…FROM HIKING THE CASCADES AND THE OLYMPICS TO STOCHASTIC VARIATIONAL INEQUALITIES

SPCOM 2015, Adelaide — Central Australia (3? C° today)

ROCKAFELLAR: CAREER

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- was ``named” in 1935 and therefore existed

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ROCKAFELLAR HIGHLIGHTS

for a testimonial confer this meeting

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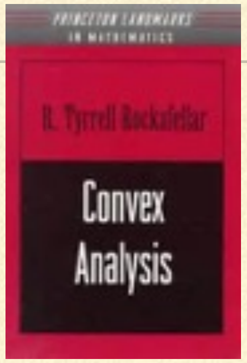
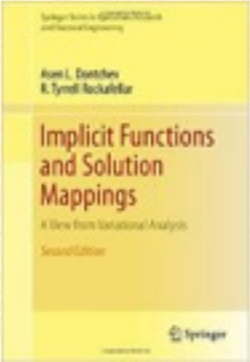
- Convex Duality (infinite dimensions), existence of dual solutions
- Normal Integrands: optimal control, stochastic programming, finance, mathematical statistics
- Monotone Operators, maximal and cyclic properties, VI's
- Augmented Lagrangians and Proximal Points: algorithmic analysis
- Subdifferential Calculus: convex, Clarke's, generalized subgradients
- Implicit functions theorems
- Risk measures, CVAR (conditional value at risk)
- Optimal control theory and HJB-properties, mathematical economics, structure reliability
- "Stochastic Variational Problems"

ROCKAFELLAR BY THE NUMBERS

Books



- Convex Analysis, Princeton Univ. Presse, 1970
- Conjugate duality and optimization, SIAM Conference B., 1974
- The theory of subgradients and its applications to problems of optimization. Convex and nonconvex functions. Heldermann, 1981
- Network Flows and Monotropic Optimization. Wiley, 1984
- Variational Analysis (with R. Wets). Springer, 1998 (3rd 2009).
- Implicit Functions and Solution Mappings (with A. Dontchev). Springer, 2009-14



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  - R.T. Rockafellar # 10! — at this meeting: only 3 in the top 100
  - … a grain of salt: John von Neumann # 350

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- # students (21+ …), # outstanding in RTR-circle, # editorships (7) … cf. CV

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- Honors: Dantzig Prize, von Neuman Prize, Lanchester Prize, Honoris Causa (4)

TERRY'S OTHER LIFE

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- mountain backpacking, sea-kayaking and landscape gardening

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- music, ballet, reading, languages … but not Seattle sailing
- traveling extensively (at the 5 or 6 continents-level per year)
- close interaction with family and friends
- … and living in one of the most beautiful areas in the world

