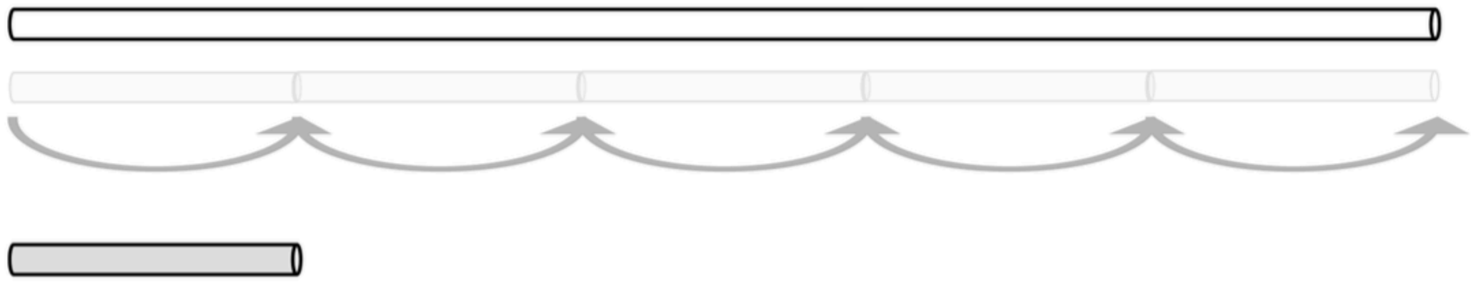

Designing for Tool Use in Mathematics Classrooms

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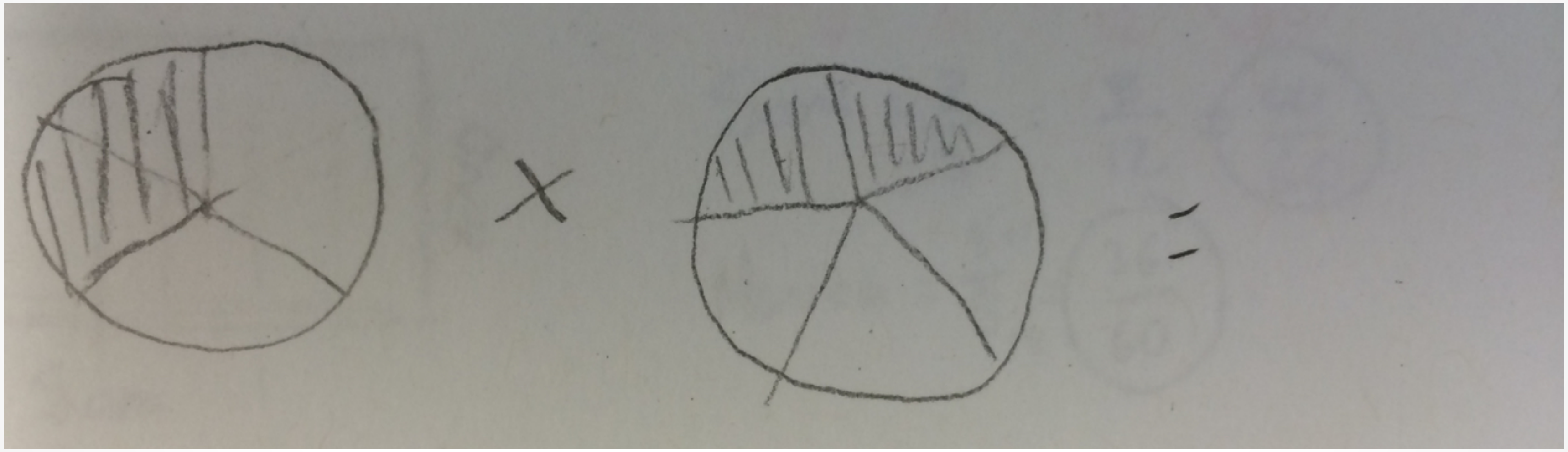
Background

