Turnpike theorems for convex problems with undiscounted integral functionals

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- Turnpike theory
- Continuous time systems
  - Undiscounted integral functionals
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- Turnpike Theorems

Turnpike Theory

Optimal control problem:

- **System:** $x_{t+1} \in a(x_t), \quad t = 0, 1, 2, \cdots .$
- **Functional** *Maximize:* $\sum_{t=0}^{T} \mathbf{u}$

  where $\mathbf{u} = u(x_t)$ *or* $\mathbf{u} = u(x_t, x_{t+1})$.

Turnpike property describes the "structure/behaviour" of optimal solutions when $T \to \infty$

- $\exists$ "turnpike set/point" that attracts all opt. solutions

- J.V. Neumann, 1932-1945 - first result obtained
  - 1932 - presented at a math.seminar at Princeton (D.Gale)
  - 1937 - published in Vienna
  - 1945 - translated into English
- P.A. Samuelson, 1948-1949 - Interpretation of Neumann's result
- 1958 - the term Turnpike was introduced in
  - *R. Dorfman, P.A. Samuelson and R.M. Solow,* Linear Programming and Economic Analysis, 1958 (Chapter 12)

- A.M. Rubinov, 1973 - Classification of the turnpike property (linear systems - Neumann-Gale model)
  - *V.L. Makarov and A.M. Rubinov,* Mathematical theory of economic dynamics and equilibria, 1973 (Russian)
  - translated into English, 1977
- L. McKenzie, 1976 - Nonlinear systems (bounded trajectories)
  - *L. McKenzie,* Turnpike Theory, Econometrica 44 (1976)

Discrete Systems: the main result

Turnpike property is true for convex problems
($graph\ a$ is convex, u is strongly concave)

Continuous time systems

System: $\dot{x} \in a(x)$

Functional: Utility fun. - $\mathbf{u}(\mathbf{t}) = u(x(t))$ or $u(x(t), \dot{x}(t))$

1. Discounted integral: $\int_0^\infty \mathbf{u}(\mathbf{t})\ e^{-rt} dt$
2. Undiscounted integral: $\int_0^T \mathbf{u}(\mathbf{t})\ dt$
3. Terminal: $\liminf_{t \to \infty} \mathbf{u}(\mathbf{t})$

Main focus: Convex Problems

- $graph\, a = \{(x, y) :\ x \in \Omega,\ y \in a(x)\} \ \Rightarrow$ is convex;
- $u \ \Rightarrow$ is strongly concave.

Some existing approaches

- Jose A. Scheinkman ($\geq$ 1976) in collaboration with W.A. Brock, A Araujo etc (Maximum Principle)
- R.T. Rockafellar (1973, 1976, 2009)
- D.E.Gusev and V.A.Yakubovich ($\geq$ 1973) (Maximum Principle)
- A.I.Panasyuk and V.I.Panasyuk (applications in engineering)
- D.A.Carlson, A.B.Haurie and A.Leizarowitz (book - 1991)
- M.Marena and L.Montrucchio
- A.J. Zaslavski (book - 2005)

Recent developments

- Long run average problem (V.Gaitsgory, 2006)
  $\lim_{T\to\infty} \frac{1}{T} \int_0^T u(x(t))dt$
- Markov Games (V.Kolokoltsov at all, 2013)
- Model predictive control (T.Damm, L.Grüne et all 2012-2014) (discrete systems)
- Time-delay systems (A.Ivanov and M.Mammadov, ≥ 2010)
- Weak stability:
  - Statistical convergence (S.Pehlivan and M.Mammadov, 2000)
  - A-Statistical convergence (P.Das, S.Dutta et all, 2014)
  - Ideal convergence (M.Mammadov and P.Szuca, 2014)

My target: to develop a complete theory for undiscounted and terminal functionals by considering

- non-convex problems
- convex problems

Today's talk: convex problems with undiscounted functionals

Most related approach: D.A.Carlson, A.B.Haurie and A.Leizarowitz (book - 1991)

- Optimality: Overtaking optimal solutions on $[0, \infty)$;
- The convex case still uses some restrictive assumptions.

