Non-Lipschitzian Reformulation Method for Constrained Optimization Problems

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1. Non-Lipschitzian Reformulation

Consider the nonlinear programming problem (NLP):

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \le 0, \quad i \in I := \{1, \dots, m\}, \\ & h_j(x) = 0, \quad j \in J := \{m+1, \dots, m+q\}, \end{aligned}$$

where $f, g_i, h_j : R^n \to R$ are assumed to be smooth functions.

We denote by C the feasible set of (NLP).

Karush-Kuhn-Tucker (**KKT**) condition for a local minimum $\bar{x}$ of (NLP), originated with Kuhn and Tucker (1951) and Karush (1939), holds if there exists a vector $\lambda \in \mathbb{R}^{m+q}$, called a KKT multiplier, such that

$$\nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in J} \lambda_j \nabla h_j(\bar{x}) = 0, \quad \lambda_i \geq 0, \ \lambda_i g_i(\bar{x}) = 0 \ \forall i \in I.$$

Second-Order Condition (**SON**), originated with Ioffe (1979), holds at a local minimum $\bar{x}$ of (NLP) if

$$\sup_{\lambda \in \mathrm{KKT}(\bar{x})} \langle w, \nabla^2_{xx} L(\bar{x}, \lambda) w \rangle \geq 0 \qquad \forall w \in \mathcal{V}(\bar{x}),$$

where $\mathrm{KKT}(\bar{x})$ is the set of all KKT multipliers at $\bar{x}$ and the critical cone $\mathcal{V}(\bar{x})$ at $\bar{x}$ is definde by

$$\mathcal{V}(\bar{x}) := \left\{ w \in R^n \;\middle|\; \begin{array}{l} \langle \nabla f(\bar{x}), w \rangle \leq 0 \\ \langle \nabla g_i(\bar{x}), w \rangle \leq 0 \;\; \forall i \in I(\bar{x}) \\ \langle \nabla h_j(\bar{x}), w \rangle = 0 \;\; \forall j \in J \end{array} \right\}.$$

Let $\phi : R^n \to R_+ \cup \{+\infty\}$ be a lower semicontinuous function such that

$$C = \{x \in R^n \mid \phi(x) = 0\}.$$

The general penalty problem is

$$\min_{x \in R^n} f(x) + \mu\phi(x).$$

Definition 1.1 *We say that the penalty function $f + \mu\phi$ is* **exact at** $\bar{x}$ *if, $f + \mu\phi$ admits a local minimum at $\bar{x}$ with some finite penalty parameter $\mu > 0$.*

Definition 1.2 *We say that the penalty term ϕ is of* **KKT-type at** $\bar{x}$ *if the* KKT *condition holds at $\bar{x}$ whenever the penalty function $f + \mu\phi$ is exact at $\bar{x}$.*

Lower order penalty: Let $0 \le p \le 1$ and $0^0 := 0$. Let

$$S(x) = \sum_{i \in I} \max\{g_i(x), 0\} + \sum_{j \in J} |h_j(x)| \quad \forall x \in R^n,$$

while the l_p penalty function associated with (NLP) is of the form

$$\mathcal{F}_p(x) := f(x) + \mu S^p(x).$$

- $p = 1$, $\mathcal{F}_1(x)$ is the classical l_1 penalty function, see Eremin (1967) and Zangwill (1967).
- $p < 1$, $\mathcal{F}_p(x)$ is referred to as the lower order l_p penalty function, first introduced in Luo et al. (1996) for the study of MPEC and was rediscovered from a unified augmented Lagrangian scheme by Huang and Yang (2003) and Rubinov and Yang (2003).

The l_p penalty problem (NLP(p)) is

$$\min_{x \in R^n} f(x) + \mu S^p(x).$$

l_1 exact penalty function implies the KKT condition and the SON condition, see Clarke (1983) and Rockafellar (1989), respectively.

Can $l_p (p < 1)$ exact penalty function be used for the KKT condition and the SON condition ?

2. First-order Necessary Conditions and KKT-type Penalty Terms

If $q = 0$ and all g_i's are concave, then the KKT condition of (NLP) holds. See Hestenes (1970).

The generalized Clarke second-order directional derivative of a $C^{1,1}$ function is defined by

$$g^{\circ\circ}(x;w) = \limsup_{y\to x,\, t\to 0+} \frac{\nabla g(y+tu)^T w - \nabla g(y)^T w}{t},$$

see Hiriart-Urruty et al (1984), Cominetti and Correa (1990), and Yang and Jeyakumar (1992).

A $C^{1,1}$ function g is concave if and only if

$$g^{\circ\circ}(x;w) \leq 0, \forall x, w \in R^n.$$

See Yang (1994).

