Tools and mathematics – tools matter

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- Mathematical and educational issues in doing a task with different tools
- Paper folding
- Jonathan M Borwein's work and tool use

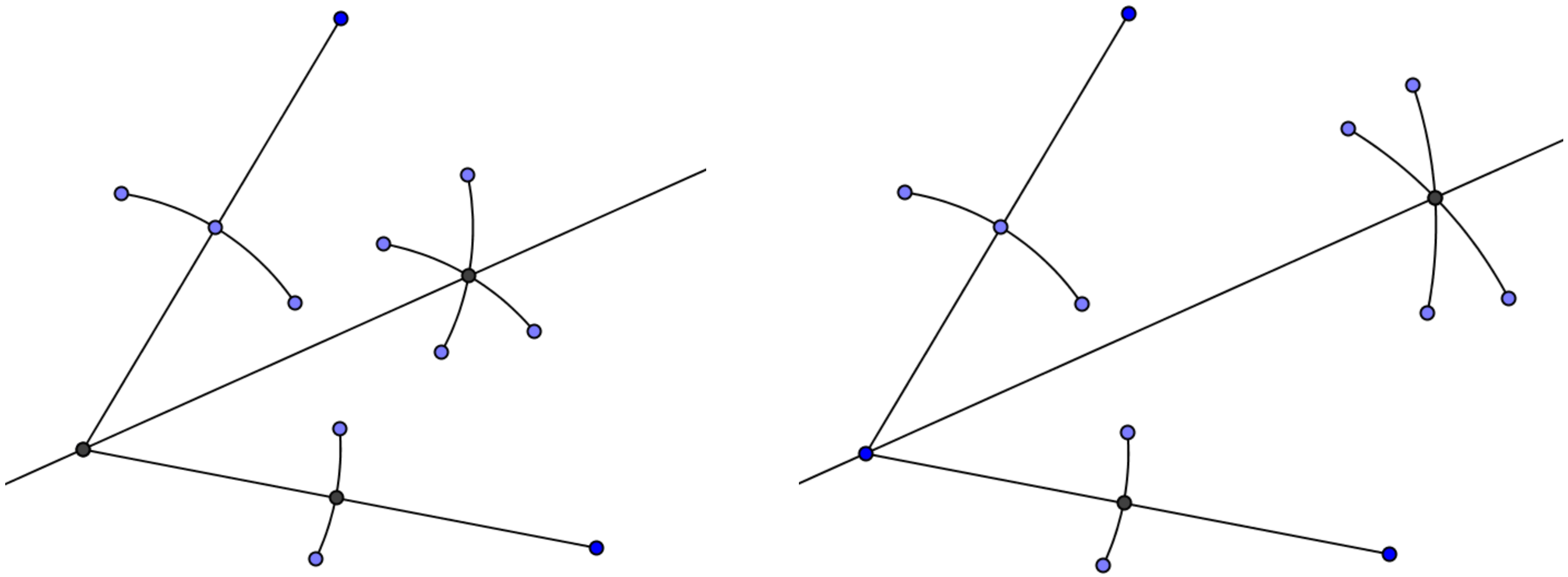
One task, four tools

Bisecting an angle using four tools

- Straight edge and compass
- Protractor
- Dynamic geometry system
- A book

Straight edge and compass

- A compass … mathematical beauty … it encapsulates the equidistant relationship between the centre of a circle and points on the circumference.
- But circles are not essential in bisecting an angle with a compass … equidistant line segments are.
- The straight edge also has a built-in mathematical feature, it is linear.



Why does the construction work?

Comments on maths, tools, actions and thought in this construction

Regarding tools:

(i) 2 (or 3 –pencil) physical tools in this construction

- Mohr-Mascheroni proved that any construction made using a compass and straight edge can be constructed using a compass alone
- Tools are rarely used in isolation

(ii) Neither tool was designed to tackle the task … but tool design is important

(iii) The user needs to know how to use the tools … these tools require quite advanced motor control

(iv) The user also needs to have an intention to use tools for particularly ends.

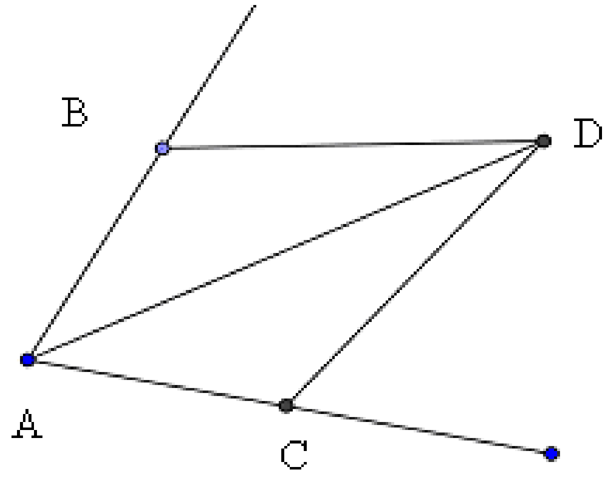
Related to this, the mind and the tools need to be co-ordinated.

Educational aspects of maths in this task.

