Phase Plotting for Hyperbolic Geometry

Scott B. Lindstrom and Paul Vrbik
CARMA
University of Newcastle

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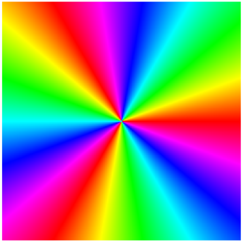
https://carma.newcastle.edu.au/scott/

Outline I

The Basics

- Phase plotting is a way of visualizing complex functions $f : \mathbb{C} \to \mathbb{C}$.
- Where $f(r_1 e^{i\theta_1}) = r_2 e^{i\theta_2}$, we plot the domain space, coloring points according to argument of image θ_2
- Top right: $z \to z$. Bottom right: $z \to z^3$.



Some Examples

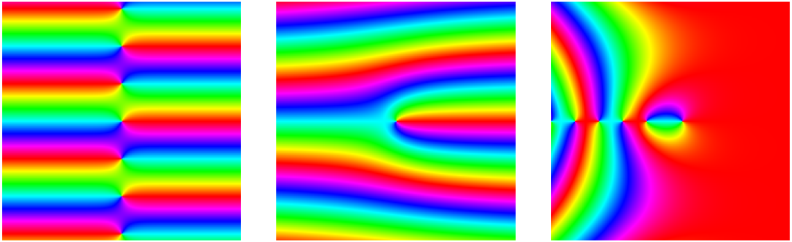


Figure: Left to right: $sinh(z), z \cdot e^z, \zeta(z)$.

Recapturing the Modulus

- We can also plot in 3d to recapture the modulus information.
- Let $f(r_1 e^{i\theta_1}) = r_2 e^{i\theta_2}$
- Again we plot over the domain space, coloring points according to argument of image θ_2
- We also give them vertical height corresponding to their modulus r_2.

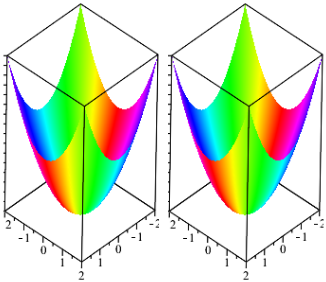


Figure: Phase plots with modulus included. Left: $z \to z$, Right: $z \to z^2$.

History

- Phase plotting is a relatively new tool.
- Recent attention
  - Elias Wegert's "Visual Complex Functions" published in 2013 [2]
  - "Complex Beauties" annual calendar (of which Jonathan Borwein was quite fond) [3]
- Wegert's Matlab code is available for download on his site.

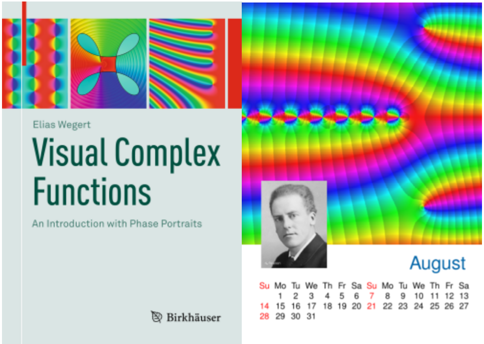


Figure: Left: Elias Wegert's "Visual Complex Functions." Right: Right: "Moment function of a 4-step Pplanar random walk" by Jonathan M. Borwein and Armin Straub from 2016 Complex Beauties calendar.

Differential Geometry

- Conformal Mappings are mappings which preserve the angles at which lines meet (and signs thereof)
- Direct Motions are mappings such that the distance between points is equal to the distance between their images.
- Parallel axiom: for a line L and point p there exists exactly one line through p which doesn't intersect L.
- Geometries which do not obey the parallel axiom:
    - Spherical Geometry (no lines through p)
    - Hyperbolic Geometry (more than one line through p)
    - Both have constant curvature (*intrinsic* property)
- The type of geometry determines how many types of direct motions there are.
- This is because conformal maps can be expressed as compositions of reflections across lines.

What Has Been Done

Phase plotting on the Riemann sphere has already been employed by Wegert.