Let

$$\begin{aligned} I(\bar{x}) : &= \{i \in I \mid g_i(\bar{x}) = 0\}, \\ I(\bar{x}, w) : &= \{i \in I \mid g_i(\bar{x}) = 0, \langle \nabla g_i(\bar{x}), w \rangle = 0\}. \end{aligned}$$

Let $L_C(\bar{x})$ be the first-order linearized tangent cone to C at $\bar{x}$ defined by

$$L_C(\bar{x}) := \left\{ w \in R^n \;\middle|\; \begin{array}{ll} \langle \nabla g_i(\bar{x}), w \rangle \leq 0 & \forall i \in I(\bar{x}) \\ \langle \nabla h_j(\bar{x}), w \rangle = 0 & \forall j \in J \end{array} \right\}.$$

By calculating the upper Dini-derivative of $\mathcal{F}_{1/2}(x)$ and using a second-order Taylor expansion, Yang and Meng (2007) showed that $S^{\frac{1}{2}}(x)$ is of KKT-type at $\bar{x}$, that is the KKT condition holds, if, for every $w \in L_C(\bar{x})$,

$$g_i^{\circ\circ}(\bar{x}, w) \le 0, \ \forall i \in I(\bar{x}, w), \ \ h_j^{\circ\circ}(\bar{x}, w) = 0 \ \forall j \in J.$$

These results have been extended to semi-infinite program and generalized semi-infinite program and the paper is submitted to a JOTA special issue dedicated to Elijah (Lucien) Polak's 85th birthday. See Yang, Chen and Zhou (2015).

By calculating the contingent directional derivative of $\mathcal{F}_{1/2}(x)$ at $\bar{x}$, Meng and Yang (2010) showed that if, for every $w \in L_C(\bar{x})$, there exists some $z \in R^n$ such that

$$\begin{aligned} &\langle \nabla g_i(\bar{x}), z \rangle + \langle w, \nabla^2 g_i(\bar{x}) w \rangle \leq 0 \quad \forall i \in I(\bar{x}, w), \\ &\langle \nabla h_j(\bar{x}), z \rangle + \langle w, \nabla^2 h_j(\bar{x}) w \rangle = 0 \quad \forall j \in J. \end{aligned}$$

then the KKT condition holds.

Meng and Yang (2015) employ the following tools from Variational Analysis, see Rockafellar and Wets (1998). For any $f : R^n \to \overline{R}$ and a point $\bar{x}$ with $f(\bar{x})$ finite,

- For any $w \in R^n$, the subderivative of f at $\bar{x}$ for w is defined by

$$df(\bar{x})(w) := \liminf_{\tau \to 0+,\ w' \to w} \frac{f(\bar{x} + \tau w') - f(\bar{x})}{\tau}.$$

- A vector $v \in R^n$ is a regular subgradient of f at $\bar{x}$, written $v \in \widehat{\partial} f(\bar{x})$, if

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\|x - \bar{x}\|).$$

- The relation between subderivative and regular subdifferential is

$$\widehat{\partial} f(\bar{x}) = \{v \in R^n \mid \langle v, w \rangle \leq df(\bar{x})(w)\ \forall w \in \mathrm{dom} df(\bar{x})\}.$$

Lemma 2.1 *Suppose that the function* $\psi : R^n \to \overline{R}$ *has a local minimum at* $\bar{x}$ *with* $\psi(\bar{x})$ *finite. Then we have*[a]

$$[\mathrm{dom} d\psi(\bar{x})]^* \subset \widehat{\partial}\psi(\bar{x}) \subset [\mathrm{ker} d\psi(\bar{x})]^*.$$

- *The first inclusion is an equality if and only if* $\widehat{\partial}\psi(\bar{x})$ *is a cone;*
- *The second inclusion is an equality if and only if*

$$[\mathrm{dom} d\psi(\bar{x})]^* = [\mathrm{ker} d\psi(\bar{x})]^*.$$

- *If the subderivative* $d\psi(\bar{x})$ *is a sublinear function as is true when* ψ *is regular at* $\bar{x}$ *(see Definition 7.25 of Rockafellar and Wets (1998)), then*

$$\mathrm{clpos}(\widehat{\partial}\psi(\bar{x})) = [\mathrm{ker} d\psi(\bar{x})]^*. \quad (1)$$

[a] The polar cone of A is defined by

$$A^* = \{z \in R^n | \langle z, x \rangle \leq 0 \; \forall x \in A\}.$$

The closure operation cannot be removed from the left-hand side of (1) even if ψ is convex.