- Draw a picture to explain $1/3 \times 2/5$

Background

- Draw a picture to explain 1/3 × 2/5



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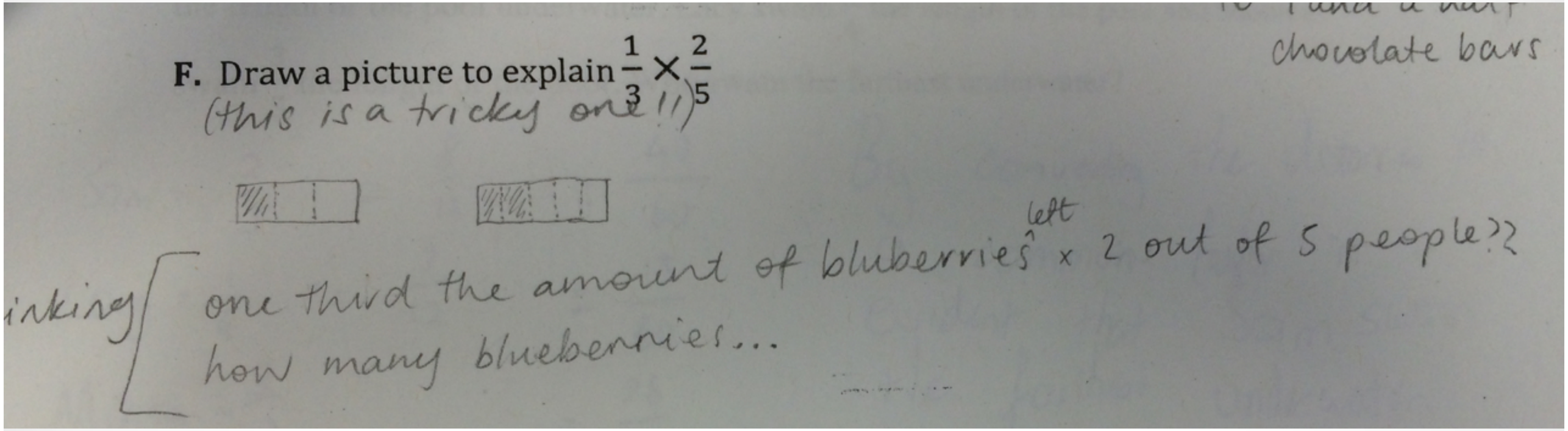
- Draw a picture to explain 1/3 × 2/5



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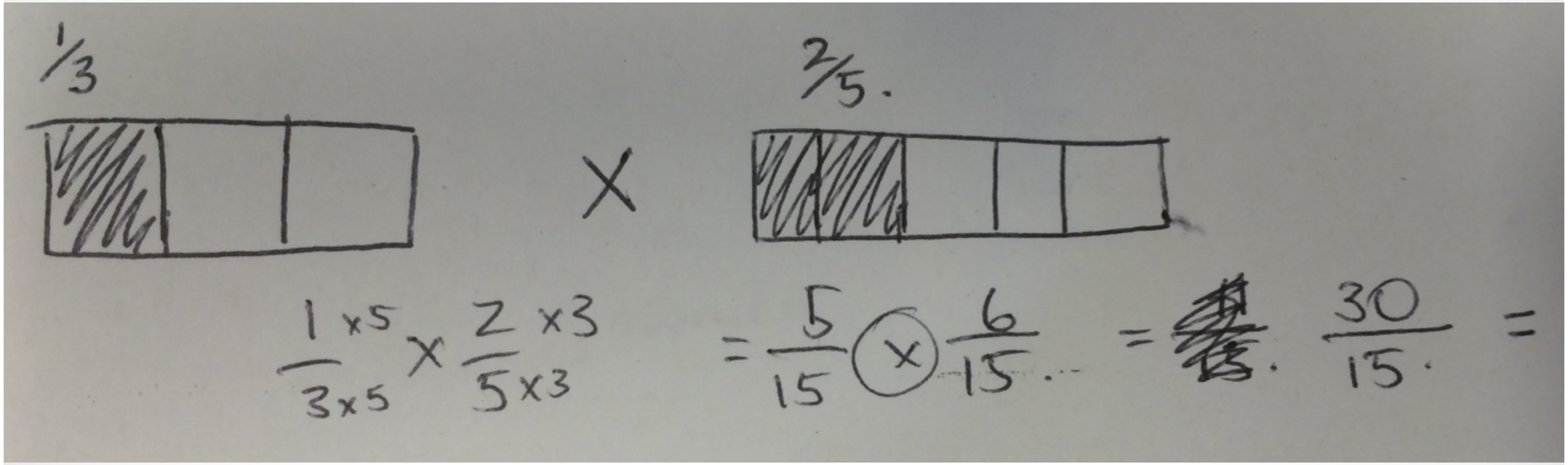