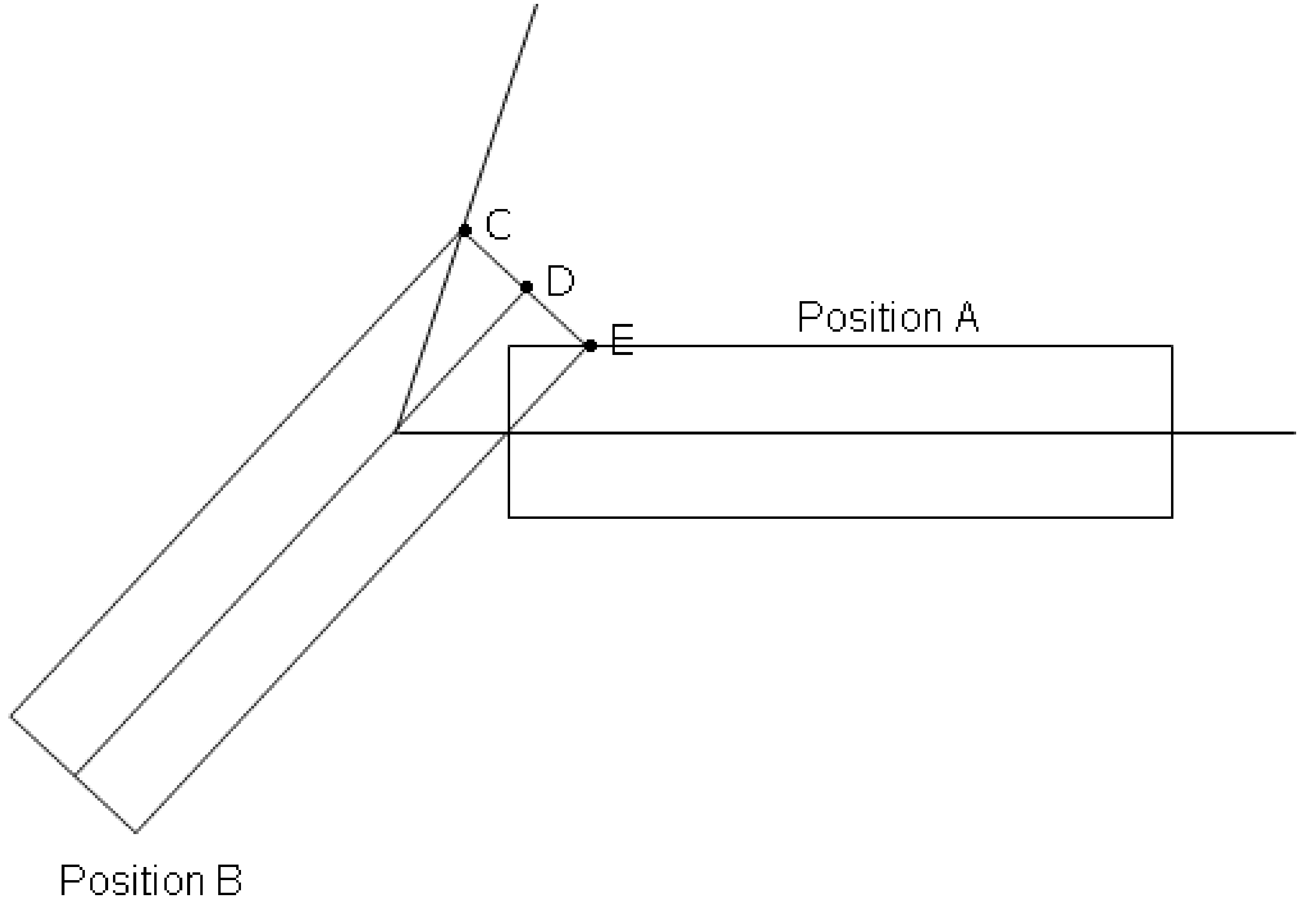
(i) The task is ‘isolated’. Why do we want to bisect an angle?

(ii) The reason why this construction bisects the angle is not clear to most people.

(iii) The mathematical relationships which the compass makes explicit and those essential for the proof are different.

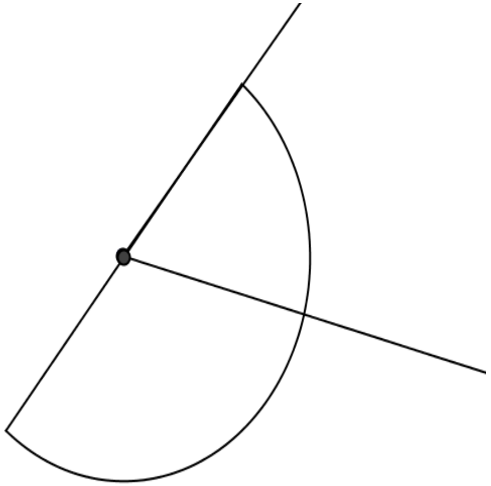


A diversion, trisecting an angle



Protractor

Generally circular in shape but, unlike the compass, it doesn't use circle properties



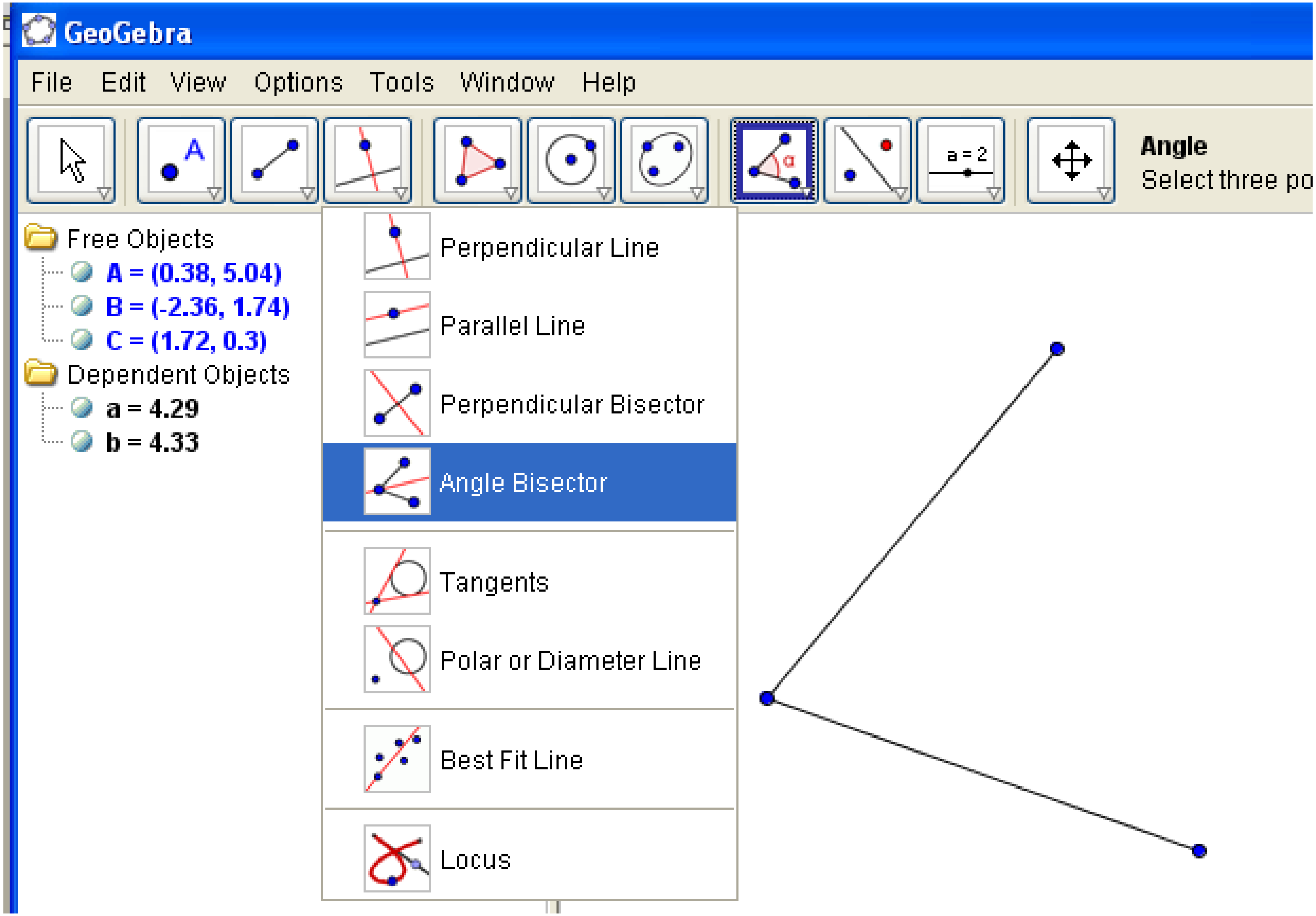
As before, the user needs to:

