On the Pataki sandwich theorem

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based on joint work with Levent Tunçel

12 July 2014, CARMA

Structured convex analysis

Recently there has been a lot of activity in convex optimisation related to the study of *structured* problems.

Structure allows for the construction of more efficient numerical methods, and for the formulation of constructive conditions for verifying regularity and well-posedness properties.

- Specialised methods, e.g. IPMs on symmetric cones.
- Special convex sets, e.g. spectrahedra.
- In-depth study of the facial structure of convex sets.

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Interior-point methods are highly successful in solving a vast number of large-scale optimisation problems, predominantly via the semi-definite programming (SDP) model. Mathematically rigorous theory allows for reliable solution of *essentially well-posed* problems; practical performance is significantly better than the complexity bounds predicted by theory.

Theory: Renegar & Gonzaga (1988-89), Nesterov & Nemirovski (1994).

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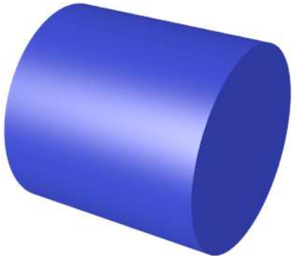
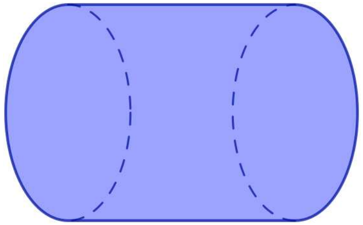
- Special convex sets, e.g. spectrahedra.

Polyhedra are the solutions to LP, spectrahedra are the solutions to SDP.



- a spectrahedron



- not a spectrahedron

C. Vinzant, *What is a spectrahedron*, Notices of the AMS, May 2014.

- In-depth study of the facial structure of convex sets.

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- Special convex sets, e.g. spectrahedra.
- In-depth study of the facial structure of convex sets.

Whenever general problems are considered, it is convenient to express certain properties in terms of facial relations on the convex sets. Niceness (facial dual completeness) and facial exposedness play key roles in characterising convex cones.

Nice and facially exposed cones

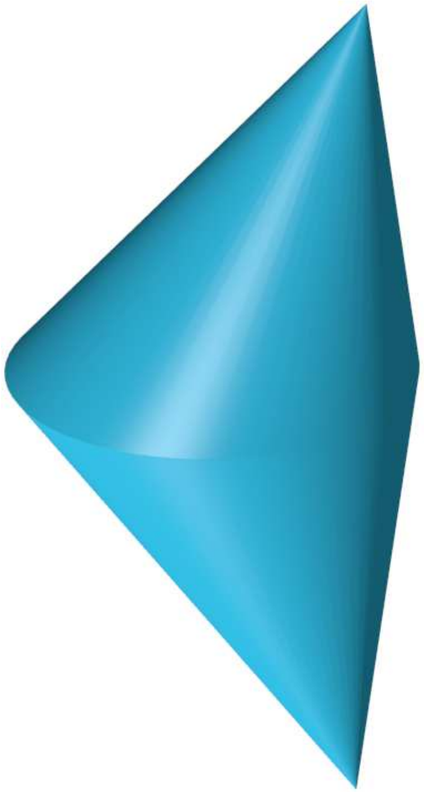
- Good geometry implies good performance of numerical methods. Indeed, niceness plays a role in the facial reduction algorithm (Borwein and Wolkowicz 1980).
- The question whether the linear image of a dual of a nice cone is closed has a simple characterization (Pataki 2007).
- Niceness plays a role in a study of the lifts of convex sets (Gouveia, Parrilo, Thomas 2012).

▶ The Pataki sandwich theorem provides (different) necessary and sufficient conditions for niceness. It is known from examples that both conditions are not tight.

▶ We strenghten the necessary condition, which is facial exposedness, adding a ‘tangential exposure’ condition.