Example 2.1 *Consider at* $\bar{x} = (0, 0)$ *the function*

$$\psi(x) = \max_{0 \le t \le 1} g(x, t),$$

where $g(x, t) = tx_1 + t^2 x_2$ *for all* $x \in R^2$ *and* $t \in R$*. The equality*

$$\mathrm{clpos}(\widehat{\partial}\psi(\bar{x})) = [\ker d\psi(\bar{x})]^* (= \{x \mid 0 \le x_2 \le x_1\})$$

holds, but when the closure operation is removed from the left-hand side, we merely have

$$\mathrm{pos}(\widehat{\partial}\psi(\bar{x})) \subsetneq [\ker d\psi(\bar{x})]^*,$$

because

$$\mathrm{pos}(\widehat{\partial}\psi(\bar{x})) = \{x \mid 0 \le x_2 \le x_1\} \backslash \{x \mid x_1 > 0, x_2 = 0\}.$$

It is well-known, see Rockafellar and Wets (1998) that, for a polyhedral set P at one of its points $\bar{x}$, there exists a neighborhood V of $\bar{x}$ such that

$$P \cap V = [\bar{x} + T_P(\bar{x})] \cap V.$$

We now introduce such a property for a convex set in an analogous way.

Definition 2.1 *We say that a convex set $C \subset \mathbb{R}^n$ admits exactness of tangent approximation (ETA, for short) at one of its points $\bar{x}$, if $\exists$ a neighbourhood V of $\bar{x}$ such that*

$$(clC) \cap V = [\bar{x} + T_C(\bar{x})] \cap V.$$

Proposition 2.1 *(Meng, Roshchina and Yang (2015))* Let $C \subset \mathbb{R}^n$ be closed and convex. The following properties are equivalent:

(i) C is locally polyhedral at every $x \in C$, i.e., $(C - \{x\}) \cap V$ is a polyhedron for some polyhedral neighbourhood V of x.

(ii) C admits ETA at every $x \in C$.

(iii) $pos(C - x)$ is closed for all $x \in C$.

We recall the **variational description of regular subgradients**:

Lemma 2.2 *(Rockafellar and Wets (1998), Proposition 8.5). A vector v belongs to $\widehat{\partial} f(\bar{x})$ if and only if, on some neighborhood of $\bar{x}$, there is a function $h \leq f$ with $h(\bar{x}) = f(\bar{x})$ such that h is differentiable at $\bar{x}$ with $\nabla h(\bar{x}) = v$. Moreover h can be taken to be continuously differentiable with $h(x) < f(x)$ for all $x \neq \bar{x}$ near $\bar{x}$.*

Remark 2.1 *This variational description is a contribution to the basics of variational analysis, as pointed out on p.347 of Rockafellar and Wets (1998).*

We can obtain from Lemmas 2.1 and 2.2 the following.

Theorem 2.1 *(Meng and Yang (2015)) Consider the following conditions:*

(i) $[\ker d\phi(\bar{x})]^* \subset L_C(\bar{x})^*$.

(ii) $\widehat{\partial}\phi(\bar{x}) \subset L_C(\bar{x})^*$.

(iii) *The penalty term* ϕ *is of* KKT-type *at* $\bar{x}$.

Then $(\mathrm{i}) \Longrightarrow (\mathrm{ii}) \Longleftrightarrow (\mathrm{iii})$.

[(iii) $\Longrightarrow$ (ii)]: Let $v \in \widehat{\partial}\phi(\bar{x})$. According to the variational description of regular subgradients in (Rockafellar and Wets, 1998, Proposition 8.5), there exist a neighborhood V of $\bar{x}$ and a continuously differentiable function $\psi : \mathbb{R}^n \to \mathbb{R}$ with $\psi(\bar{x}) = \phi(\bar{x}) = 0$ and $\nabla\psi(\bar{x}) = v$ such that

$$\psi(x) \leq \phi(x), \qquad \forall x \in V.$$

Set $f = -\psi$. Clearly,

$$f(x) + \phi(x) = -\psi(x) + \phi(x) \geq 0 = f(\bar{x}) + \phi(\bar{x}), \qquad \forall x \in V.$$

That is, the penalty function $f + \phi$ admits a local minimum at $\bar{x}$. Since ϕ is a KKT-type penalty term at $\bar{x}$, the KKT condition holds at $\bar{x}$. Thus $-\nabla f(\bar{x}) \in L_C(\bar{x})^*$. Thus $v \in L_C(\bar{x})^*$, and

$$\widehat{\partial}\phi(\bar{x}) \subseteq L_C(\bar{x})^*.$$

Theorem 2.2 *(Meng and Yang (2015)) Let* $0 \le p < 1$. *Consider the following conditions:*

(i) $[\mathrm{ker} dS^p(\bar{x})]^* = L_C(\bar{x})^*$.