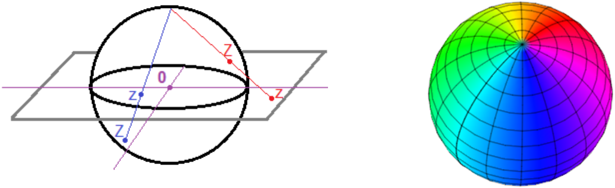


Figure: Left: The construction of the Riemann Sphere with stereographic projection. Right: phase plotting for a Möbius transformation (direct motion) on the Riemann Sphere.

New in this Work

- We extend the notion of phase plotting to surfaces useful in visualizing hyperbolic geometry:
  - Pseudosphere
  - Poincaré Disc
  - Beltrami Half-Sphere
  - Klein Disc
- For the task, we had to redefine the hsv coloring rules for different representations of hyperbolic space.
- We did so using *Maple*.
  - We exploited *Maple*'s texture plotter in order to cover 3d objects with colors.
  - This generates much nicer shapes than simply coloring individual points in space.

Pseudosphere

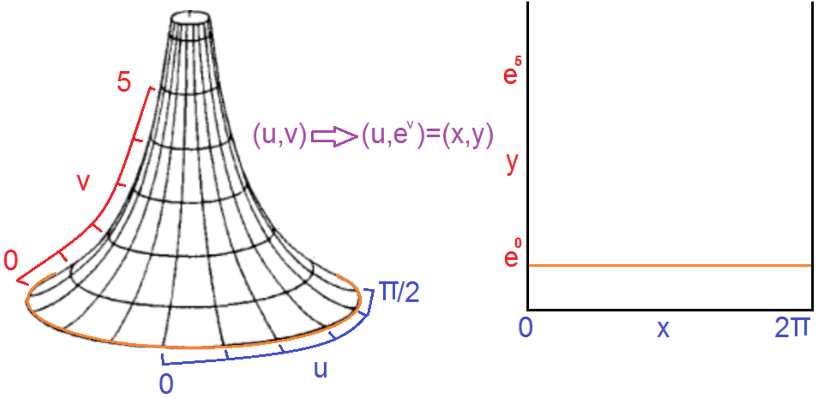


Figure: The conformal map from the pseudosphere to the hyperbolic upper half plane.

Pseudosphere

- The map is between the pseudosphere and a small area of the upper half plane
- If we colored according to the planar phase plotting rules, problems:
  - Fewer colors for visualization
  - Coloring would be tied to Euclidean geometry rather than Hyperbolic geometry, warping perspective.
  - Unable to tell if points mapped out of visible region.
- Solution: defined a new coloring scheme unique to hyperbolic space.

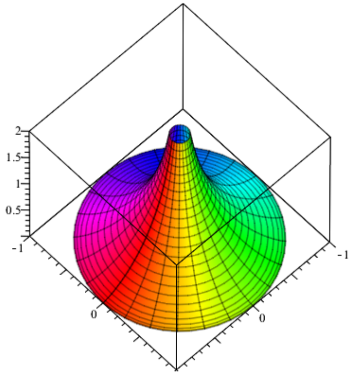


Figure: Colors change along tractrices rather than Euclidean subspaces.

