right\}$$

By Theorem 1.1 we solve the equivalent conic optimization problem:

$$\begin{array}{ll}
\min\limits_{x, v_0, v, V, u, Y, s, t, \lambda, \nu} & 58x + v_0 + 23000v_1 + 9000v_2 + 533 \times 10^9 V_1 + 82 \times 10^9 V_2 \\
\text{s. t.} & \sum_{i=1}^{3} u_i - 130y_1^0 - 100y_2^0 + 25000\lambda_1 + 10000\lambda_2 + \lambda_3 - 21000\nu_1 \\
& -8000\nu_2 + \nu_3 - v_0 \leq 0, \\
& \left\| \begin{pmatrix} v_1 + 130y_1^1 + 100y_2^1 + \lambda_1 - \nu_1 \\ V_1 - u_1 \end{pmatrix} \right\| \leq V_1 + u_1, \\
& \left\| \begin{pmatrix} v_2 + 130y_1^2 + 100y_2^2 + \lambda_2 - \nu_2 \\ V_2 - u_2 \end{pmatrix} \right\| \leq V_2 + u_2, \\
& \left\| \begin{pmatrix} v_3 + 130y_1^3 + 100y_2^3 + \lambda_3 - \nu_3 \\ V_3 - u_3 \end{pmatrix} \right\| \leq V_3 + u_3, \\
& y_1^0 + y_2^0 + y_3^0 = 0, \; .3y_1^0 + .5y_2^0 + y_4^0 = 0, \; -x + 1.5y_1^0 + y_2^0 = 0, \\
& y_1^1 + y_2^1 + y_3^1 = 1, \; .3y_1^1 + .5y_2^1 + y_4^1 = 0, \; 1.5y_1^1 + y_2^1 = 0, \\
& y_1^2 + y_2^2 + y_3^2 = 0, \; .3y_1^2 + .5y_2^2 + y_4^2 = 1, \; 1.5y_1^2 + y_2^2 = 0, \\
& y_1^3 + y_2^3 + y_3^3 = 0, \; .3y_1^3 + .5y_2^3 + y_4^3 = 0, \; 1.5y_1^3 + y_2^3 1 = 1
\end{array}$$

$$y_1^1 + y_2^1 + y_3^1 = 1, \ 0.3y_1^1 + 0.5y_2^1 + y_4^1 = 0, \ 1.5y_1^1 + y_2^1 = 0,$$
$$y_1^2 + y_2^2 + y_3^2 = 0, \ 0.3y_1^2 + 0.5y_2^2 + y_4^2 = 1, \ 1.5y_1^2 + y_2^2 = 0,$$
$$y_1^3 + y_2^3 + y_3^3 = 0, \ 0.3y_1^3 + 0.5y_2^3 + y_4^3 = 0, \ 1.5y_1^3 + y_2^3 = 1,$$
$$-21000s_1^1 - 8000s_2^1 + s_3^1 + 25000t_1^1 + 10000t_2^1 + t_3^1 - y_1^0 \le 0,$$
$$-21000s_1^2 - 8000s_2^2 + s_3^2 + 25000t_1^2 + 10000t_2^2 + t_3^2 - y_2^0 \le 0,$$
$$-21000s_1^3 - 8000s_2^3 + s_3^3 + 25000t_1^3 + 10000t_2^3 + t_3^3 - y_3^0 \le 0,$$
$$-21000s_1^4 - 8000s_2^4 + s_3^4 + 25000t_1^4 + 10000t_2^4 + t_3^4 - y_4^0 \le 0,$$

$$s_1^1 - t_1^1 - y_1^1 \le 0, \ s_2^1 - t_2^1 - y_1^2 \le 0, \ s_3^1 - t_3^1 - y_1^3 \le 0,$$
$$s_1^2 - t_1^2 - y_2^1 \le 0, \ s_2^2 - t_2^2 - y_2^2 \le 0, \ s_3^2 - t_3^2 - y_2^3 \le 0,$$
$$s_1^3 - t_1^3 - y_3^1 \le 0, \ s_2^3 - t_2^3 - y_3^2 \le 0, \ s_3^3 - t_3^3 - y_3^3 \le 0,$$
$$s_1^4 - t_1^4 - y_4^1 \le 0, \ s_2^4 - t_2^4 - y_4^2 \le 0, \ s_3^4 - t_3^4 - y_4^3 \le 0,$$
$$x \ge 0, \ V, u, \lambda, \nu \ge 0, \ s^k, t^k \ge 0, \ k = 1, 2, 3, 4.$$

The numerical results show that $x = 31,500$, max-worst cost = -940,770.

(Gao, S, Wu) Based various levels of knowledge about moments, different steel purchasing levels are calculated respectively and the comparison is shown in the table below.

Number of moments known	Steel purchased
1	30500
2	30500
3	30500
4	30500
5	27861
6	17876
7	17876

It is interestingly noted adding one extra moment information may have no value in the sense of making SOAD decision . When cost of evaluating uncertainty moments is high, dealing with low level of knowledge can be sufficient to worst-case decision making.

2 The chance constraint: A non-convex case

$$\mathbb{P}(a(\tilde{z})^T x \geq b(\tilde{z})) \geq 1 - \epsilon \tag{14}$$

For mathematicians,

$$\mathbb{E}\left(\mathbf{1}_{a(\tilde{z})^T x \geq b(\tilde{z})}\right) \geq 1 - \epsilon \tag{15}$$

It is easy to see that $\mathbf{1}_{a(\tilde{z})^T x \geq b(\tilde{z})}$ is not bilinear in $(x, \tilde{z})$ so the convex SOAD framework does not apply.

For financial experts,

$$\mathbb{P}[a(\tilde{z})^T x \geq b(\tilde{z})] \geq 1-\epsilon \iff \mathrm{VaR}_{1-\epsilon}\left(b(\tilde{z}) - a(\tilde{z})^T x\right) \leq 0.$$

Define (Rockafellar and Uryasev)

$$\mathrm{CVaR}_{1-\epsilon}(X) = \min_{\beta}\left\{\beta + \frac{1}{\epsilon}\mathbb{E}[(X-\beta)_+]\right\}, \tag{16}$$

where X is a single random variable. If $X = v(x, \tilde{z})$ is bilinear in $(x, \tilde{z})$ (or convex in x), then $\mathrm{CVaR}(v(x, \tilde{z}))$ is convex in x.

It can be shown that

$$\mathrm{VaR}_{1-\epsilon}(X) \leq \mathrm{CVaR}_{1-\epsilon}(X).$$

Thus, CVaR is a convex upper bound of VaR. In fact, it is best c.u. of it (Nemirovski and Shapiro).

By bounding $\mathbb{E}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})_+)$ one can reduce the case to LP, SOCP, or SDP. Our computational test shows that even the LP bounding method is satisfactory for a resource allocation problem.

See details in *Operations Research* 58 (2010) 470-485.

Thank You!