(ii) $\widehat{\partial} S^p(\bar{x}) = L_C(\bar{x})^*$.

(iii) S^p *is a* KKT-type *penalty term at* $\bar{x}$.

Then $(\mathrm{i}) \Longrightarrow (\mathrm{ii}) \Longleftrightarrow (\mathrm{iii})$.

• In the case of $p = 0$, (i) and (ii) are equivalent, and moreover Theorem 2.2 recovers a well-known result that the GCQ, i.e.

$$T_C(\bar{x})^* = L_C(\bar{x})^*$$

is the weakest one ensuring KKT conditions, as $\mathrm{ker} dS^0(\bar{x}) = T_C(\bar{x})$.

• In the case of $0 < p < 1$, we are not aware of the equivalence of (i) and (ii), although they are the same in many situations.

The degenerate KKT multiplier set at $\bar{x}$ is defined by

$$\mathrm{KKT}_0(\bar{x}) := \left\{ \rho \;\middle|\; \begin{array}{l} \sum_{i\in I} \rho_i \nabla g_i(\bar{x}) + \sum_{j\in J} \rho_j \nabla h_j(\bar{x}) = 0 \\ \rho_i \geq 0 \quad \forall i \in I(\bar{x}),\;\; \rho_i = 0 \; \forall i \in I \backslash I(\bar{x}) \end{array} \right\}$$

The second subderivative of f at $\bar{x}$ for v and w is defined by, see Rockafellar and Wets (1998)

$$d^2 f(\bar{x}|v)(w) := \liminf_{\tau\to 0+,\, w'\to w} \frac{f(\bar{x}+\tau w') - f(\bar{x}) - \tau\langle v, w\rangle}{\frac{1}{2}\tau^2}.$$

By definition, it is straightforward to verify that

$$d^2 S(\bar{x}|0)(w) = 2[dS^{\frac{1}{2}}(\bar{x})(w)]^2.$$

By a direct calculation using the chain rule for second subderivatives of piece-wise linear-quadratic functions [a], we have

$$dS^{\frac{1}{2}}(\bar{x})(w) = +\infty \quad \forall w \notin L_C(\bar{x}),$$

and if $w \in L_C(\bar{x})$, we have

$$\begin{aligned} & dS^{\frac{1}{2}}(\bar{x})(w) \\ = \; & \sqrt{\frac{1}{2} d^2 S(\bar{x}|0)(w)} \\ = \; & \frac{\sqrt{2}}{2} \sqrt{\max_{\rho \in \mathrm{KKT}_0(\bar{x}),\, \|\rho\|_\infty = 1} \left\langle \left[\sum_{i \in I} \rho_i \nabla^2 g_i(\bar{x}) + \sum_{j \in J} \rho_j \nabla^2 h_j(\bar{x}) \right] w, w \right\rangle}, \\ & \widehat{\partial} S^{\frac{1}{2}}(\bar{x}) = \{ v \mid \langle v, w \rangle \le dS^{\frac{1}{2}}(\bar{x})(w) \quad \forall w \}. \end{aligned}$$

But we cannot give an explicit formula for $\widehat{\partial} S^{\frac{1}{2}}(\bar{x})$.

[a] See Chapter 13 of Rockafellar and Wets (1998).

Proposition 2.2 $S^{\frac{1}{2}}$ *is of KKT-type at* $\bar{x}$ *if one of the two following conditions is satisfied:*

(i) *For every* $w \in L_C(\bar{x})$*, it follows that*

$$\max_{\lambda \in \mathrm{KKT}_0(\bar{x})} \left\{ \sum_{i \in I} \lambda_i \langle w, \nabla^2 g_i(\bar{x}) w \rangle + \sum_{j \in J} \lambda_j \langle w, \nabla^2 h_j(\bar{x}) w \rangle \right\} = 0. \quad (2)$$

(ii) $\ker dS^{\frac{1}{2}}(\bar{x}) = L_C(\bar{x})$.

Moreover, we have (i) $\Longleftrightarrow$ (ii).