Turnpike Theorems

Problem (P):

System: $\dot{x} \in a(x), \quad x(0) = x^0,$

Maximize: $J_T(x(\cdot)) = \int_0^T u(x(t))\, dt$

- $a : \Omega \rightrightarrows R^n$ has compact images, is continuous in the Hausdorff metric
- $u : \Omega \to R^1$ is continuous
- $X_T \neq \emptyset$ denotes the set of trajectories on the interval $[0, T]$
- Ω is bounded and $x(t) \in \operatorname{int} \Omega, \quad \forall t \in [0, T],\ x(\cdot) \in X_T,\ T > 0$
- $M \triangleq \{x \in \Omega,\ 0 \in a(x)\}$ - is the set of stationary points
- $x^* \in M$ is optimal stationary point if $u(x^*) = \max_{x \in M} u(x)$
- Given $T > 0$, trajectory $x(\cdot)$ is called
  - optimal if $\quad J_T(x(\cdot)) = J_T^* \triangleq \sup J_T(x(\cdot))$
  - $\xi-$optimal if $\quad J_T(x(\cdot)) \geq J_T^* - \xi; \quad$ where $\xi \geq 0$.

Main Assumptions

A1 (Exist. "good" sol-s): $\exists\, b < +\infty$, for every $T > 0$ $\quad \exists\, x(\cdot) \in X_T$:

$$J_T(x(\cdot)) \;\geq\; u^* T - b.$$

A2 (Convex Problem):

- $graph\, a$ is convex, compact
- u is concave (not necessarily strictly)
- $\forall\; x_1, x_2 \in \Omega, \quad \alpha \in (0,1),$ one of the following holds:

$$u(\alpha\, x_1 + (1-\alpha)\, x_2) \;>\; \alpha\, u(x_1) + (1-\alpha)\, u(x_2);$$

$$\mathrm{int}\; a(\alpha\, x_1 + (1-\alpha)\, x_2) \;\supset\; \alpha\, a(x_1) + (1-\alpha)\, a(x_2).$$

A3: There exists $x' \in \Omega$ such that $u(x') > u^*$.

A4: There exists $\tilde{x} \in M$ such that $0 \in \mathrm{int}\; a(\tilde{x})$.

Theorem 3.1: Assume that Assumptions **A1-A4** hold. Then there exists a unique optimal stationary point x^* and

(1) - Upper bound for $J_T(x(\cdot))$: there exists $C < +\infty$ such that

$$\int_0^T u(x(t))\,dt \;\leq\; u^*\,T + C$$

for all $T > 0$ and for all trajectories $x(\cdot) \in X_T$;

(2) - Turnpike property: (given any $\xi \geq 0$): for every $\varepsilon > 0$, there exists $K_\varepsilon < +\infty$ s.t.

$$\mathrm{meas}\{t \in [0,T] : \; ||x(t) - x^*|| \geq \varepsilon\} \;\leq\; K_\varepsilon$$

for all $T > 0$ and for all ξ-optimal trajectories $x(\cdot) \in X_T$;

(3): if $x(\cdot)$ is an optimal trajectory and $x(t_1) = x(t_2) = x^*$, then

$$x(t) = x^*, \quad \forall t \in [t_1, t_2].$$

Two special cases.

- **Utility function u is strictly concave:**

  A3 can be eliminated: $\exists\ x' \in \Omega$ such that $u(x') > u^*$.

  Theorem 3.2: Assume that function u is strictly concave and Assumptions **A1, A2, A4** hold. Then there exists a unique optimal stationary point x^* and all the assertions **(1)-(3)** of Theorem 3.1 are valid.

- **Mapping a is strictly convex:**

  A4 can be eliminated: $\exists\ \tilde{x} \in M$ such that $0 \in \operatorname{int}\, a(\tilde{x})$

  Theorem 3.3: Assume that mapping a is strictly convex, Assumptions **A1, A2, A3** hold. Then there exists a unique optimal stationary point x^* and the assertions **(2)** and **(3)** of Theorem 3.1 are valid.

THANK YOU