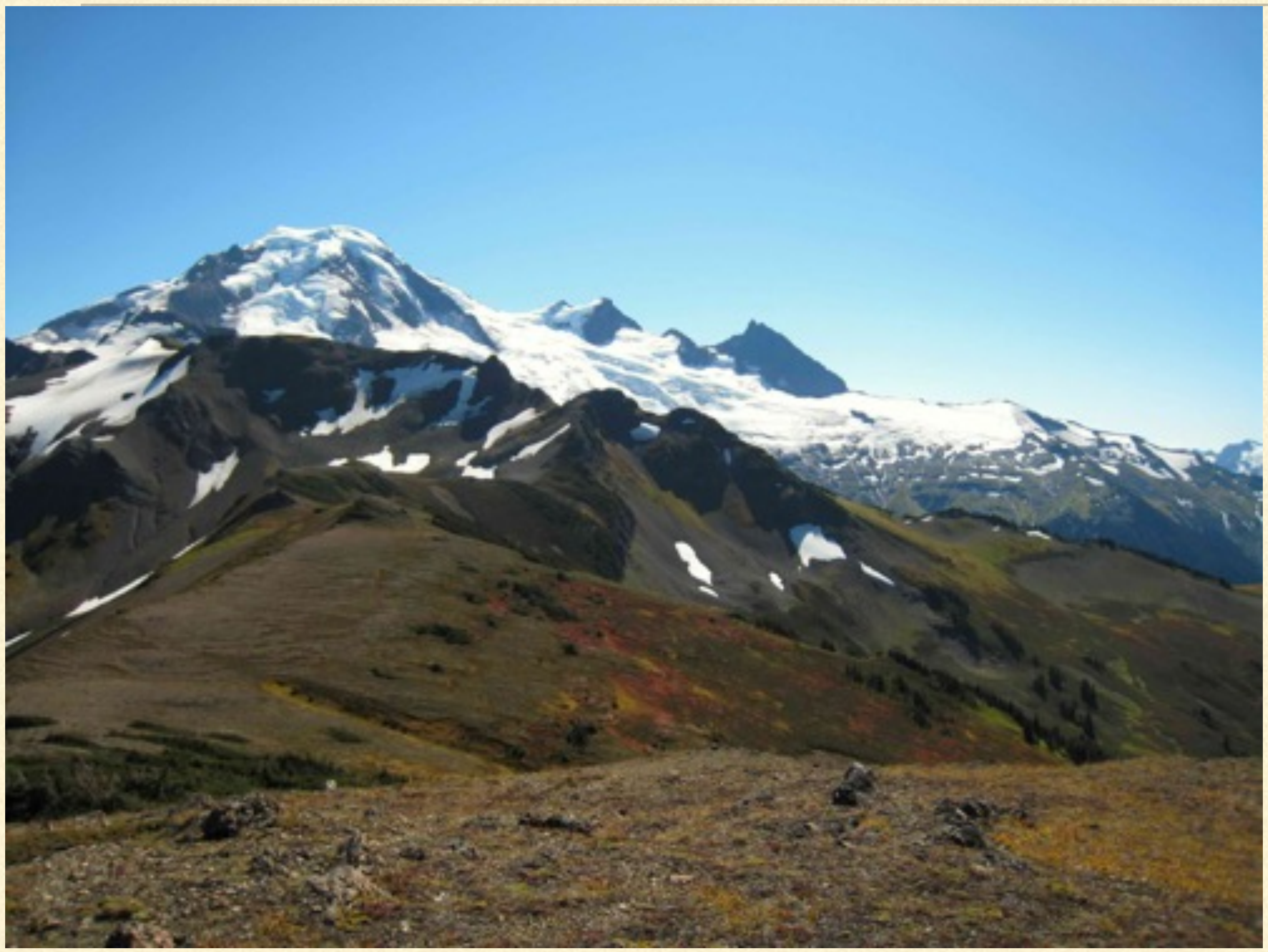
THE OLYMPICS

THE OLYMPICS

THE OLYMPICS

THE CASCADES

California to Canada



THE CASCADES

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ROGER J-B WETS

Wikipedia page

ROGER J-B WETS

ROGER J-B WETS   and TERRY

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TERRY & ROGER

- a near miss 1962 Mathematical Programming Symposium
- 1964: an introduction to conjugacy

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TERRY & ROGER

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  - and steady visitors/lecturers: Ky Fan, E. Asplund, … + optimizers

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- [1967 two duality theorems (Pacific J. Mathematics & JMAA)]

TERRY & ROGER: STOCHASTIC PROGRAMMING DUALITY

- ’69-70 Stochastic programs with fixed recourse: the equivalent deterministic problem. *SIAM Review*.
- ’71 Roger → Chicago, Lexington
- Nov. 71 Cologne to Univ. Bonn. (P. Vogel)
- ’74 results at “Control & Optimization” Paris meeting (A. Bensoussan & J.-L. Lions) mostly 2-stage recourse problem

STOCHASTIC OPTIMIZATION ERA

1971 - 2015

STOCHASTIC PROGRAMS WITH RECOURSE

$$\min \langle c, x^1 \rangle + \mathbb{E}^P\{\langle q_\xi, x^2_\xi \rangle\}$$
$$\text{such that } \; Ax^1 = b, \;\; x^1 \geq 0,$$
$$\forall\, \xi : W_\xi x^2_\xi = d_\xi - T_\xi x^1, \; x^2_\xi \geq 0$$

decision x^1, observe ξ, recourse x^2_ξ

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a more comprehensive formulation:

$$\min \; \mathbb{E}\Big\{ f(\boldsymbol{\xi}; x^1, x^2_\xi) \Big\} \;\; \text{such that}$$
$$x^1 \in C^1, \quad x^2_\xi \in C^2(\boldsymbol{\xi}; x_1)\; P\text{-a.s.}$$

KINKS — CHALLENGES

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- an infinite dimensional optimization problem
- choice of 'manageable' spaces for y and dual variables, perturbations
- constraint qualification!
- induced constraints: $Ax = b, x \geq 0$ does not imply $\exists$ feasible recourse $\forall \xi$
- to be approximated to rely on finite-dimensional optimization

THE "SUM" OF CONVEX FUNCTIONS

'81 IIASA, Vienna

$(\Xi, \mathcal{A}, P),\ \mathcal{G} \subset \mathcal{A},\ \ \text{(potentially } \mathcal{G} = \{\emptyset, \Xi\})$

$f : \Xi \times \mathbb{R}^n \to \overline{\mathbb{R}}$, a convex random lsc function

$Ef(x) = \mathbb{E}\{f(\boldsymbol{\xi}, x)\} < \infty$ on $\mathcal{L}^\infty_n(\mathcal{G})$

$\exists\, x \in \mathcal{L}^\infty_n : Ef(x)$ finite and norm-continuous

$\xi \mapsto \operatorname{cl}\operatorname{dom} f(\xi, \cdot)$ is $\mathcal{G}$-measurable

$\forall x \in \mathcal{L}^\infty_n(\mathcal{G})$, one has

$\partial \mathbb{E}^{\mathcal{G}} f\big(\cdot, x(\cdot)\big) = \mathbb{E}^{\mathcal{G}} \partial f\big(\cdot, x(\cdot)\big)$ P-a.s.

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Interchange of Minimization and Expectation
Interchange of Minimization and Subdifferentiation

NONANTICIPATIVITY AS A CONSTRAINT

'75 - Lexington Conference on Stochastic Optimization

$$\min E\left\{f(\xi;x^1, x^2_\xi)\right\}$$

$$x^1 \in C^1 \subset \mathbb{R}^{n_1},$$

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$$\min E\left\{ f(\xi; x^1_\xi, z^1_\xi) - \left\langle \bar{w}_\xi, x^1_\xi \right\rangle \right\}$$
$$x^1_\xi \in C^1,\; x^2_\xi \in C^2(\xi; x^1_\xi)$$

$\forall \xi \in \Xi$:
$$\min f(\xi; x^1, x^2) - \left\langle \bar{w}_\xi, x^2 \right\rangle$$
$$x^1 \in C^1,\; x^2 \in C^2(\xi; x^1)$$

FINDING $\overline{w}_\xi$: Progressive Hedging Algorithm

'86 Beijing, Academia Sinica

0. w_ξ^0 such that $\mathbb{E}\{w_\xi^0\} = 0,\ \nu = 0.$ Pick $\rho > 0$

1. for all ξ :

$$(x_\xi^{1,\nu}, x_\xi^{2,\nu}) \in \arg\min f(\xi; x^1, x^2) - \langle w_\xi^\nu, x^1 \rangle$$

$$x^1 \in C^1 \subset \mathbb{R}^{n_1},\ x^2 \in C^2(\xi, x^1) \subset \mathbb{R}^{n_2}$$