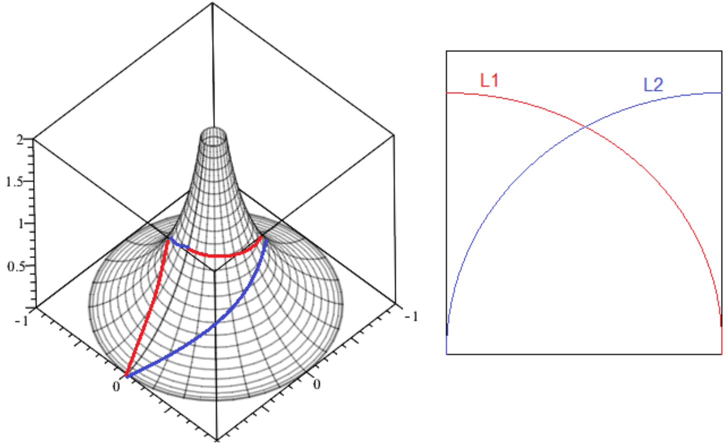
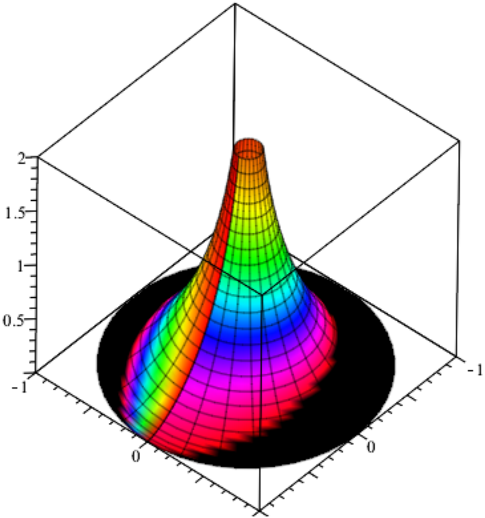
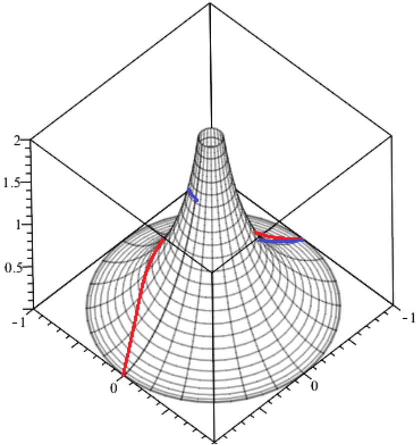


Figure: Computing a direct motion (h-rotation) in hyperbolic space. Here $M = I_{L_2} \circ I_{L_1}$ where L_1 and L_2 correspond to circles in $\mathbb{C}$ centered at 0 and 2π with radius 2π. The Möbius transformation is $4 * \pi^2/(2 * \pi - z)$.

Pseudosphere: h-rotation

- The regions sent out of view are the regions we expected to be sent out of view.
- The rainbow spectrum is now rotated, as hyperbolic space has been rotated.
- Notice how non-tractrix lines are now visible!





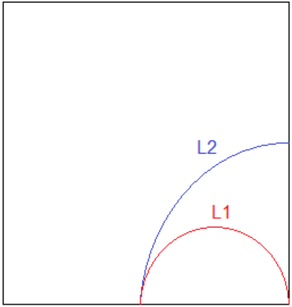
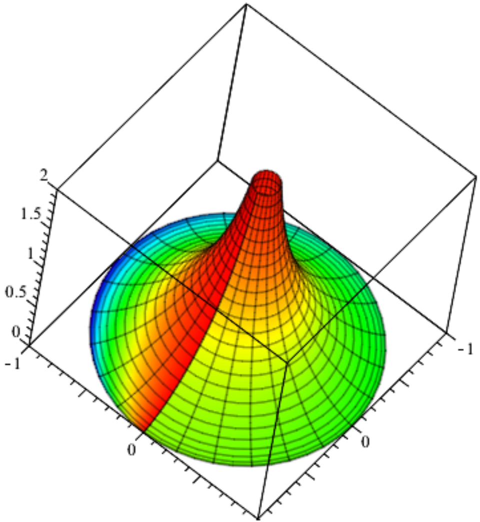


Figure: Direct motion: a *limit* rotation. $M = I_{L_2} \circ I_{L_1}$ where L_1 and L_2 correspond to circles in $\mathbb{C}$ centered at $\frac{3}{2}\pi$ and 2π with radii $\frac{1}{2}\pi, \pi$ respectively. The Möbius transformation is $\pi^2/(-z+2\pi)$.

Pseudosphere: limit rotation

- The center of the rotation is at the right rear
- Much of foreground is green; these points have all been pulled towards the right rear.
- Only some points starting inside the circle for L_1 are mapped to the left rear.