Background

- Write a problem scenario (word problem) that would lead to a division 2/3 ÷ 1/2

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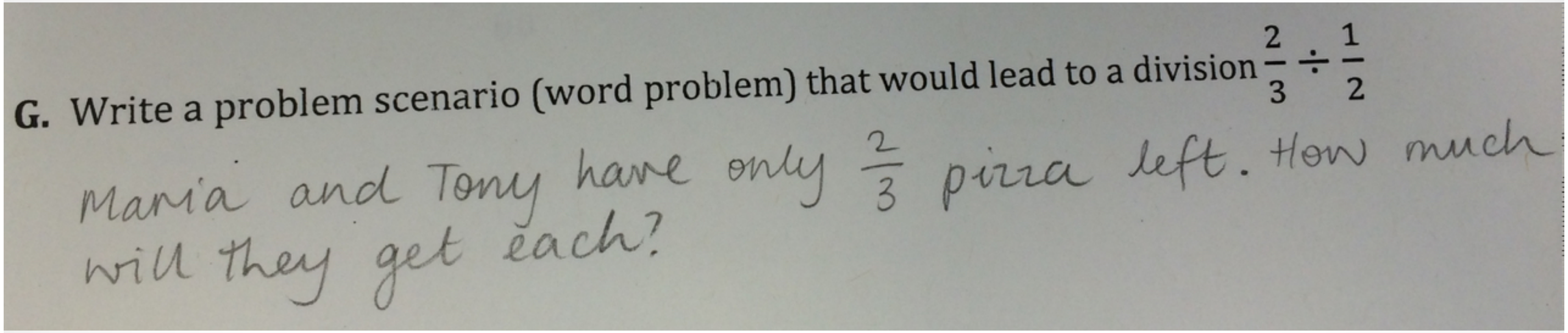


G. Write a problem scenario (word problem) that would lead to a division $\frac{2}{3} \div \frac{1}{2}$

Maria and Tony have only $\frac{2}{3}$ pizza left. How much will they get each?

Background

- Write a problem scenario (word problem) that would lead to a division 2/3 ÷ 1/2

Johny had $\frac{2}{3}$ of a cake. This is only $\frac{1}{2}$ of what his Mum ~~cooked~~ had left from baking on the weeked. how much cake was there before Johny took his share.

Background

- Draw a picture to explain 1/3 × 2/5
- Write a problem scenario (word problem) that would lead to a division 2/3 ÷ 1/2

- What's going on?

Background

All example student responses are based on

- **cutting and/or sharing food** situation

- Learning as situated

How we teach,

especially models and tools we use

shape learning

- What gets learned and what does not
- What becomes easy and what remains difficult