2. $\overline{x}^{1,\nu} = \mathbb{E}\{x_\xi^{1,\nu}\}$. Stop if $\left| x_\xi^{1,\nu} - \overline{x}^{1,\nu} \right| = 0$ (approx.)

otherwise $w_\xi^{\nu+1} = w_\xi^\nu + \rho\left[x_\xi^{1,\nu} - \overline{x}^{1,\nu} \right]$, return to 1. with $\nu = \nu + 1$

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Convergence: add a proximal term

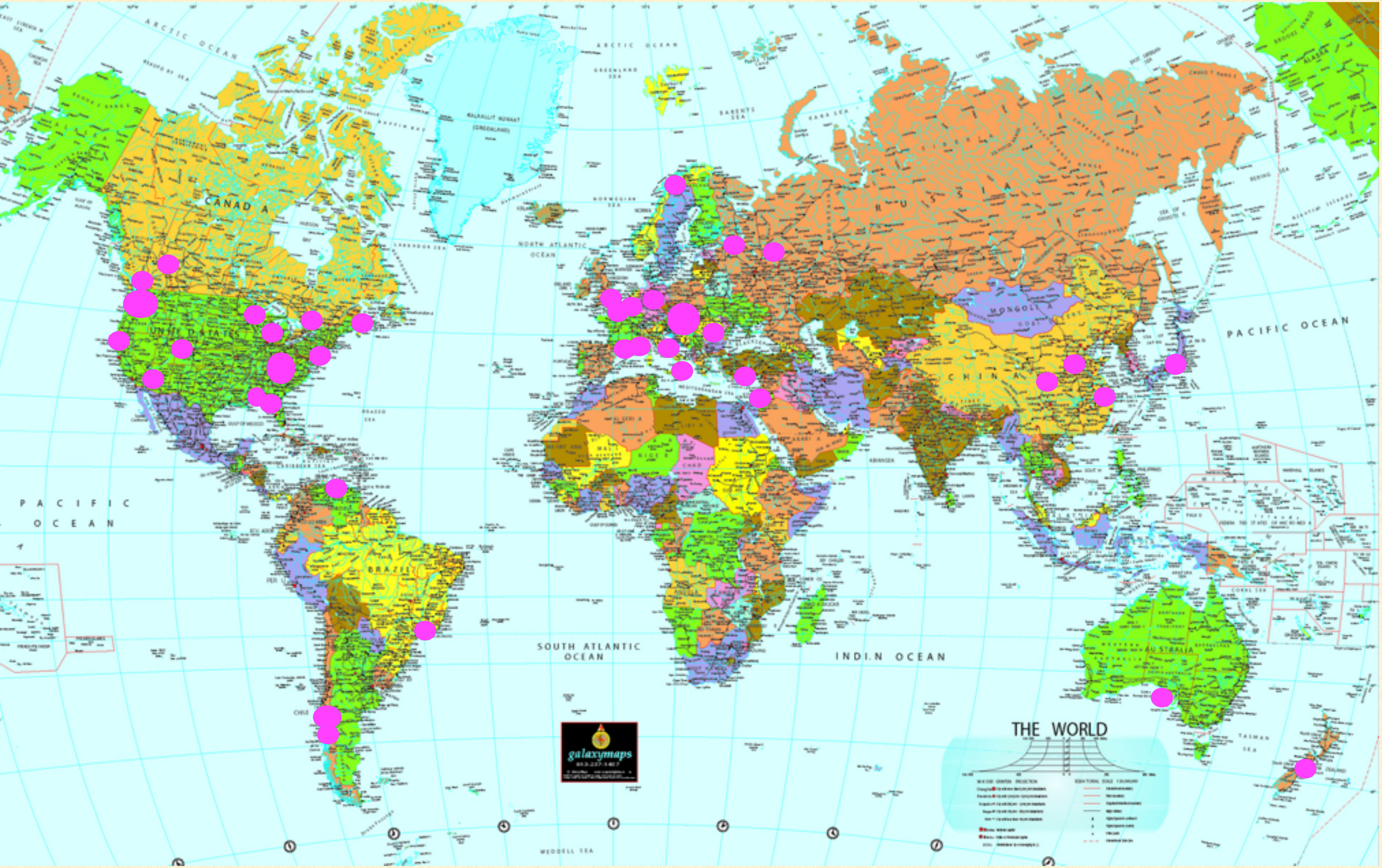
$$f(\xi; x^1, x^2) - \langle w_\xi^\nu, x^1 \rangle - \frac{\rho}{2}\left| x^1 - \bar{x}^{1,\nu} \right|^2$$

linear rate in $(x^{1,\nu}, w^\nu)$ ... eminently parallelizable

STOCHASTIC EQUILIBRIUM AND VARIATIONAL INEQUALITIES

’05-’15(+?) Santiago, Whidbey Island, St Petersburg, Adelaide
(mostly with Alejandro Jofré)