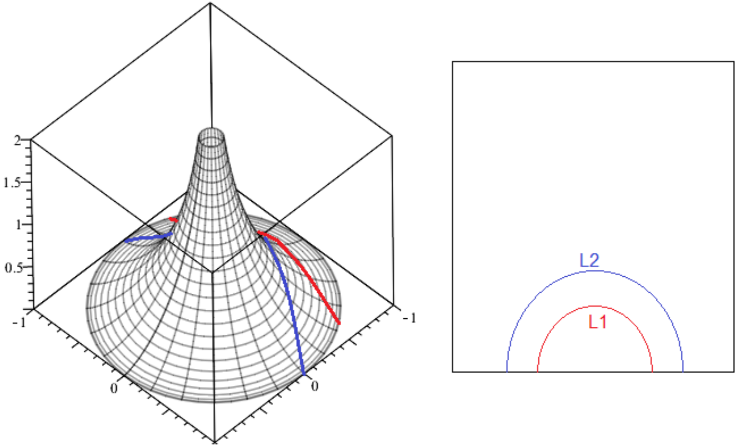
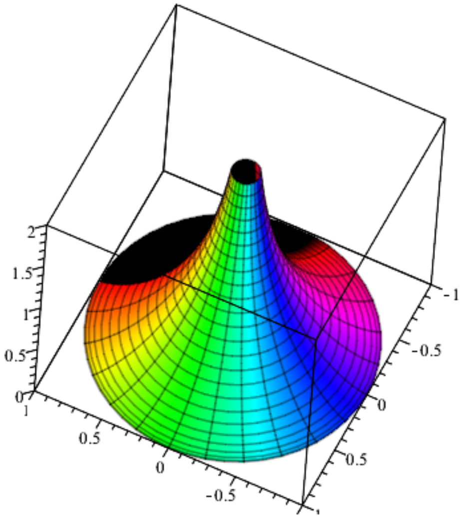


Figure: Direct motion: an h-translation. $M = I_{L_2} \circ I_{L_1}$ where L_1 and L_2 correspond to circles in $\mathbb{C}$ centered at π with radii $\frac{1}{3}\pi$, $\frac{1}{2}\pi$ respectively. The Möbius transformation is $\frac{9}{4}z - \frac{5}{4}\pi$.

Pseudosphere: limit rotation

- We see tractrices sent to tractrices
- Some points are translated out of view.
- Space appears to contract, but has not actually done so. If we made our translation by reflecting across tractrix lines, this effect would not be visible.



Pseudosphere: Tractrix "Height"

- One can use a simple "hack" of the interface to determine the tractrix height of the image points.
- Simply compose the map:

$$F(z) = \frac{2\pi}{\alpha} \cdot \log(\Im(z)) + \exp(1) \cdot i$$

on the motion in question.

- Here the color spectrum begins at tractrix edge; α is chosen to be the tractrix height at which it terminates.

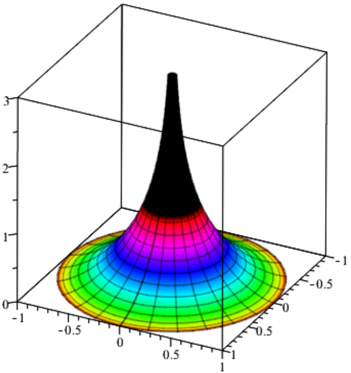


Figure: F composed on identity map where $\alpha = 2$.

Poincaré Disc

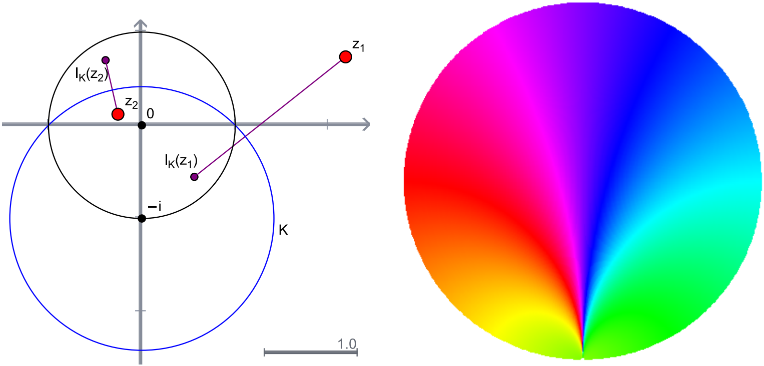


Figure: Left: Construction of the Poincaré Disc. Right: phase plotting on Poincaré Disc as defined by our rule.