Background

All example student responses are based on

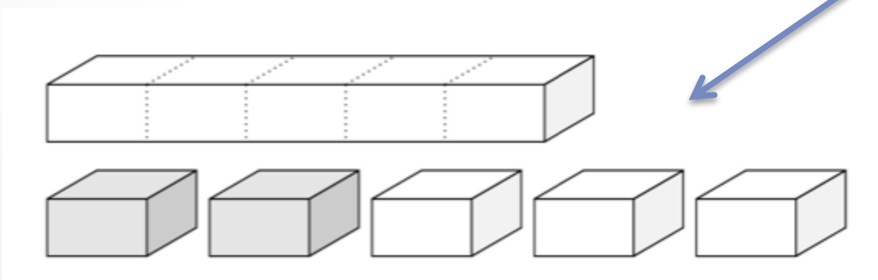
- **cutting and/or sharing food** situation
- (part-whole model of fraction)

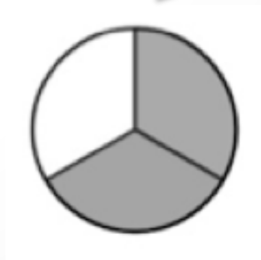
Equal partitioning models

Because:

that's how we teach fractions

- Is this really at the heart of the problem? *and if so*
- What *else* can we do?





Is this really at the heart of the problem?

Realistic Mathematic Education (e.g., Gravemeijer, 1994)

Instructional theory that orients our work

Choosing a viable starting point for instruction

- serve as paradigmatic cases in which to "anchor students' increasingly abstract mathematical activity" (Cobb, et al., 1997, p. 159)

3 fraction-related images likely to emerge as a result of equal partitioning approach

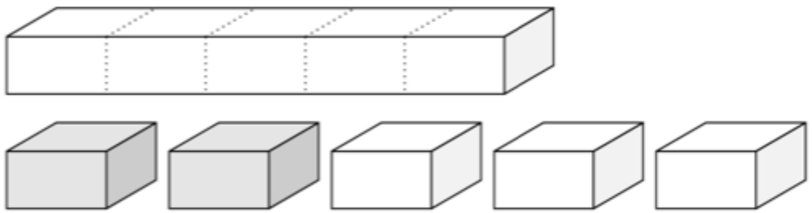
and

These make it particularly challenging to develop a comprehensive understanding of rational numbers

Is this really at the heart of the problem?

What students do:

- Acting on objects – “fractions are things”



Fraction as fracturer (Freudenthal, 1983)

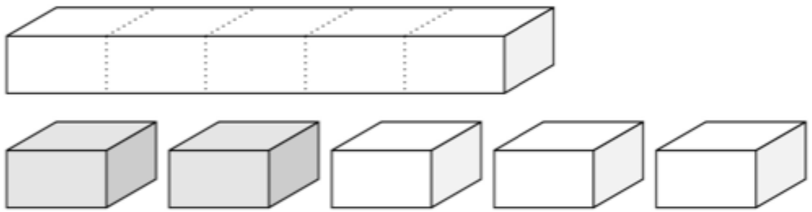
It is tempting to associate fractions with situations that involve the physical and irreversible transformation of an object

Reciprocal relations not in the picture

Is this really at the heart of the problem?

What students do:

- So many out of so many



  - *Additive* reasoning (**2 out of 5** vs. 2/5ths as large as)
  - Leads to ***proper fractions only***

- Fractions included in a whole
  - ✓ "The number of boys is what fraction of the number of children?"
  - ✗ "The number of boys is what fraction of the number of girls?"

(Thompson & Saldanha, 2003)

What else can we do?

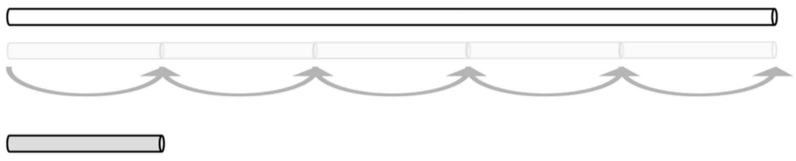
Fraction as comparer (Freudenthal, 1983)

- Main referent is not objects, but magnitude values (e.g., lengths, volumes, masses)
- Derived from *measurement*, instead of *partitive division*
- Unit fractions: numbers that account for the size of a magnitude value, relative to that of a magnitude value of reference:

  A is 1/n as large as B when B is n times as large as A

What else can we do?