our "Working" places



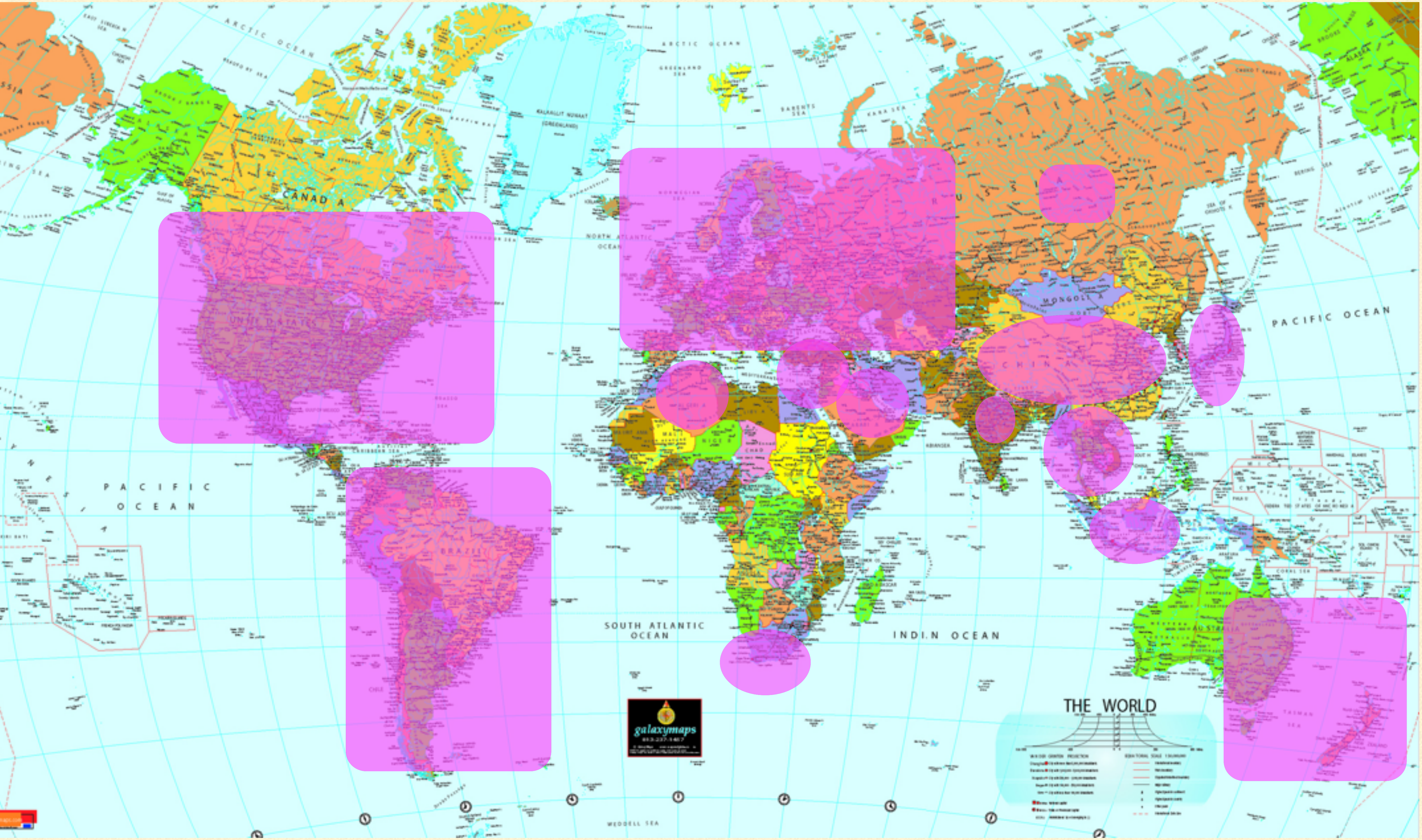

Seattle 2003


Minneapolis 2010


St Petersburg 2011


Rio de Janeiro 1981

… could have been "working" places



VARIATIONAL ANALYSIS ERA

1979-1998 2001-2015, also via third parties


Summer 1979 — Planning
“our” Stochastic Programming Book

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What should be include in the Appendix?


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MPS ’79 , Montreal, IIASA’s invitation


Summer 1979 — Planning
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What should be include in the Appendix?
MPS ’79 , Montreal, IIASA’s invitation
Laxenburg Fall 1980 — Chapter 1

convexity, duality
measurable & integration issues
thirst for an overarching theory