Poincaré Disc

- We adopt a new plotting rule
- Still colors tractrix generators in a single color
  - Pre-images of h-lines are still h-lines
  - Consistent with Pseudosphere
- The trick is subtle.
- Where T is inversion map f user function, HSV map for p in disc is: $\frac{1}{2\pi} arg \circ T \circ \Re e \circ f \circ T$.

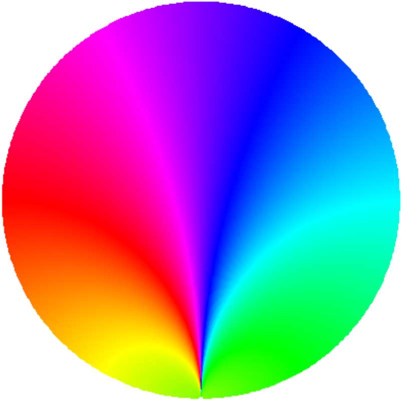


Figure: Phase plotting on Poincaré Disc.

Poincaré Disc

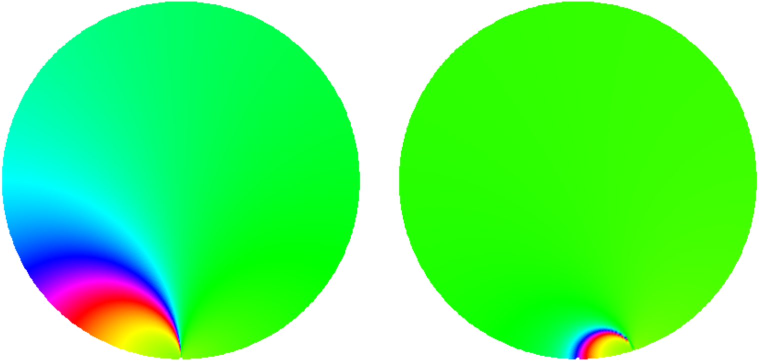


Figure: Left: h-translation $z \to z + 2$. Right: h-rotation $z \to (4\pi^2)/(2\pi - z)$ corresponding to inversion in 2 circles radius 2π centered at $0, 2\pi$. Notice that the preimages of lines are still lines.

Poincaré Disc

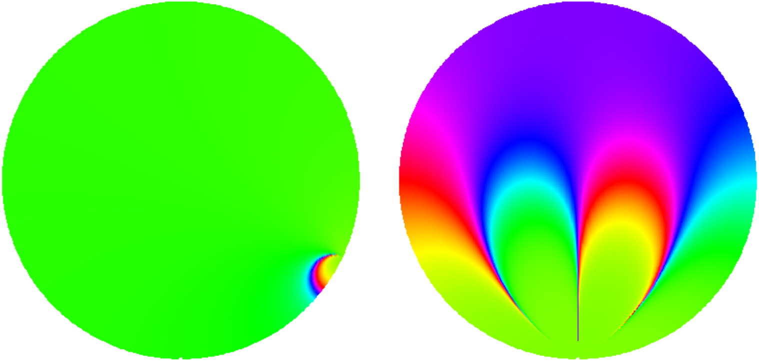


Figure: Left: a *limit* rotation $z \to \pi(2\pi - 3z)/(\pi - 2z)$ corresponds to inversion in circles of radius π centered at 0 and 2π. Right: a map which is not a direct motion: $z \to z^3$.