Fraction as comparer (Freudenthal, 1983)



UNIT FRACTION

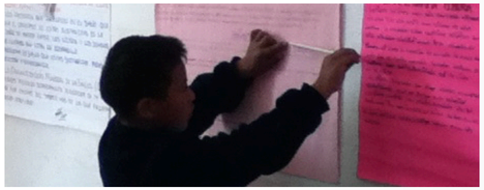
- not included in the reference unit
- size determined by how many iterations it takes to measure the reference unit
- can be iterated & the accumulated length is in relation to the length of the reference unit
- Symbol system progressively developed from kids' activity

Teaching fractions as measures

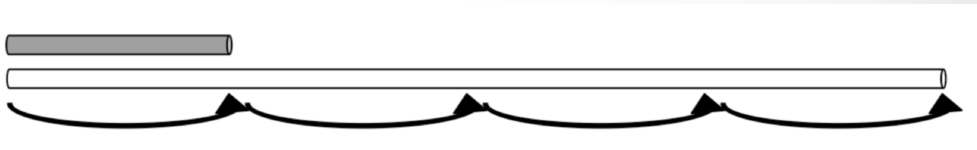
- Design research (classroom design experiments)

Fractions as measures

- sequence
  - o Measuring with body parts (hand length)
    - Need to standardize unit of measure
  - o Standard unit of measure: “the stick”
    - Need to account for remainders



  - o Producing subunits of measure
    - unit fraction lengths: *smalls*
    - construed as divisiors in measurement division: reference unit divided whole number of times with no reminder



Stick length ÷ ? length = 4

- (Stick thing ÷ 4 equal parts = ?, quotient in a partitive division)
-

Fractions as measures

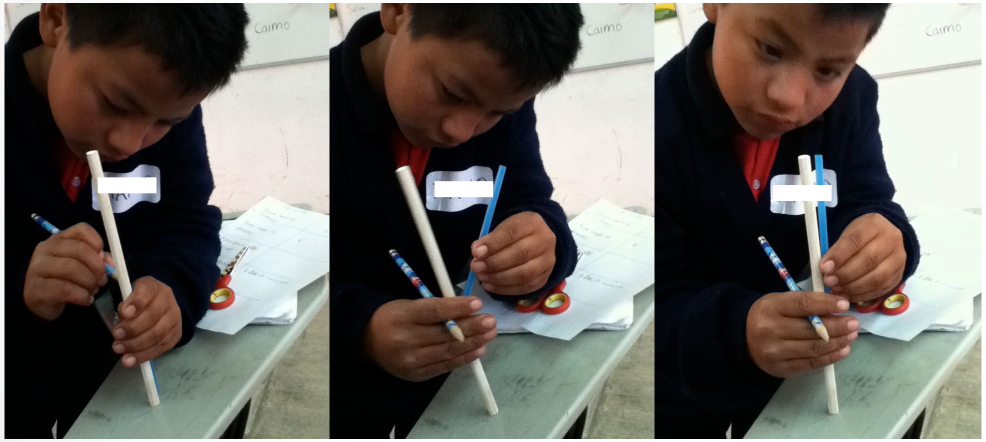
- Sequence

This is roughly where my presentation stopped
 - if you want a glimpse of what we managed to do with kids in classrooms so far, read on

I included
- a sketchy overview of what we did,
- mathematical practices that were established in the classroom,
- some typical examples of ways students reasoned
- indications from exploring of where we plan to take this next
- References and our papers from this work

Fractions as measures

- sequence
  - Producing subunits of measure
    - 1/2, 1/3, 1/4, …, 1/10



First mathematical practice in the classroom

reasoning about the inverse order relation of unit fractions

- The More Times It Fits, the Smaller It Has To Be

Fractions as measures

- sequence
  - Common fraction: iteration of a subunit certain number of times
    - the fraction 7/4 would account for a length that corresponded to

      7 iterations of a subunit of length 1/4

Denominator - the length of a *small*, relative to the length of the reference unit