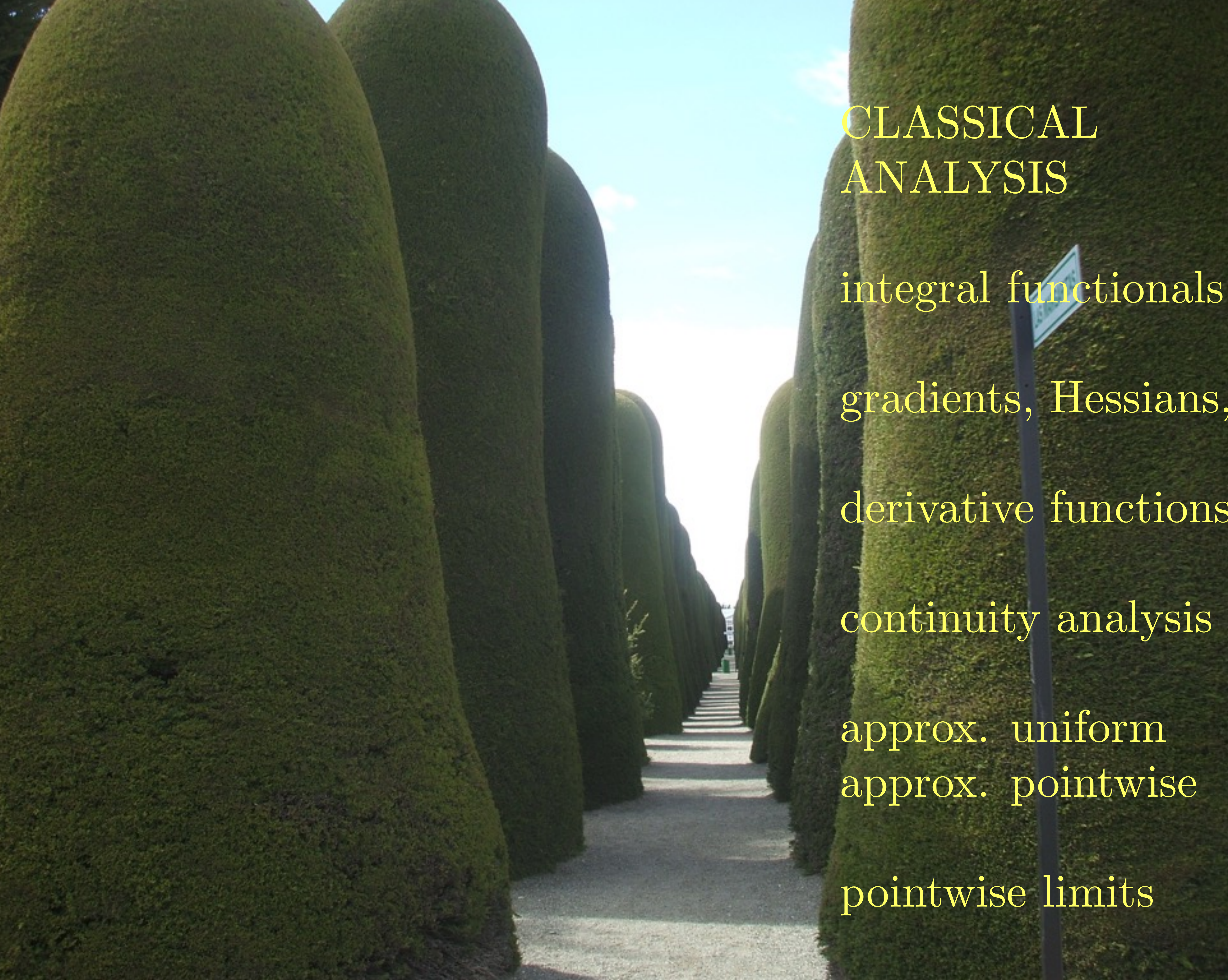
subdifferentiability ∂f

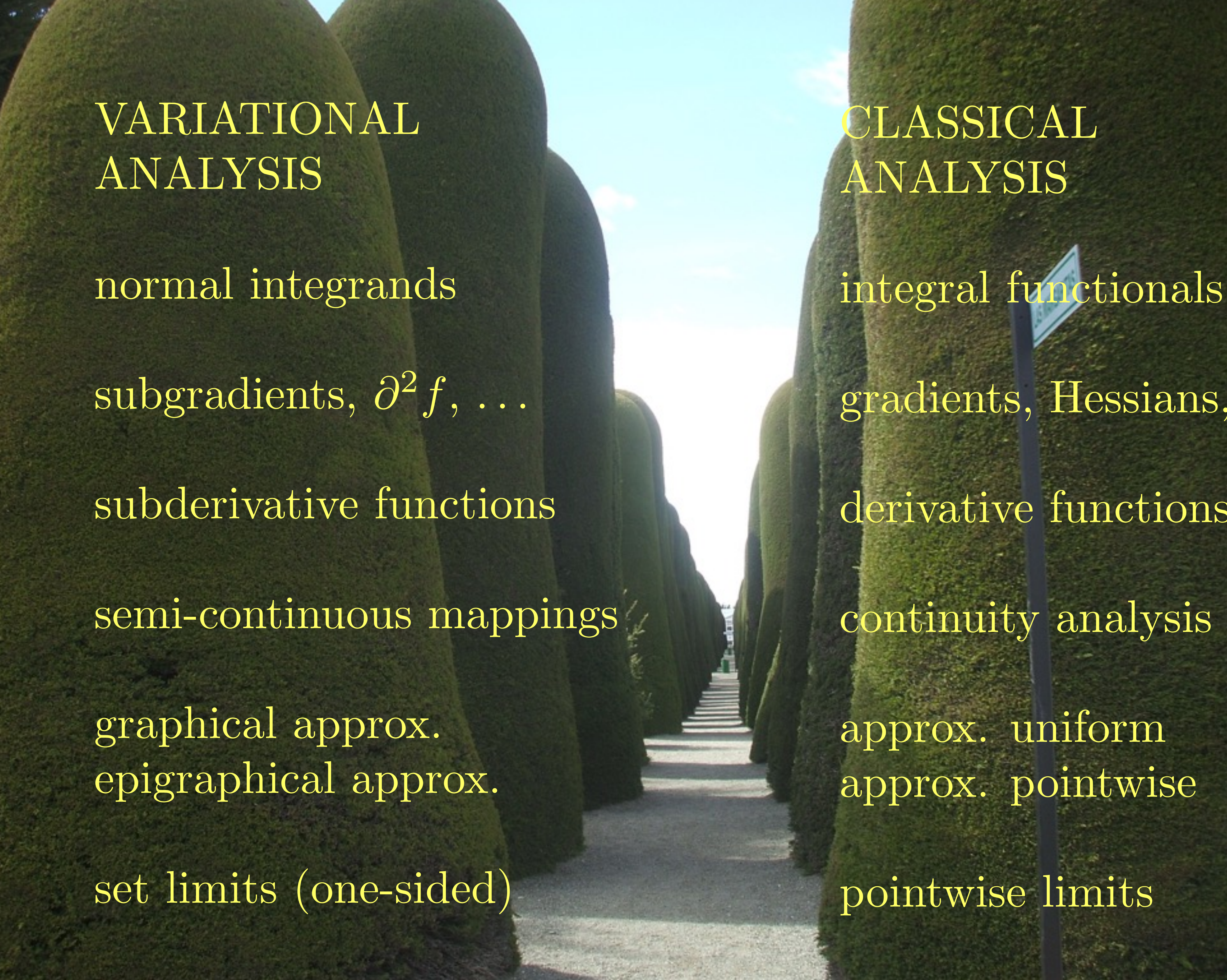
. . .

monotone operators
variational geometry
. . .