- Condition (2) is newly obtained, and we have

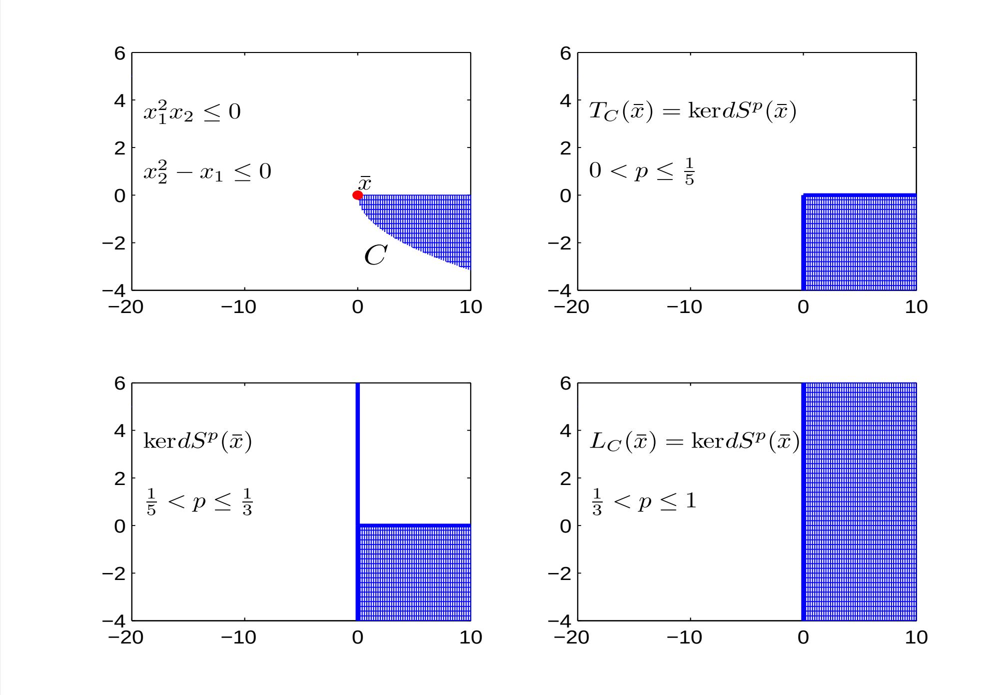
$$\text{MFCQ} \Longrightarrow (2),$$

because the MFCQ at $\bar{x} \Longleftrightarrow \mathrm{KKT}_0(\bar{x}) = \{0\}$.

Example 2.2 *Let* $\bar{x} = (0,0)$ *and let*

$$C = \left\{ x \in R^n \,\middle|\, \begin{array}{c} x_1^2 x_2 \le 0 \\ x_2^2 - x_1 \le 0 \end{array} \right\}.$$

- *(2) holds and* $\mathrm{KKT}_0(\bar{x}) = R_+ \times \{0\}$.



$T_C(\bar{x}) = R_+ \times (-R_+)$, $L_C(\bar{x}) = R_+ \times R$, and

$$\ker dS^p(\bar{x}) = \begin{cases} R_+ \times (-R_+) & \text{if } 0 < p \le \frac{1}{5}, \\ R_+ \times (-R_+) \cup \{0\} \times R_+ & \text{if } \frac{1}{5} < p \le \frac{1}{3}, \\ R_+ \times R & \text{if } \frac{1}{3} < p \le 1. \end{cases}$$

3. Second-order Necessary Conditions via Exact Penalty Functions

Denote the set of all KKT multipliers at $\bar{x}$ by $\mathrm{KKT}(\bar{x})$ and the critical cone at $\bar{x}$ by

$$\mathcal{V}(\bar{x}) := \left\{ w \in R^n \;\middle|\; \begin{array}{l} \langle \nabla f(\bar{x}), w\rangle \leq 0 \\ \langle \nabla g_i(\bar{x}), w\rangle \leq 0 \quad \forall i \in I(\bar{x}) \\ \langle \nabla h_j(\bar{x}), w\rangle = 0 \quad \forall j \in J \end{array} \right\}.$$

The second-order necessary condition (for short, SON), originated with Ioffe (1979), holds at a local minimum $\bar{x}$ of (NLP) if

$$\sup_{\lambda \in \mathrm{KKT}(\bar{x})} \langle w, \nabla^2_{xx} L(\bar{x}, \lambda) w\rangle \geq 0 \qquad \forall w \in \mathcal{V}(\bar{x}),$$

where the convention $\sup \emptyset := -\infty$ is used.

The parabolic subderivative of f at $\bar{x}$ for w with respect to z is defined by, see Rockafellar and Wets (1998)

$$d^2 f(\bar{x})(w \mid z) := \liminf_{\tau \to 0+,\, z' \to z} \frac{f(\bar{x} + \tau w + \frac{1}{2}\tau^2 z') - f(\bar{x}) - \tau df(\bar{x})(w)}{\frac{1}{2}\tau^2}.$$