Numerator - a number that accounted for how many iterations of the length of the *small* accumulated into the length represented by the fraction

Fractions as measures

- sequence

  - Equivalences with the reference unit (the stick)
    - $4 \times 1/4 = 1$ (was not obvious & required support)

    Once established:
    - Compare any fraction with 1
    - Use 1 as a benchmark in comparisons
    - Improper ←→ mixed fractions

Quantitative: the relative size of a fraction representing a length that was enough, or not enough, to "fill" (cover) the length of the reference unit (see some examples of reasoning below)

Second mathematical practice

Reasoning about Fraction Comparisons

99/100 and 5/5 (translated from Spanish)

Marisol: I think that 5 smalls of five is bigger because 99 smalls of one hundred is smaller because it is not enough to fill the stick.

Teacher: It is not enough to fill the stick. Carlos?

Carlos: 99 smalls of one hundred is not going to be enough to fill the stick
because it is missing one small for it to be 100 smalls of one hundred,
and 5 (smalls) of five do fill the stick.

Second mathematical practice

Reasoning about Fraction Comparisons

12/13 and 6/5

> **Eduardo:** Because you need 13 smalls of thirteen to fill the stick, and with 12 it's not enough. And in the other you need 5, but they are 6 and it even goes further.

- The iteration of a small of five (1/5) more than five times did not become a troublesome issue for any of the students
- Kids construed the entities that unit fractions quantify as being *separate* from the reference unit and, thus, **could be iterated *unrestrictedly***

Fractions as measures

- Symbol system

Need to distinguish between '4 sticks' and 'small of four':

So4 or **S4** each meaning "small of four"

↓

$\boxed{4}$ meaning "small of four"

$\boxed{4}\ \boxed{4}\ \boxed{4}$ meaning "three smalls of four" - additive, intuitive, kids were later asked to work with 17 smalls of 4, so that they recognise this as no longer practical

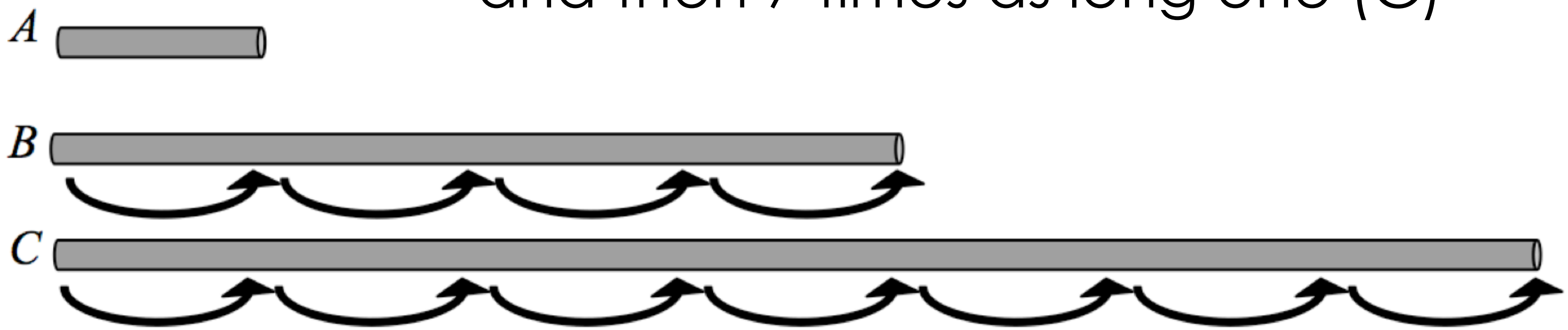
$$\boxed{4}\ \boxed{4}\ \boxed{4} \rightarrow 3\boxed{4} \rightarrow \begin{matrix}3\\ \boxed{4}\end{matrix} \rightarrow \frac{3}{4}$$

Fractions as measures

- Continuing the development of the sequence involves trials of ideas in 1-on-1 setting
- Analysis of Pedro's reasoning
- Pedro
  - Year 5 student, urban public school in Mexico
  - Weekly 1h sessions
  - Sessions 8-20: *Fractions as Measures* sequence (above)
    - o This took a while
  - Analysis of Pedro's reasoning in session 21

Reasoning about Reciprocal Relation

Pedro was asked to create a straw as long as his little finger (A), then 4 times as long one (B), and then 7 times as long one (C)



Reasoning about Reciprocal Relation: *B* is the stick

P: *B* will be one so *C* will be two (chuckling)?