"variational" approximation

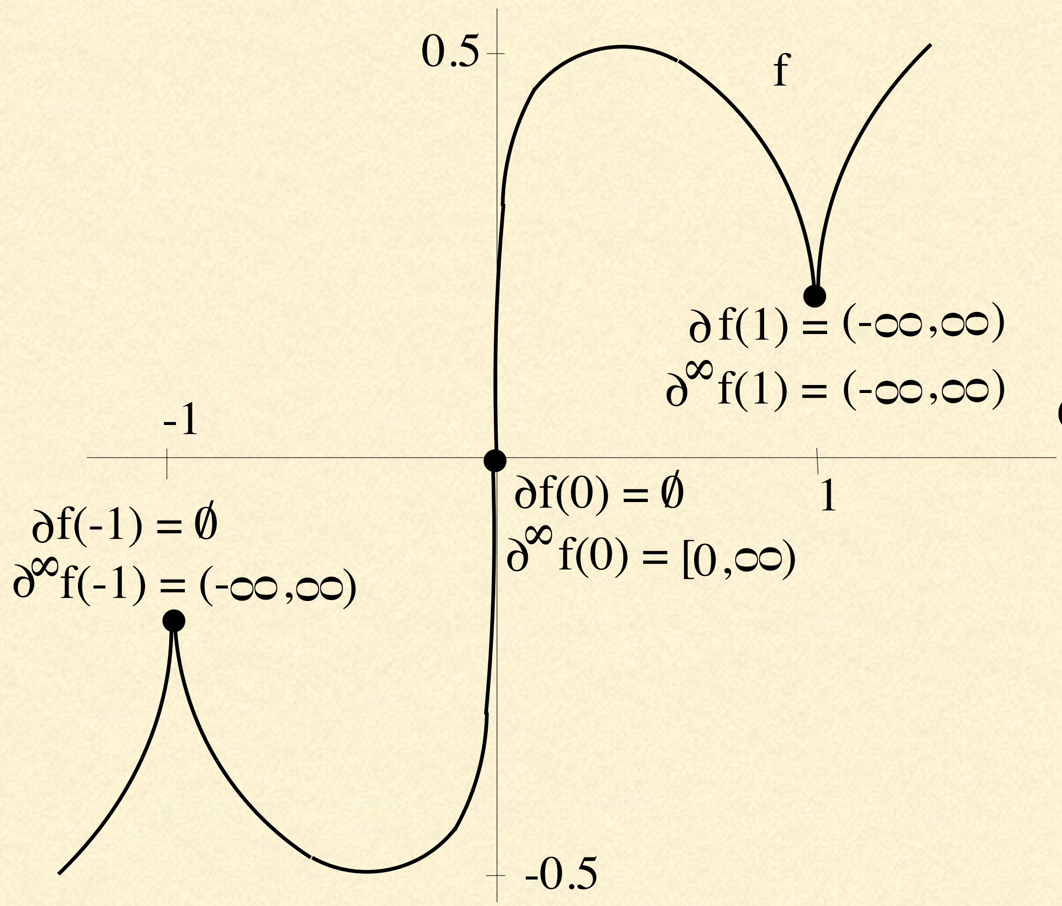
hyperspace topology

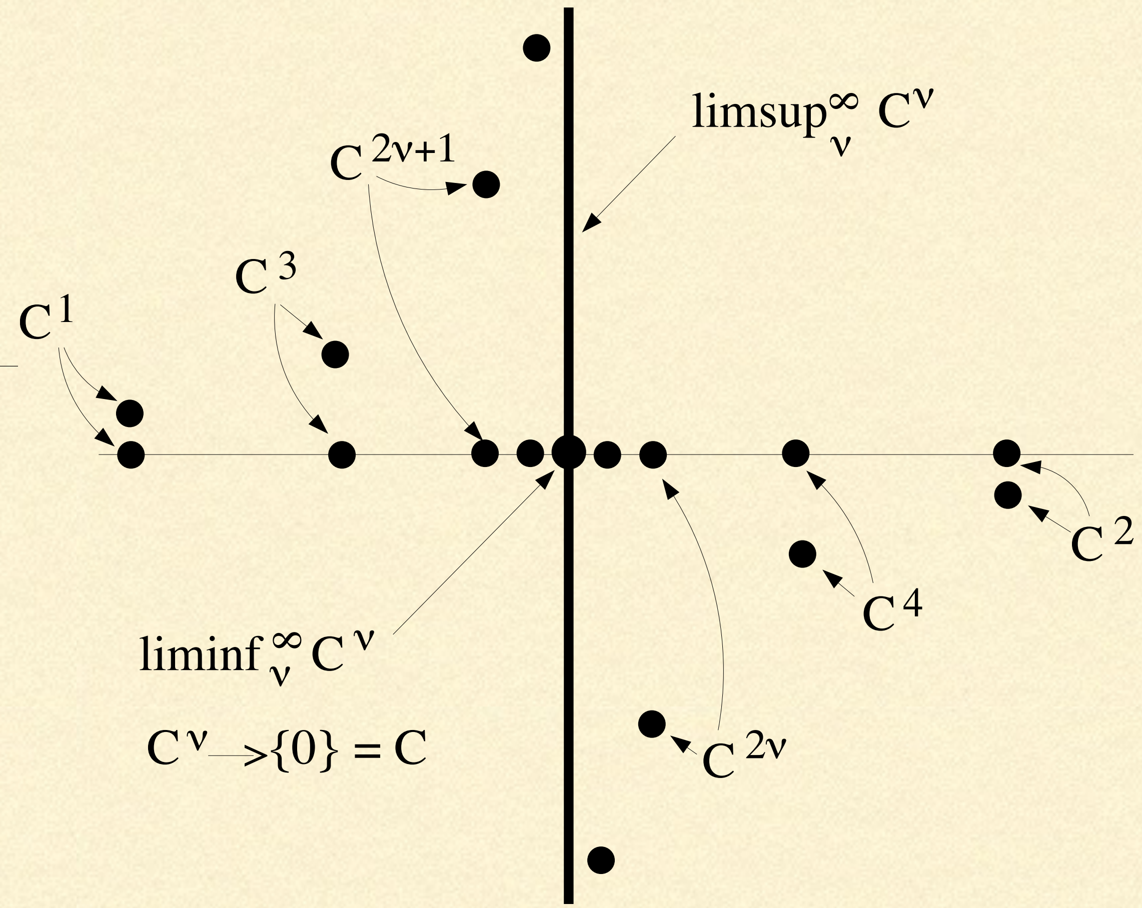

CLASSICAL
ANALYSIS
integral functionals
gradients, Hessians, . . .
derivative functions
continuity analysis
approx. uniform
approx. pointwise
pointwise limits