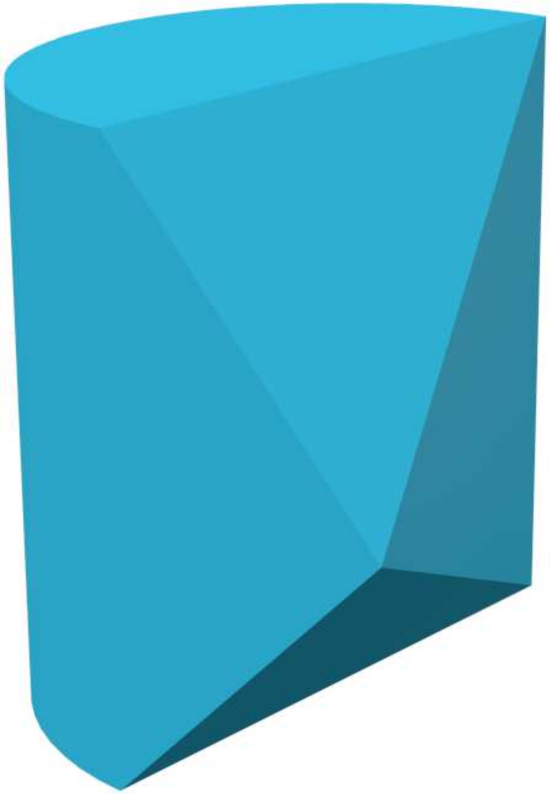
Some Weird sets

Closed convex sets can have counterintuitive facial structure.



"Classic example"

Extreme points not closed;
counterintuitive facial structure.



Barker, 1978

All extreme points are exposed,
but there are unexposed faces.

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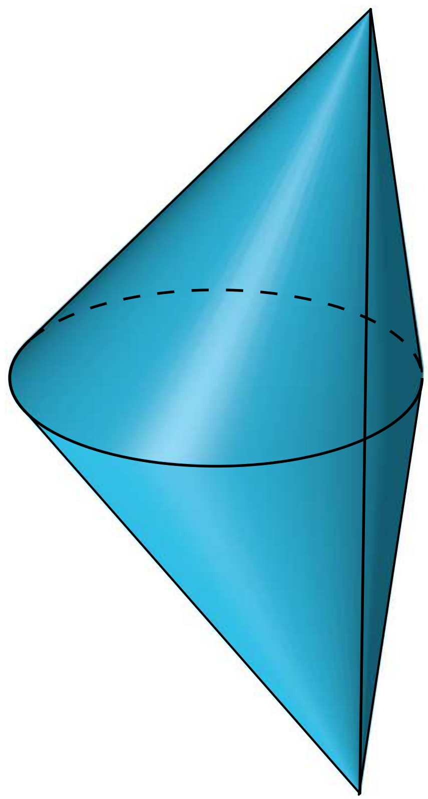

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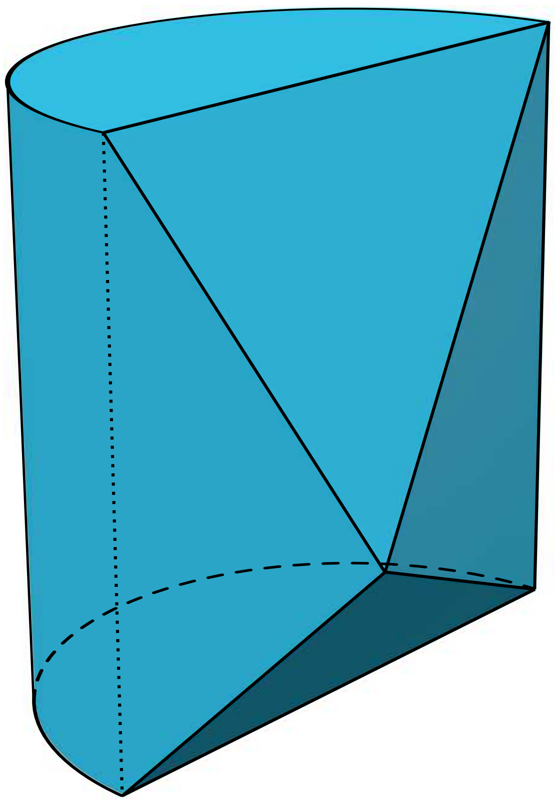

