Special Semester in Computation and Visualisation

University of Newcastle

Title: Introduction to Experimental Mathematics

Room: V205, Mathematics Bldg, University of Newcastle

Time: 9-12, 1-2, Tuesday 3rd October

Abstract: The talks will be organized as follows:

1. **What is experimental mathematics?** (15 minutes)

   This overview briefly summarizes what is meant by ?experimental mathematics?, as pioneered in large part by the late Jonathan Borwein. We also explain why experimental mathematics offers a unique opportunity to involve a much broader community in the process of mathematical discovery and proof ? high school students, undergraduate students, computer scientists, statisticians and data scientists. It also presents opportunities for outreach to the public in a way that traditional mathematics has not.

2. **Pi and normality: Are the digits of π "random"?** (50 minutes)

   In this talk we review the history of π, including recently discovered formulas such as the Borwein quartic formula (each iteration of which roughly quadruples the number of correct digits). We then describe the Bailey-Borwein-Plouffe (BBP) formula (which permits one to directly calculate base-16 or binary digits of π beginning at an arbitrary starting point), which was discovered by a computer program, arguably the first major success of the experimental paradigm in modern mathematics. We then explain why the existence of BBP-type formulas for π and other mathematical constants has an interesting implication for the age-old question of whether and why the digits of π and other constants are "random" ? i.e., the property that every m-long string of base-b digits appears, in the limit, with frequency $1/b^m$. By extending these techniques, and by using a ?hot spot lemma? proved using ergodic theory methods, we are able to prove normality for a large class of specific explicit constants (sadly not yet including π), and also to present specific examples of why normality in one number base does not necessarily imply normality in other bases.

3. **High-precision arithmetic and PSLQ** (30 minutes)

   This talk describes the mathematics and computational techniques employed to compute with numbers of very high numeric precision – typically thousands or millions of digits. One key technique is the usage of fast Fourier transforms to accelerate multiplication, typically by a factor of many thousands. Other algorithms permit one to evaluate the common transcendental functions (e.g., cos, sin, exp, log, etc.) to high precision. The talk then discusses the PSLQ algorithm, which is one of the key tools of experimental mathematics, and gives a variety of examples of this techniques in use.

4. **Experimental mathematics and integration** (50 minutes)

   One of the most common applications of the experimental methodology in mathematics is to computationally evaluate a definite integral to high precision and then use the PSLQ algorithm to recognize its value in terms of well-known mathematical constants and formulas. The key challenge here is to compute integrals (finite or infinite interval; real line or multidimensional) to very high precision– typically hundreds or thousands of digits. Fortunately, some rather effective algorithms, notably as the tanh-sinh scheme, are known for this purpose. The talk then presents several examples of this methodology in action, including the evaluation of Ising integrals and box integrals. We also present some examples showing how these methods can fail unless performed carefully.

5. **Ancient Indian mathematics** (40 minutes)

   It has been commonly thought that our modern system of positional decimal arithmetic with zero arose in the 13-th century, with the writings of Fibonacci. In fact, it arose at least 1000 years earlier, possibly before 0 CE. One of the most interesting early artifacts exhibiting decimal arithmetic in use is the Bakhshali manuscript, an ancient Indian mathematical treatise that was discovered in 1881 near Peshawar (then in India, but now in Pakistan). In the early 20-th century a British scholar assigned the Bakhshali manuscript to the 13-th century, because he was convinced that it was derived from Greek mathematics, but others have argued that it was several centuries older. In September 2017, the Bodelian Library in London, which houses the manuscript, announced the results of radiocarbon tests which show that at least part of the Bakhshali manuscript dates back to 300 CE or so. This talk describes the Bakhshali manuscript in detail, including examples of solutions of linear equations, second-degree Diophantine equations, arithmetic progressions, and iterative approximations of square roots. The talk then mentions some analysis, by the present author and the late Jonathan Borwein, on the square root methods in the Bahkshali manuscript and other ancient Indian documents.

6. **Computational and experimental evaluation of large Poisson polynomials** (50 minutes)

   In some earlier studies of lattice sums arising from the Poisson equation of mathematical physics, it was proven that

$$\sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{(m^2+n^2)}$$

   is always $\pi log A$, where A is an algebraic number. By means of some very large computations with the PSLQ algorithm, polynomials associated with A were computed for numerous rational arguments x and y. Based on early results, Jason Kimberley of the University of Newcastle, Australia, conjectured a number-theoretic formula for the degree of A in the case $x = y = 1/s$ for some integer s. In a subsequent study, co-authored with Jonathan Borwein, Jason Kimberley and Watson Ladd, the Poisson polynomial problem was addressed with significantly more capable computational tools. As a result of this improved capability, we confirmed that Kimberley's formula holds for most integers s up to 52, and also for $s = 60$ and $s = 64$. As far as we are aware, these computations, which employed up to $64,000$-digit precision, producing polynomials with degrees up to 512 and integer coefficients up to 10^{229}, constitute the largest successful integer relation computations performed to date. Finally, by applying some advanced algebraic techniques, we were able to prove Kimberley's conjecture and also affirm the fact that when s is even, the polynomial is palindromic.