VARIATIONAL ANALYSIS	CLASSICAL ANALYSIS
normal integrands	integral functionals
subgradients, $\partial^2 f$, ...	gradients, Hessians, ...
subderivative functions	derivative functions
semi-continuous mappings	continuity analysis
graphical approx. epigraphical approx.	approx. uniform approx. pointwise
set limits (one-sided)	pointwise limits

COSMIC SPACE





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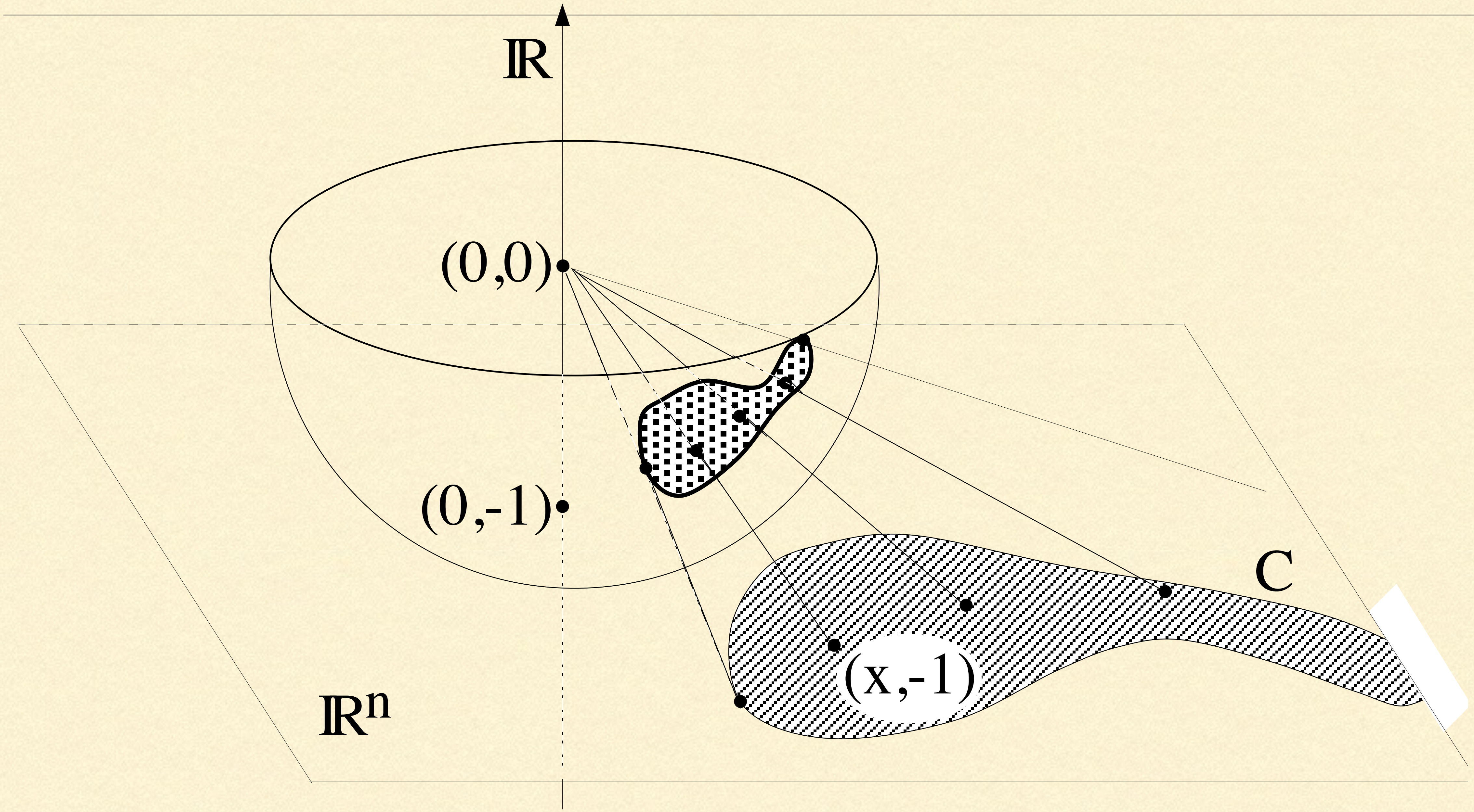