- perform physical actions to enact the task
- know how to use the tool
- have an intention to use it for a purpose
- and mind-hand co-ordination

Educational aspects of maths in this task

- The incorporation of arithmetic into a geometric task … which is due to the tool
- Reading a scale brings makes the solution an approximation … with a compass too but in principle it is ‘ideal’ … aesthetics!
- No problem in trisecting or n-secting an angle
- Transparency of the solution

A dynamic geometry system



- It is possible to co-ordinate the use of some DGSs with other digital tools but, in educational use, a DGS is generally used as a self contained system
- The G in DGS is a surface feature, the system is digital
- The way *GeoGebra* bisects the angle is not transparent to the user (a 'black box' … Jon B 'white box' … "under the hood")

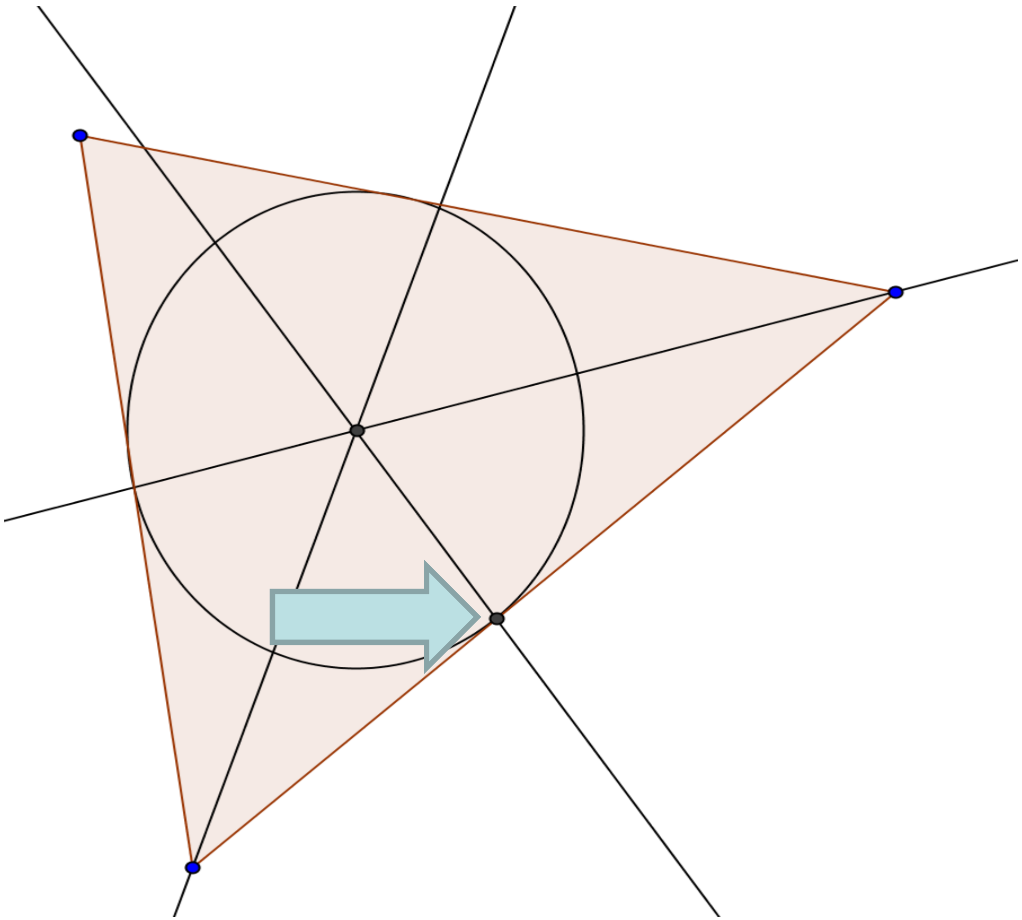
Another transparency issue - the teacher can choose what functions are available for the student or not (e.g. hide the angle-bisector command) … but students can be very inventive and students may measure the angle, draw an approximate bisecting line and then move the line until the second angle measures half the first one.

What educational value in this enactment of the task? … lots of clicking … but …

- Potential epistemic value (as opposed to pragmatic value) through ‘dragging’ …
- If this task is a sub-task of a larger task (e.g. constructing the inscribed circle to a triangle), the speed and accuracy with which angle bisectors can be constructed aid the user in not getting bogged down in a sub-task of a larger task