T: Let's focus on *A* first.

P: *A* will be two.

T: Let's see, why two?

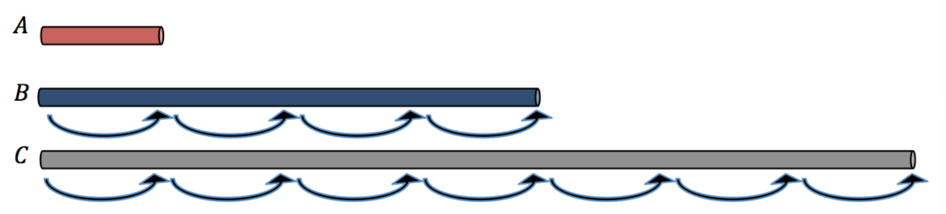
P: No, *A* is going to be four (showing four fingers).

T: And what is bigger, one or four?

P: Four.

T: So is this longer than this (placing straw *A* next to straw *B*).

P: No. (Pause). Then it would be smaller? No? (Looking at the teacher).



Reasoning about Reciprocal Relation: *B* is the stick

T: Let's see. If this is your stick (pointing at straw *B*), what is this (holding straw *A*)?

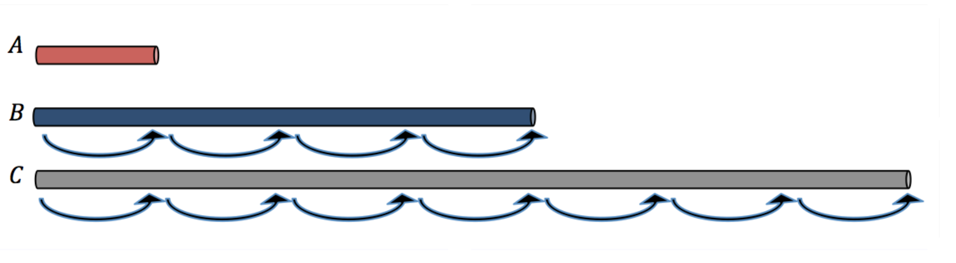
P: A fourth.

T: Ok. Why a fourth?

P: Because *B* is divided into four (gesturing with his hand along straw *B*), and since I have a fourth, then it is one, two, three, four (taking straw *A* and iterating it along straw *B* as he counted).

T: Ok, write it in the table (Pedro's record of reciprocal comparisons).

A	1	1/4		
B	4	1		
C	7			



Reasoning about Reciprocal Relation: *B* is the stick

P: Then, C would be bigger than *B* (looking at straws *B* and C)

T: Ok.

P: Four (likely meaning the length of straw *B*), they would be (touching straw C, closing his eyes and pausing to think) a seventh?

T: A seventh? What is bigger, a whole (referring to straw B) or a seventh (referring to straw C)?

P: A whole.

T: So this one (touching straw *B*) is longer than this one (touching straw C)?

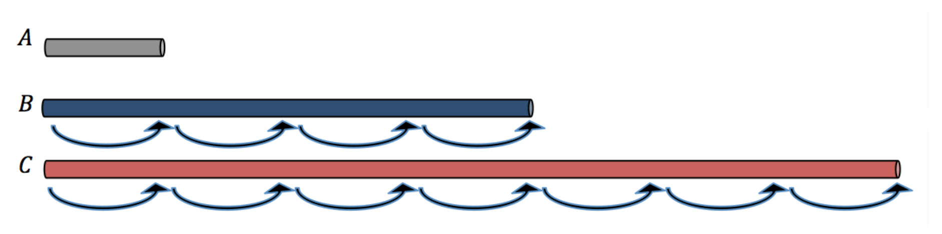
P: Oh, no.