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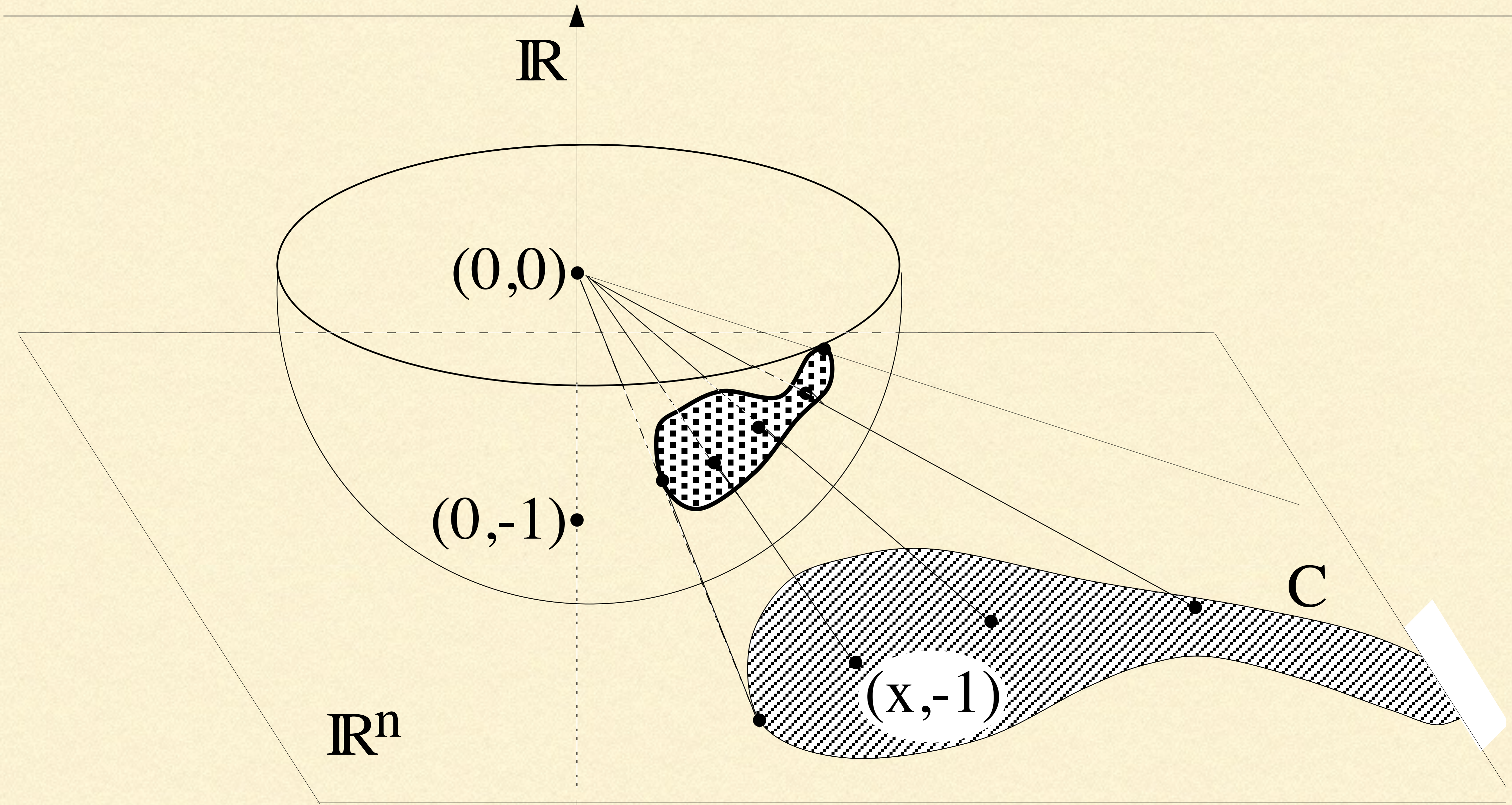

total convergence: set convergence of points and "dir" points

EPI-CONVERGENCE

f^ν epi-convergence to f
implies convergence of minimizers (roughly)

$\pm$ the only convergence notion
with this property

$\text{e-}\liminf f^\nu \quad \text{e-}\limsup f^\nu \quad \text{e-}\lim f^\nu$

APPROXIMATION & SUBDIFFERENTIATION

$f : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad f(\bar{x})$ finite

$\Delta_\tau f(\bar{x})(w) := \frac{f(\bar{x}+\tau w)-f(\bar{x})}{\tau}$ for $\tau > 0$

subderivative of f at $\bar{x}$,

$$df(\bar{x}) \;:=\; \text{e-lim}\inf_{\tau \searrow 0} \Delta_\tau f(\bar{x})$$

regular subderivative of f at $\bar{x}$

$$\hat{d}f(\bar{x}) \;:=\; \text{e-lim}\sup_{\tau \searrow 0 x \underset{f}{\to} \bar{x}} \Delta_\tau f(x)$$

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epi-derivative ('88 Rock.)

$\text{e-lim}_{\tau \searrow 0 \;\; w' \to w} \Delta f(\bar{x})(w')$

regular subderivative of f at $\bar{x}$

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A "UNIQUE" SUBGRADIENT

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'73 Clarke: convexification, …, Rubinov, …

A “UNIQUE” SUBGRADIENT

Minsk! ’76-88
Mordukhovich
& Kruger

’73 Clarke: convexification, …, Rubinov, …

A "UNIQUE" SUBGRADIENT

$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $f(\bar{x}) \in \mathbb{R}$

v is a *regular subgradient*, $v \in \hat{\partial} f(\bar{x})$, if

$$f(x) \ \geq \ f(\bar{x}) + \langle v, x - \bar{x} \rangle + o\big(|x - \bar{x}|\big)$$

v is a *subgradient*, $v \in \partial f(\bar{x})$, if

$$\exists\, x^\nu \underset{f}{\to} \bar{x}, \ \ v^\nu \in \hat{\partial} f(x^\nu) \text{ with } v^\nu \to v$$

v is a *horizon subgradient*, $v \in \partial^\infty f(\bar{x})$, if

$$\exists\, x^\nu \underset{f}{\to} \bar{x}, \ \ v^\nu \in \hat{\partial} f(x^\nu) \text{ with } v^\nu \to \operatorname{dir} v \text{ or } v = 0$$

Minsk! '76-88
Mordukhovich
& Kruger

'73 Clarke: convexification, …, Rubinov, …

… WOULD NOW INCLUDE

- epi/hypo-convergence (saddle points), lopsided convergence
- much more about 2nd order differentiability
- variational inequalities, equilibrium problems, MPEC
- Fitzpatrick functions, …


Stochastic Variational Analysis ?
Dec. '14, Santiago