Poincaré Disc

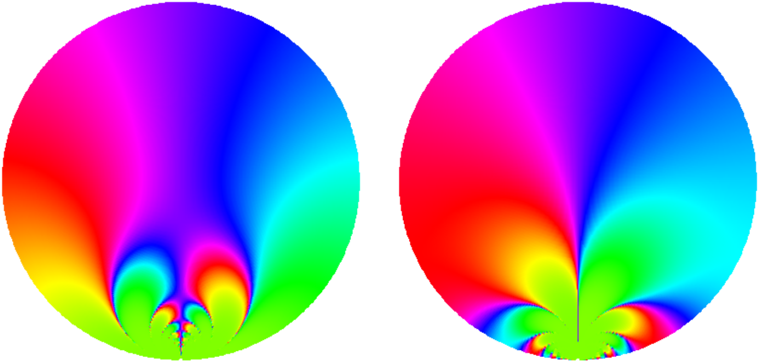


Figure: Two more maps which are not direct motions. Left: $z \to sinh(z)$. Right: $z \to sin(z)$.

Beltrami Half-Sphere

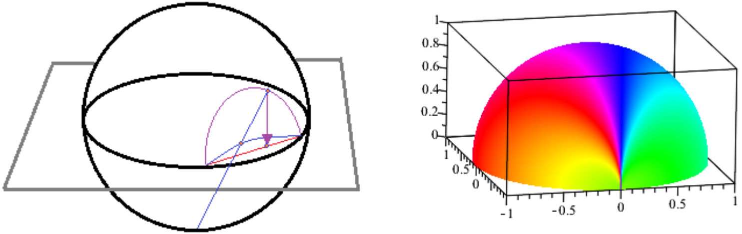


Figure: Left: Construction of the Beltrami Half-Sphere (first step) and Klein Disc (second step). Right: phase plotting on Beltrami Half-Sphere for $z \to z$.

- The Beltrami half-sphere is constructed via a lower stereographic projection of the Pöincare disc.
- Phase plotting rule is inherited from Pöincare disc.

Beltrami Half Sphere

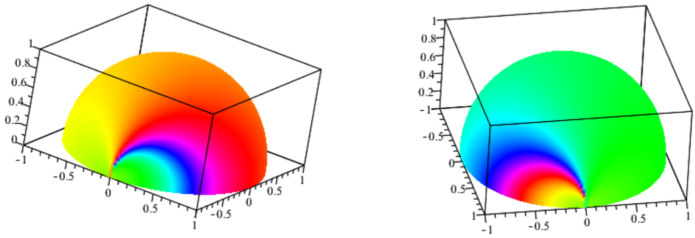


Figure: Left: h-translation $z \to z - 2$. Right: h-rotation $z \to (z-3)/(z-1)$ corresponding to inversion in circles of radius 2 centered at $-1, 1$.

- Notice how lines in hyperbolic space are now semi-circles orthogonal to unit circle.
- Hyperbolic subspaces are hemispheres orthogonal to unit circle.

Beltrami Half Sphere

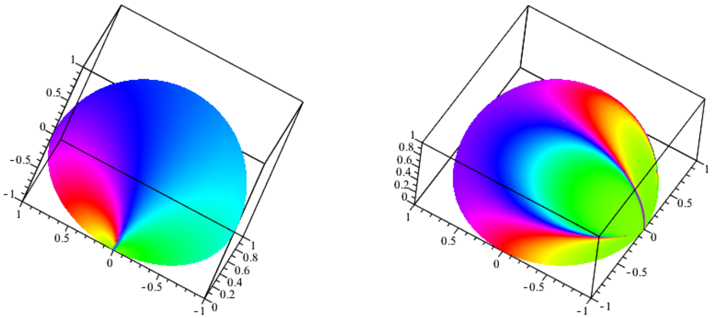


Figure: Left: a *limit* rotation $z \to z/(z+1)$ corresponding to inversion in two circles of radius 2 centered at $-2, 2$. Right: a map which is not a direct motion: $z \to z^3$.