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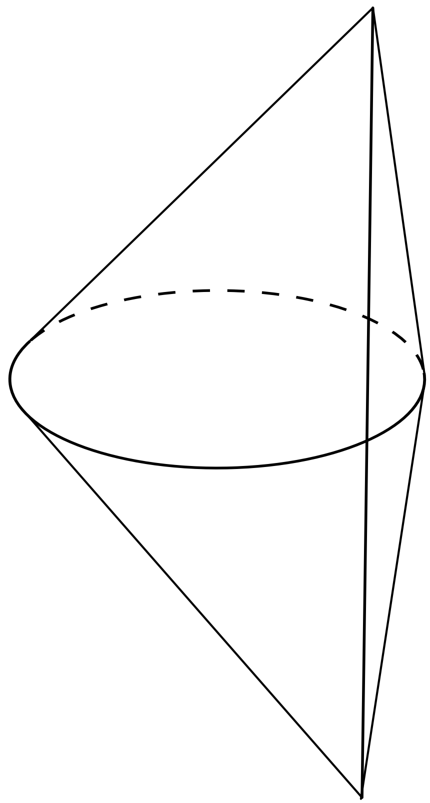

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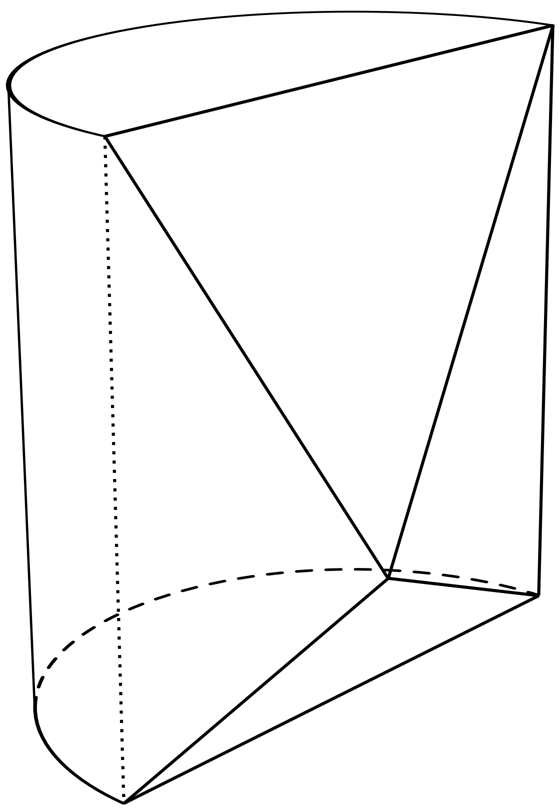

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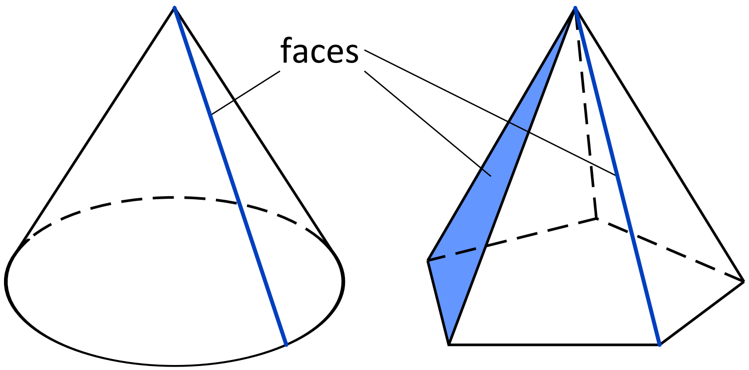
Nice (facially dual complete) cones

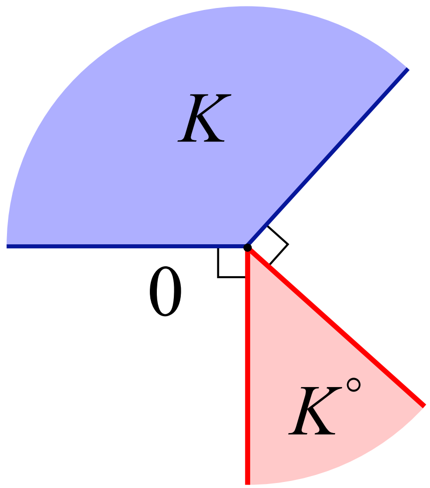
A closed convex cone K is *nice* if for every face F of K the set $K^\circ + F^\perp$ is closed, where $F^\perp$ is the orthogonal complement to $\operatorname{span} F$, and K° is the polar (normal) cone to K.