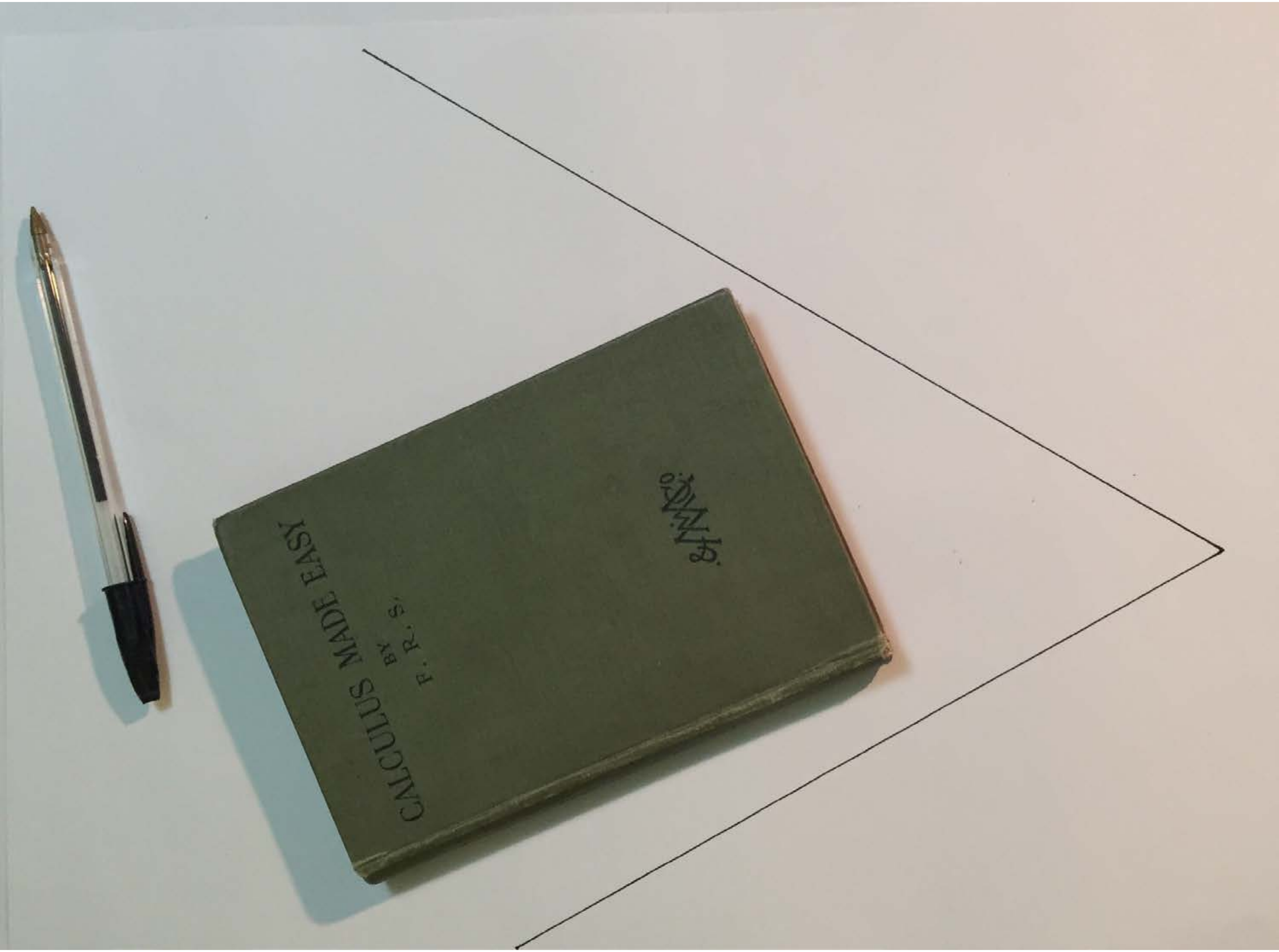
Also …

More maths in the DGS enacted larger task?



Realising epistemic values requires linking tool-actions with mathematical relationships