Klein Disc

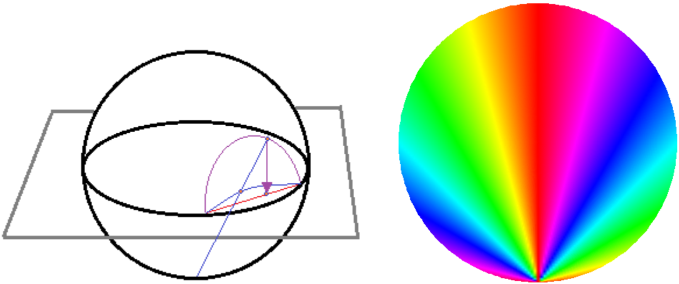
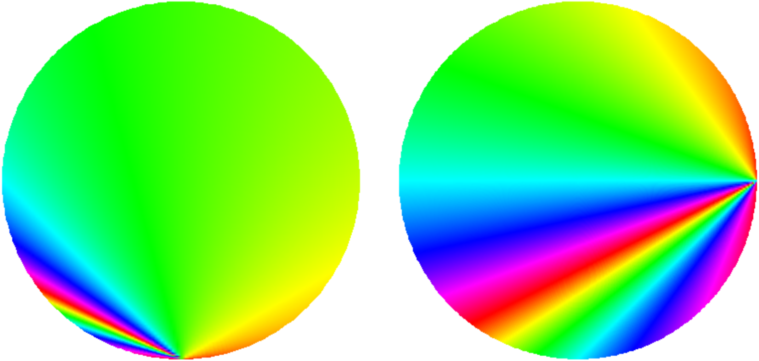


Figure: Left: Construction of the Beltrami Half-Sphere (first step) and Klein Disc (second step). Right: phase plotting on Klein Disc for $z \to z$.

- The Klein Disc.
- Phase plotting rule is again inherited from Pöincare disc.

Klein Disc



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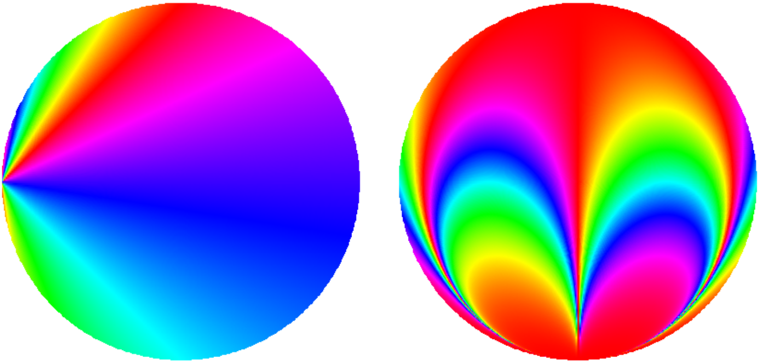
Klein Disc



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Klein Disc

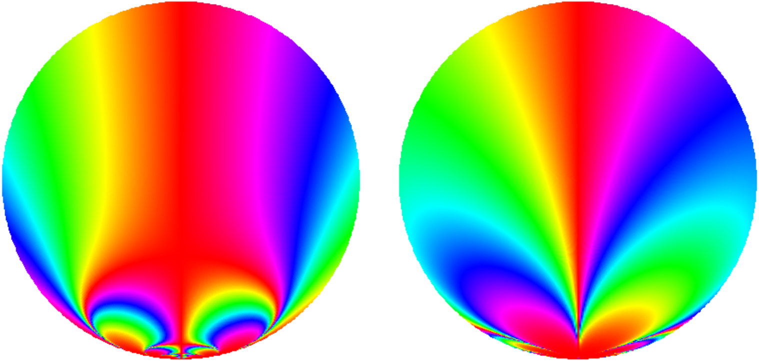

Figure: Two more maps which are not direct motions. Left: $z \rightarrow sinh(z)$. Right: $z \rightarrow sin(z)$.

References I

[1] Tristan Needham, *Visual Complex Analysis*.

[2] Elias Wegert, *Visual Complex Functions*.

[3] Complex Beauties Calendar `http://www.mathe.tu-freiberg.de/fakultaet/information/math-calendar-2016`.