T: So how can C be a seventh of *B*?

P: Oh no. Then it would be one whole (closing one eye and pausing to think) three (short pause) fourths?

T: Why?

P: Because there are four here (gesturing with his hand along straw *B*), and there are four here (gesturing with his hand in the same way along part of straw C),



Reasoning about Reciprocal Relation: *B* is the stick

P: Then, C would be bigger than *B* (looking at straws *B* and C)

T: Ok.

P: Four (likely meaning the length of straw *B*), they would be (touching straw C, closing his eyes and pausing to think) a seventh?

T: A seventh? What is bigger, a whole (referring to straw B) or a seventh (referring to straw C)?

P: A whole.

T: So this one (touching straw *B*) is longer than this one (touching straw C)?

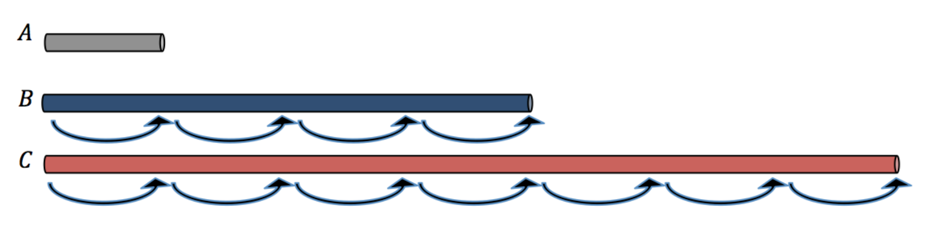
P: Oh, no.

T: So how can C be a seventh of *B*?

P: Oh no. Then it would be one whole (closing one eye and pausing to think) three (short pause) fourths?

T: Why?

P: Because there are four here (gesturing with his hand along straw *B*), and there are four here (gesturing with his hand in the same way along part of straw C), but there are three more here (pointing at the rest of the length of straw C), so a whole has been formed, with three fourths (added to it).



Reasoning about Reciprocal Relation: C is the stick

P: Then, if C is one, it (meaning *A*) would be (pause) one seventh?

T: Are you just guessing?

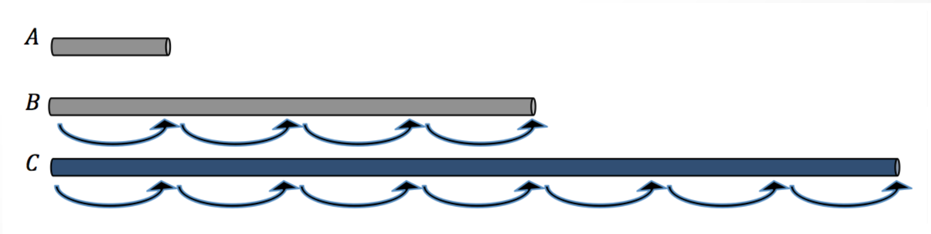
P: No. That one would actually be a seventh (pointing at *A*).

T: A seventh, why?

P: Because it fits seven times in C. And *B* would have four sevenths.

T: Ok. Why?

P: Because here (aligning straws *B* and *C*) if you measure it (meaning "with *A*"), there are four here (touching the *B* straw) and seven here (touching the *C* straw). But if you join them, there are four sevenths here (touching the *B* straw). So it is four sevenths.



Reasoning about Reciprocal Relation

- compared the lengths of three other straws without physically creating them
  - (1, 3, and 10)
- established that his age (10 years) was 10/13 of his sister's, and his sister's age was his plus 3/10 of his age
- determined
  - the fraction of the student population of his school, in his classroom (32/407), and
  - the size of the school population relative to the number of students in his classroom (407/32)

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Our papers

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