A book … not by imitating a compass!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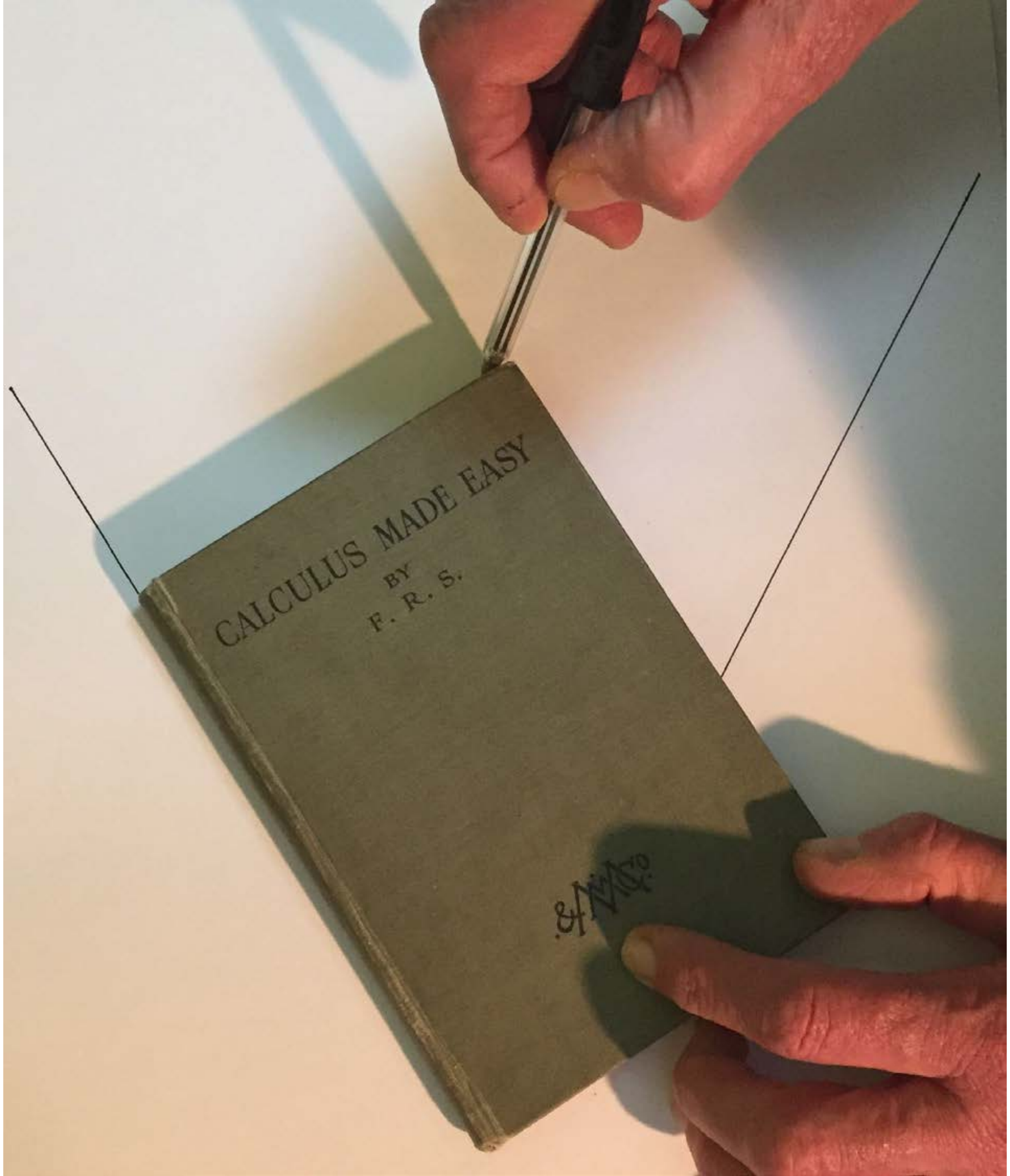
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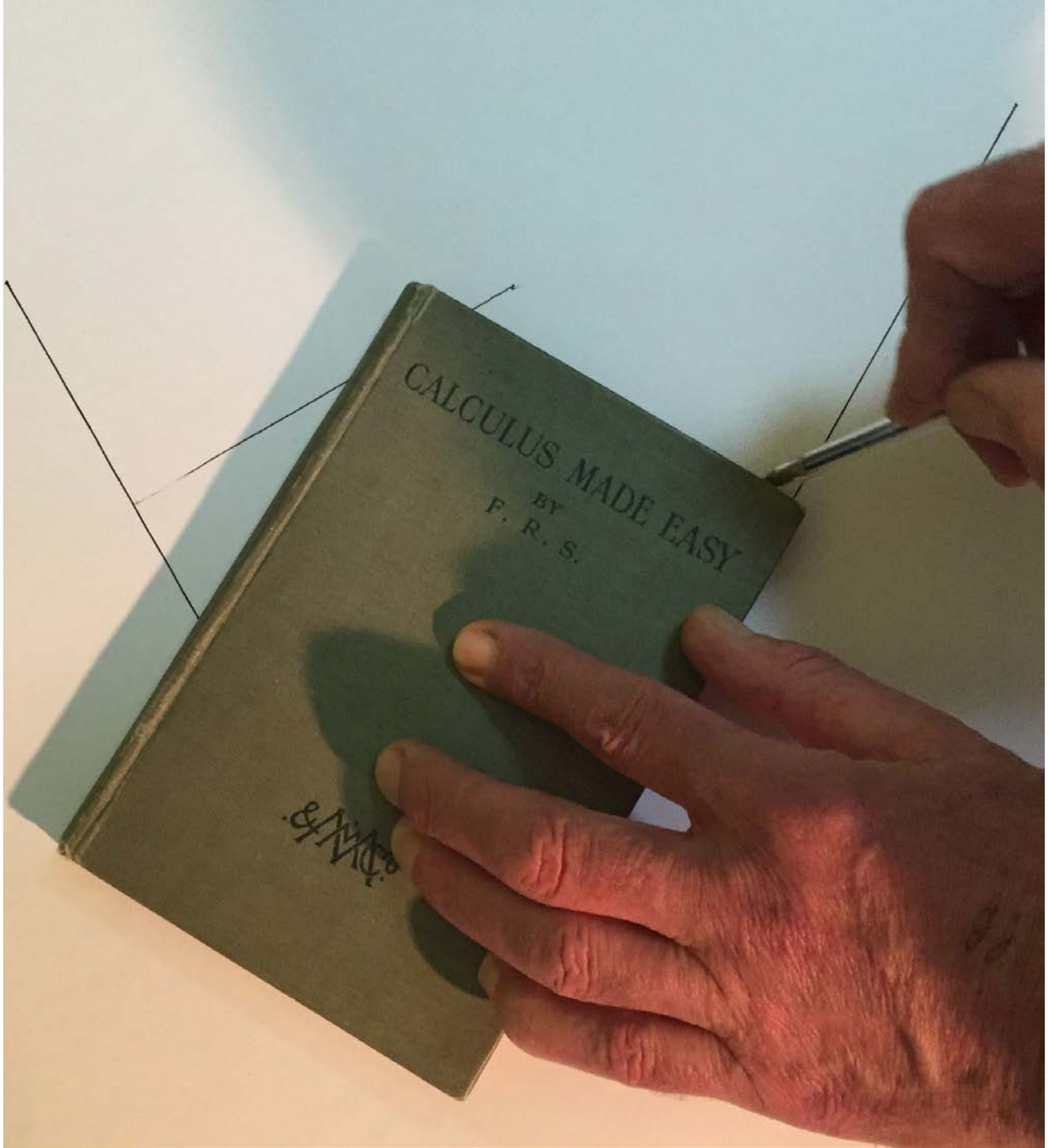
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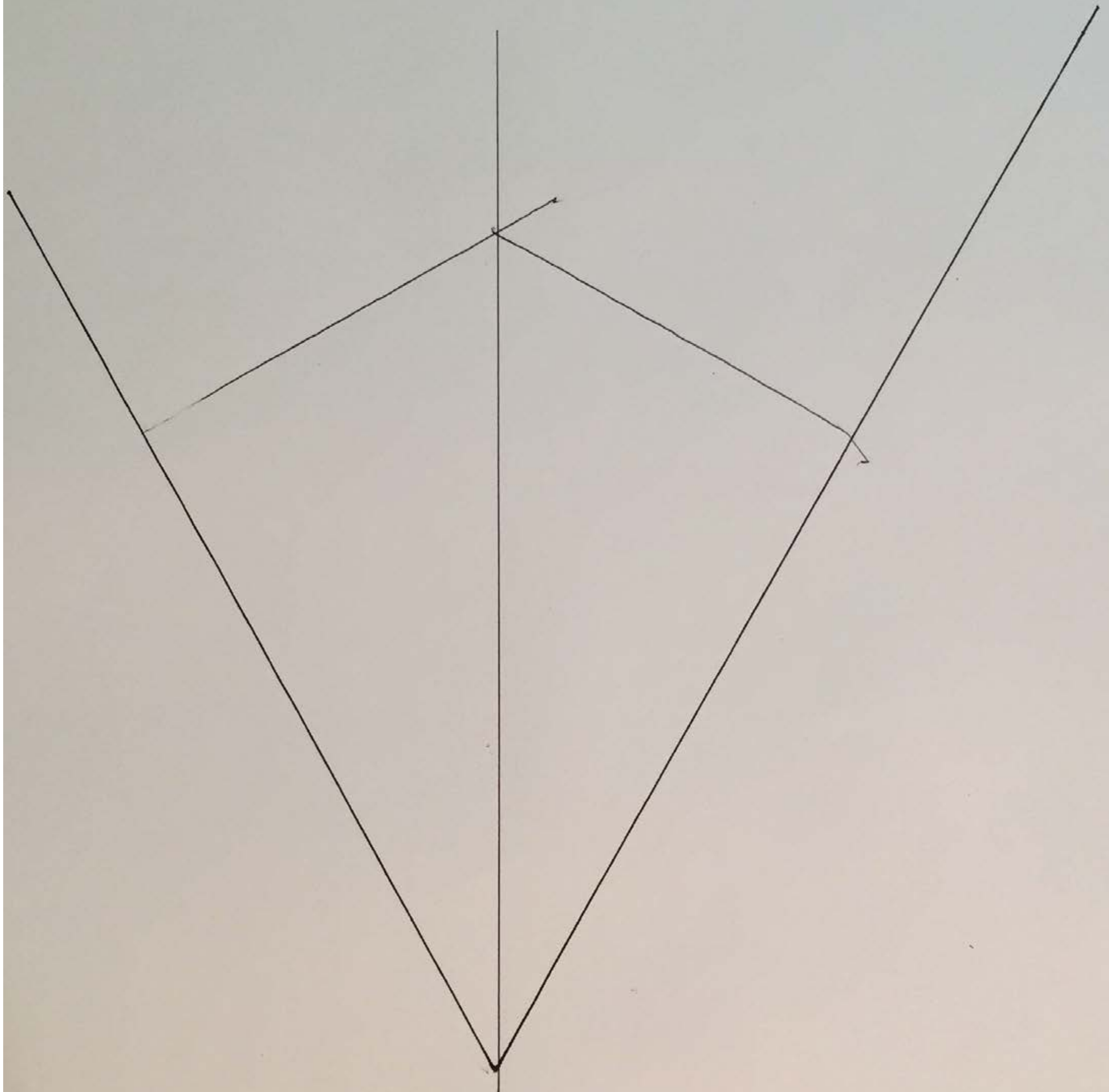