For any w and z, let

$$I(\bar{x}, w) := \{i \in I(\bar{x}) \mid \langle w, \nabla g_i(\bar{x})\rangle = 0\},$$
$$I(\bar{x}, w, z) := \{i \in I(\bar{x}, w) \mid \langle z, \nabla g_i(\bar{x})\rangle + \langle w, \nabla^2 g_i(\bar{x}) w\rangle = 0\},$$

and let the second-order linearized tangent set to C at $\bar{x}$ in the direction $w \in L_C(\bar{x})$ be given by

$$L_C^2(\bar{x} \mid w) := \left\{ z \,\middle|\, \begin{array}{ll} \langle \nabla g_i(\bar{x}), z\rangle + \langle w, \nabla^2 g_i(\bar{x}) w\rangle \leq 0 & \forall i \in I(\bar{x}, w) \\ \langle \nabla h_j(\bar{x}), z\rangle + \langle w, \nabla^2 h_j(\bar{x}) w\rangle = 0 & \forall j \in J \end{array} \right\}.$$

Let $w \in T_A(\bar{x})$. The second-order tangent set to A at $\bar{x}$ is

$$T_A^2(\bar{x} \mid w) := \{z | \exists t_k \downarrow 0 \text{ and } z_k \to z \text{ such that } \bar{x} + t_k w + \frac{1}{2} t_k^2 z_k \in A \text{ for all } k\}.$$

- l_1 exactness $\Longrightarrow$ (SON). See Corollary 4.5 of Rockafellar (1989), where more general results on second-order necessary conditions were obtained for a convex composite optimisation problem by virtue of twice epi-derivative and a basic constraint qualification.

For the l_1 exact penalty function, we can show

$$L_C^2(\bar{x} \mid w) = \ker d^2 S(\bar{x})(w \mid \cdot) \quad \forall w \in L_C(\bar{x}),$$

by applying a second-order Taylor expansion.

On the other hand, Kawasaki (1988) introduced the following second-order Guinard constraint qualification (SGCQ)

$$L_C^2(\bar{x} \mid w) = \mathrm{clconv}[T_C^2(\bar{x} \mid w)] \quad \forall w \in \mathcal{V}(\bar{x}).$$

As we have

$$T_C^2(\bar{x} \mid w) = \ker d^2 S^0(\bar{x})(w \mid \cdot), \quad \forall w \in T_C(\bar{x}),$$

the SGCQ reduces to

$$L_C^2(\bar{x} \mid w) = \mathrm{clconv}[\ker d^2 S^0(\bar{x})(w \mid \cdot)] \quad \forall w \in \mathcal{V}(\bar{x}).$$

So, we are looking at the following second-order constraint qualification:

$$L^2_C(\bar{x} \mid w) \subset \operatorname{clconv}[\ker d^2\phi(\bar{x})(w \mid \cdot)] \qquad \forall w \in \mathcal{V}(\bar{x}).$$

Theorem 3.1 *(Meng and Yang (2015)) Let $\bar{x}$ be a local minimum of* (NLP). *Suppose that the penalty function $f + \mu\phi$ is exact at $\bar{x}$. If*

$$L_C^2(\bar{x} \mid w) \subset \mathrm{clconv}[\mathrm{ker} d^2\phi(\bar{x})(w \mid \cdot)] \qquad \forall w \in \mathcal{V}(\bar{x}), \tag{3}$$

then the SON *condition holds, and in particular when $L_C^2(\bar{x} \mid w) = \emptyset$, the supremum in the* SON *condition is $+\infty$.*

Let $\bar{x} \in C$ and let $\phi = S^p$.

We shall give sufficient conditions in terms of the original data for the inclusion

$$L_C^2(\bar{x} \mid w) \subset \ker d^2 S^p(\bar{x})(w \mid \cdot) \qquad \forall w \in L_C(\bar{x}) \tag{4}$$

to hold, which is slightly stronger than (3) since in general $\ker d^2 S^p(\bar{x})(w \mid \cdot)$ is not a closed and convex set and $\mathcal{V}(\bar{x}) \subsetneq L_C(\bar{x})$.

Theorem 3.2 *(Meng and Yang (2015)) Let $\bar{x}$ be a local minimum of* (NLP). *Suppose that the l_p penalty function is exact at $\bar{x}$. If, in addition, one of the following conditions is satisfied:*

(i) $p \in (\frac{2}{3}, 1]$,

(ii) $p = \frac{2}{3}$ *and, for every $z \in L_C^2(\bar{x} \mid w)$, it follows that*

$$\begin{cases} \langle w, \nabla^2 g_i(\bar{x}) z \rangle + \dfrac{1}{3} g_i^{(3)}(\bar{x})(w, w, w) \leq 0 & \forall\, i \in I(\bar{x}, w, z), \\ \langle w, \nabla^2 h_j(\bar{x}) z \rangle + \dfrac{1}{3} h_j^{(3)}(\bar{x})(w, w, w) = 0 & \forall\, j \in J, \end{cases}$$