Do users see how the construction works?

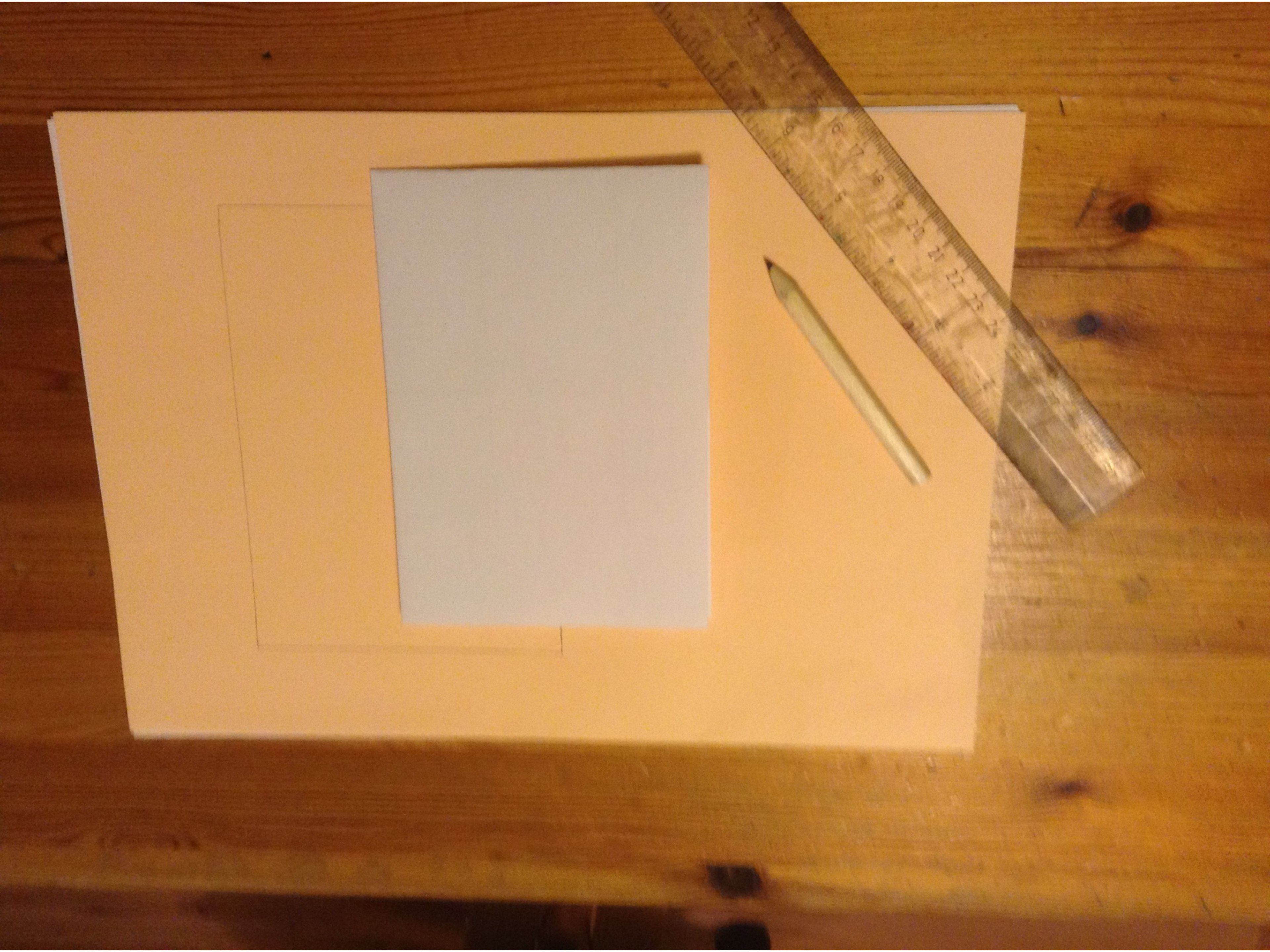
In my experience … generally ‘yes’

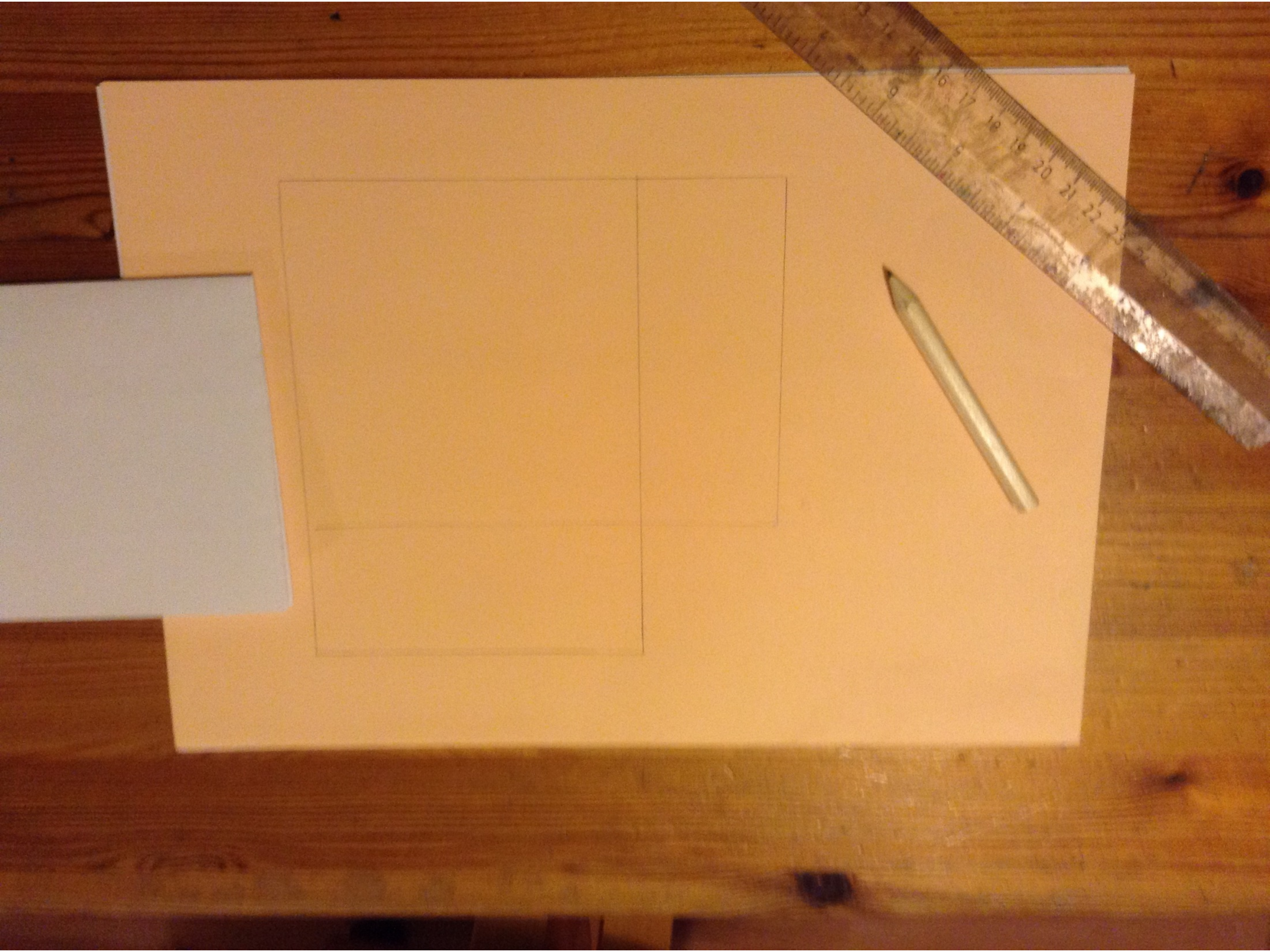
Paper folding (1)