(iii) $p \in [0, \frac{2}{3})$, $q = 0$ *(i.e., there is no equality constraint) and, for every $z \in L_C^2(\bar{x} \mid w)$ with $(w, z) \neq 0$, it follows that*

$$\langle w, \nabla^2 g_i(\bar{x}) z \rangle + \frac{1}{3} g_i^{(3)}(\bar{x})(w, w, w) < 0 \qquad \forall\, i \in I(\bar{x}, w, z),$$

then (4) holds and so does the SON *condition.*

4. Interior-Point ℓ_p-Penalty Method

Consider the $l_p(p \leq 1)$ penalty problem of the following form with inequality constraints only

$$\min_{x \in R^n} \rho f(x) + \sum_{i \in I} (\max\{g_i(x), 0\})^p.$$

We introduce the following p-order relaxation constrained problem[a]:

$$\text{(RCP)} \quad \min_{x,s} \phi_p(x, s; \rho) := \rho f(x) + \sum_{i \in I} s_i$$

$$\text{s.t. } s_i \geq 0 \text{ and } s_i^{1/p} - g_i(x) \geq 0, \; i \in I.$$

- (RCP) shares the same differentiability as (NLP);
- It can be shown that the l_p penalty problem for (RCP) is always exact. For the case $p = 1$, see Curtis (2010).

[a]See Tian, Yang and Meng (2014), Interior-point $l_{1/2}$ penalty function method, *JIMO (to appear)*

The primal-dual interior-point method is used to solve the p-order relaxation problem (RCP), which is to solve a sequence of logarithmic barrier subproblems

$$\text{(LBCP)} \quad \min_{x,s} \rho f(x) + \sum_{i\in I} s_i - \mu \sum_{i\in I} \log s_i - \mu^{1/p} \sum_{i\in I} \log\big(s_i^{1/p} - g_i(x)\big)$$
$$\text{s.t. } s_i > 0 \text{ and } s_i^{1/p} - g_i(x) > 0, \; i \in I,$$

where $\mu > 0$ is the barrier parameter.

- Barrier parameter $\mu^{1/p}$ is set for the term $\sum_{i\in I} \log\big(s_i^{1/p} - g_i(x)\big)$, which provides better numerical results than μ and can be justified by the first-order conditions.

The first-order necessary conditions of the barrier subproblem (LBCP) are given as follows

$$\rho\nabla f(x) + A(x)y = 0, \tag{5a}$$

$$e - 1/pYs^{1/p-1} - u = 0, \tag{5b}$$

$$Y\left(s^{1/p} - c(x)\right) - \mu^{1/p}e = 0, \tag{5c}$$

$$Us - \mu e = 0. \tag{5d}$$

where $y, u \in \mathbb{R}^m$ are Lagrange multipliers, $Y := \text{diag}(y)$, $U := \text{diag}(u)$ and $A(x) := [\nabla g_1(x), \cdots, \nabla g_m(x)]$.

• Modified Newton method is used for finding a search direction, see Benson, Shanno and Vanderbei (2004).