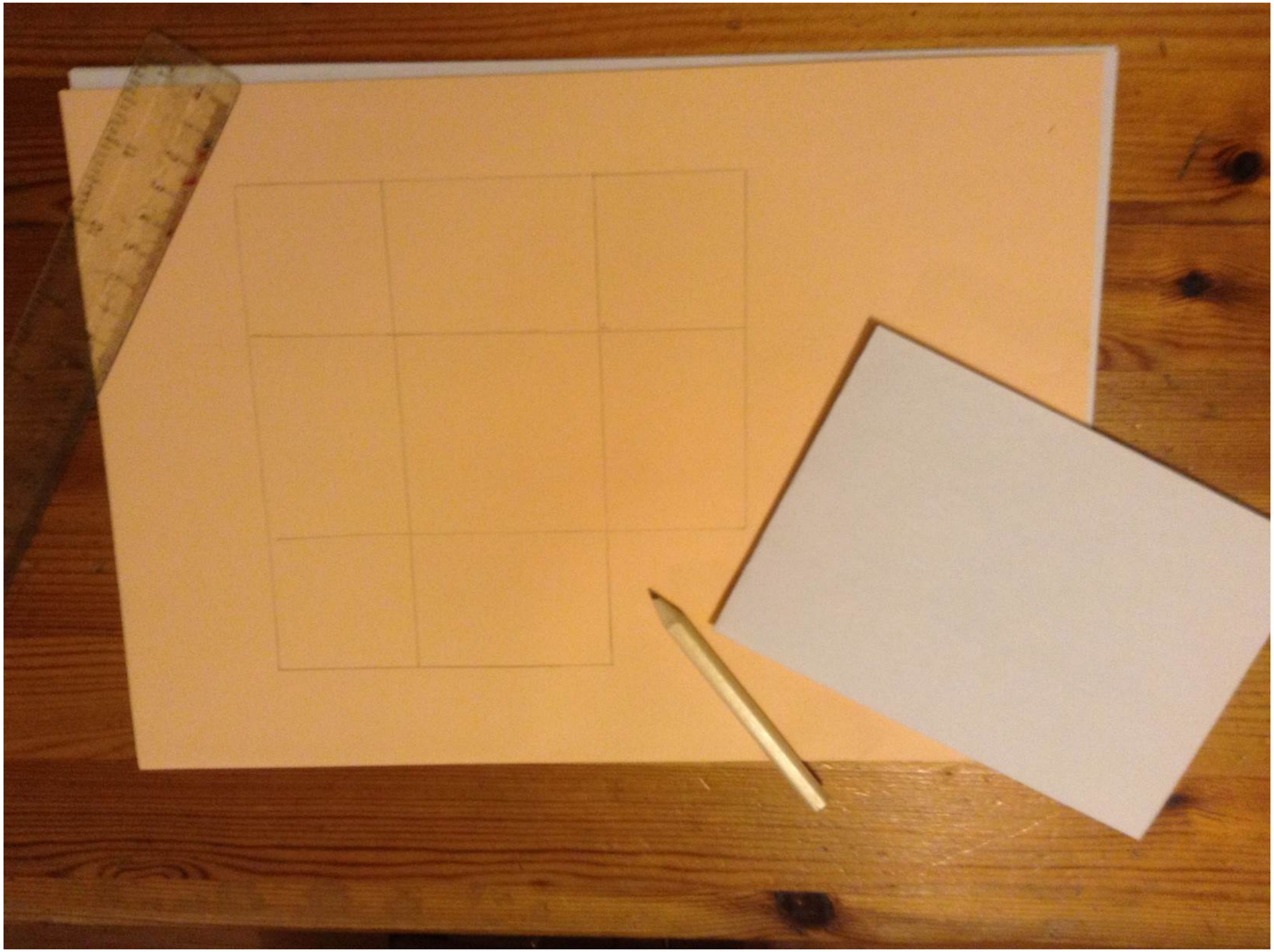
The ‘magic’ of A-sized paper

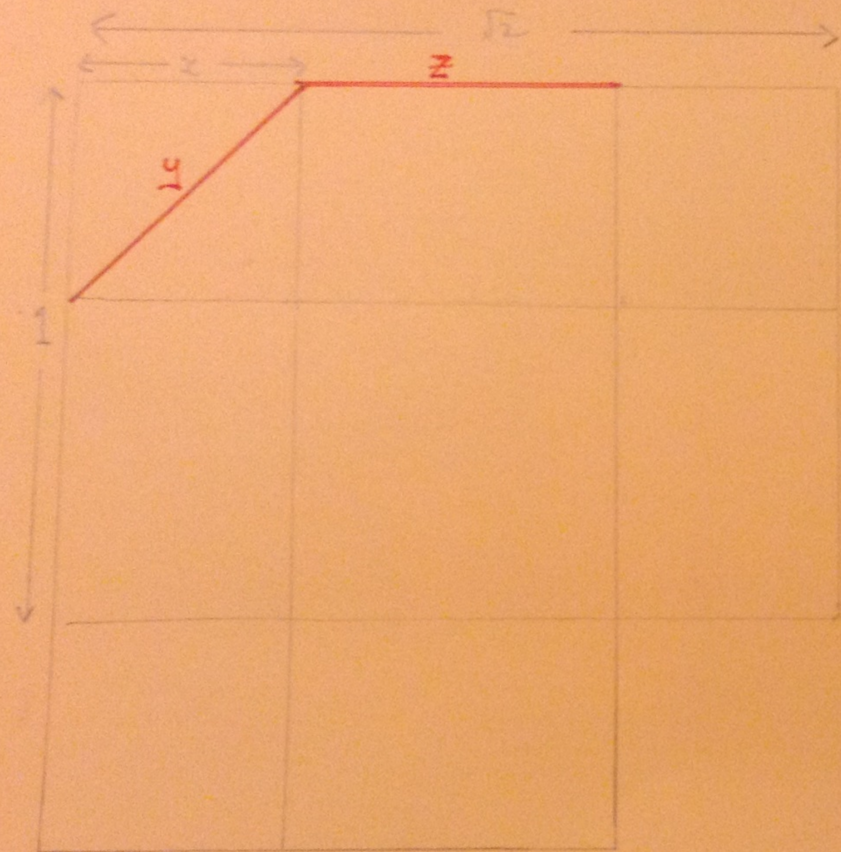
(ratio of their sides is $2:\sqrt{2}$)

There are constructions (e.g. an octogan) which cannot be done with any old book … but an octogan can be constructed with an A-sized book





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Paper folding (2)

Bisecting an angle by folding paper

Jonathan M Borwein

I'd read some of Jon's non-maths research writings before meeting him

I met Jon at the ICMI proof conference in Taiwan in 2009. His presentation was

Digitally-Assisted Discovery and Proof

My thoughts "This guy has a lot of interesting ideas"

Luc and I drafted the structure of the book in 2012 and we wanted input from a research mathematician … partly to keep our ideas in check

I suggested Jon to Luc. Luc agreed and I approached Jon

3 years, 10 skypes, 100s of e-mails and many comments on sketches and drafts later …… and we had the tool book

It was nice to find I had personal things in common with Jon … Canada … an interest in evolution … politics

Jon's chapter in the tool book was important in itself … and to show that tools matter to a research mathematician

But the e-dialogue between Jon, Luc and me was also very important …

... as I wrote in

Kortenkamp, U., Monaghan, J., & Trouche, L. (2016). Jonathan M Borwein (1951–2016): exploring, experiencing and experimenting in mathematics–an inspiring journey in mathematics. *Educational Studies in Mathematics*,*93*(2),131-136

I will write on Jon and the culture of mathematics as this is part of his legacy and it is important and relevant to ESM readers. This is a topic that grew in my mind as we wrote ‘the tool book’ and I learnt much from Jon on this by working with him on the book.

Jon's 8 uses of computers

1. Gaining insight and *intuition*
2. *Discovering* new relationships
3. *Visualizing* math principles
4. *Testing* & especially *falsifying* conjectures
5. *Exploring* a result to see if it *merits* proof
6. *Suggesting* approaches for formal proof
7. *Computing* replacing lengthy hand derivations
8. *Confirming* analytically derived results

“All of these uses play a central role in my daily research life”

All of these uses could be school maths uses. And Jon had much to say on school maths, e.g. his comment on 7:

“My preference on tests, rather than banning calculators or computers, is to adapt the questions to make them computationally aware “

But Jon agreed that we must be careful … the uses of computers in research maths do not automatically transfer to school maths … Luc will have more to say on this in his presentation

Jon, Luc and I not alone in saying tools matter:

Bosch, M., & Chevallard, Y. (1999). La sensibilité de l’activité mathématique aux ostensifs. Objet d’étude et problématique. *Recherches en didactique des mathématiques*, *19*(1), 79-124.

Western culture establishes … a structural opposition between activities considered to be ‘manual’ and activities considered to be ‘intellectual’ … prioritises activities of the mind over the work of the hand …

It goes without saying that … ‘mathematics’ is considered to be of the first type of activities, that is, working ‘with the head’ with notional tools, reasoning, ideas, insights and very little material elements. ….

… In fact, the few material instruments used in school mathematics are generally regarded as simple ‘aids’, sometimes as an indispensable aid but not actually a part of the activity itself.

But Jon, Luc and I wrote a book that documents how much tools matter to mathematics