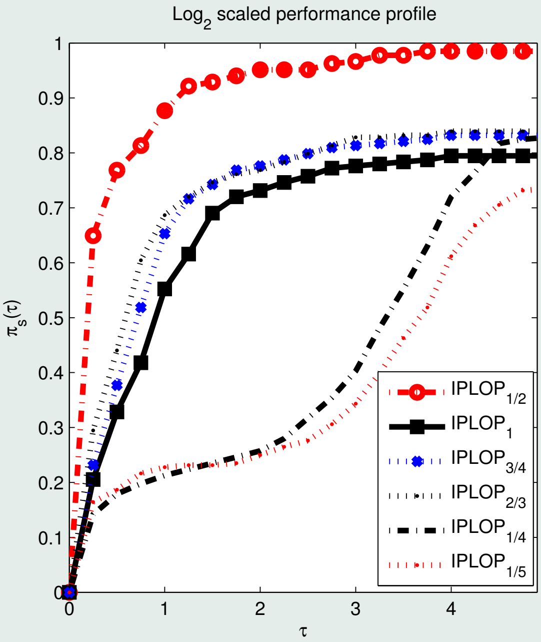
Numerical Experiments

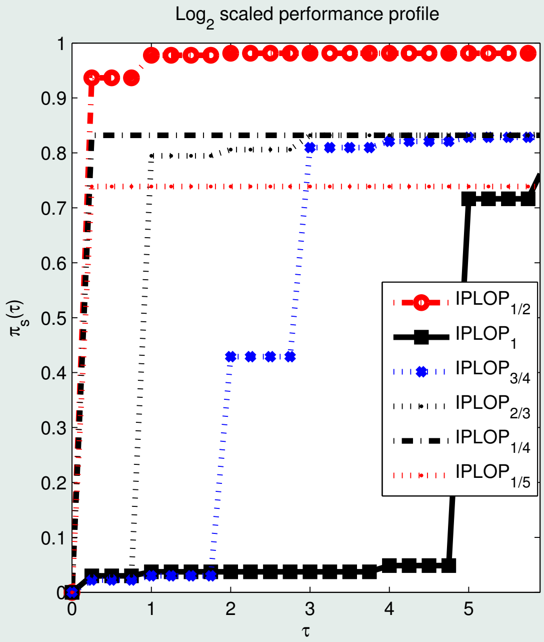
- We refer to our algorithm as IPLOP method, which stands for the Interior-Point Lower-Order Penalty method;
- We use 266 inequality constrained problems from the CUTEr collection, COPS, MITT and GLOBAL Library test sets as our test problems;
- The existing interior-point ℓ_1-penalty method (PIPAL-a and PIPAL-c methods in PIPAL1.0 developed by Curtis (2010)) is used to compare the performance with the proposed method;

Using the performance profiles of Dolan and Moré (2002), we plot the following figures. For example, the plots $\pi_s(\tau)$ in the left one denote the scaled performance profile

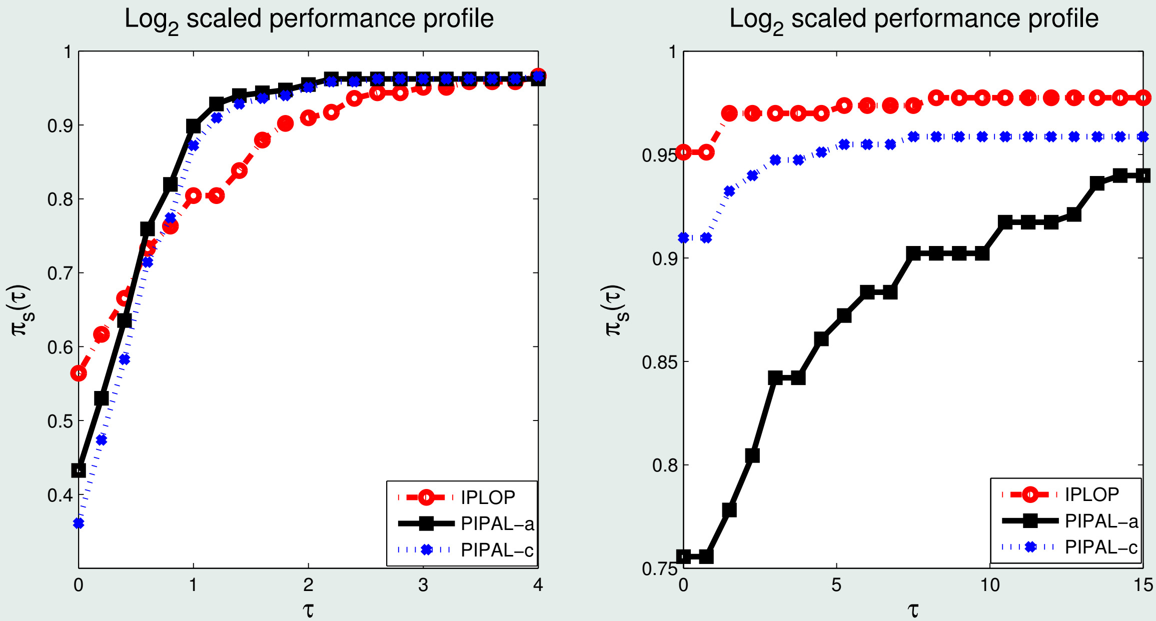
$$\pi_s(\tau) := \frac{\text{no. of problems } \hat{p} \text{ where } \log_2(r_{\hat{p},s}) \leq \tau}{\text{total no. of problems}}, \ \tau \geq 0,$$

where $\log_2(r_{\hat{p},s})$ is the scaled performance ratio between the iteration number to solve problem $\hat{p}$ by solver s over the fewest iteration number required by the solvers of the IPLOP method with different p. It is clear that $\pi_s(\tau)$ is the probability for solver s that a scaled performance ratio $\log_2(r_{\hat{p},s})$ is within a factor $\tau \geq 0$ of the best possible ratio.





- The left one is plotted by the the number of iterations;
- The right one is plotted by the values of $\frac{1}{\rho}$.



- The left one is plotted by the the number of iterations;
- The right one is plotted by the values of $\frac{1}{\rho}$.

5. Conclusions

In this talk, we partly answer the question as whether and how **optimality conditions** of NLPs can be derived from exactness of penalty functions.

- We define KKT-type penalty terms, and give their characterizations and some sufficient conditions.
- We derive the SON condition from exactness of penalty functions.
- We design an interior point l_p penalty method.

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