A nonempty convex subset F of K is a *face* of K if $x + y \in F$ with $x, y \in K$ imply $x, y \in F$.



A polar (normal) cone of K is

$$K^\circ = \{y \mid \sup_{x \in K} \langle x, y \rangle \leq 0\}$$



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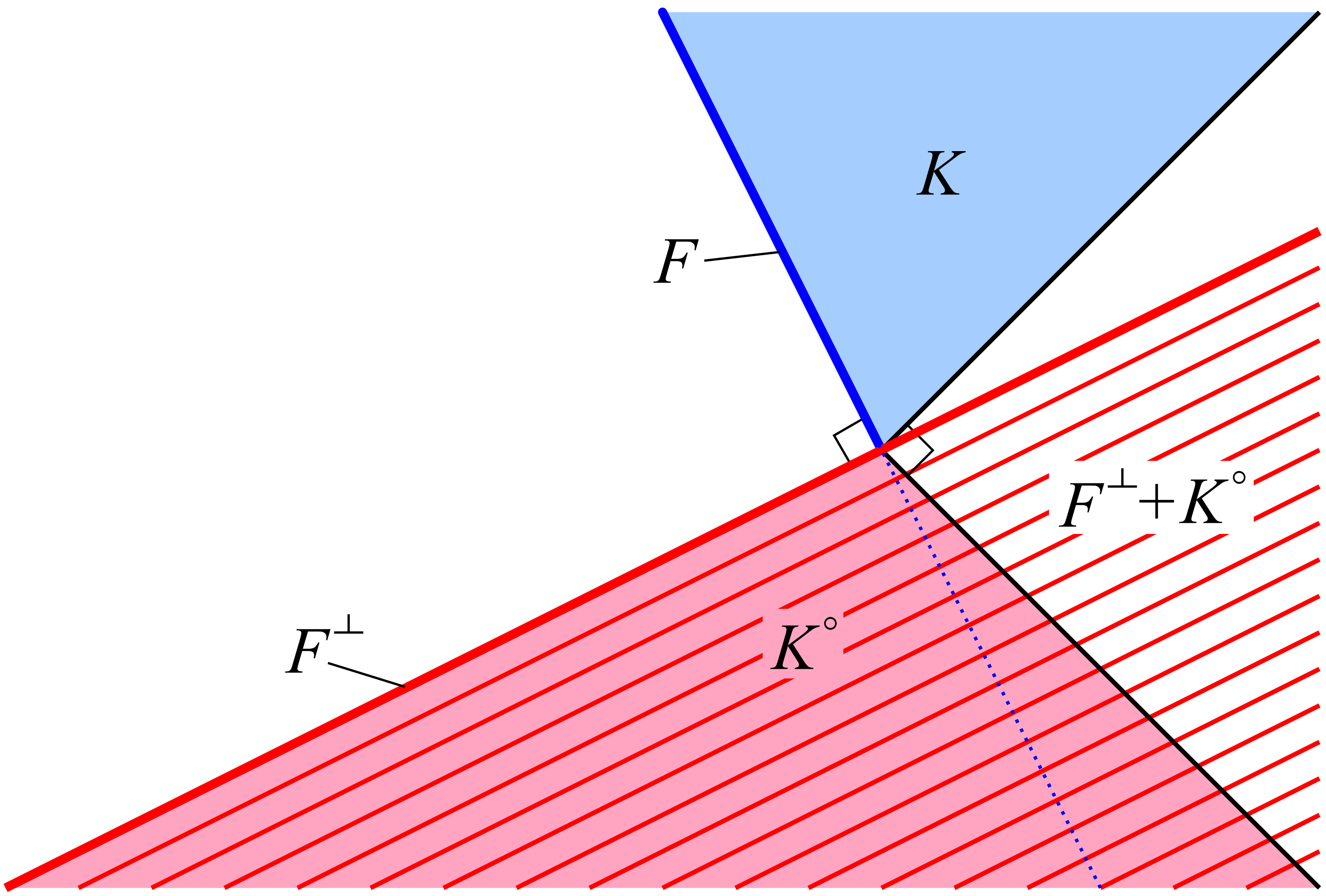

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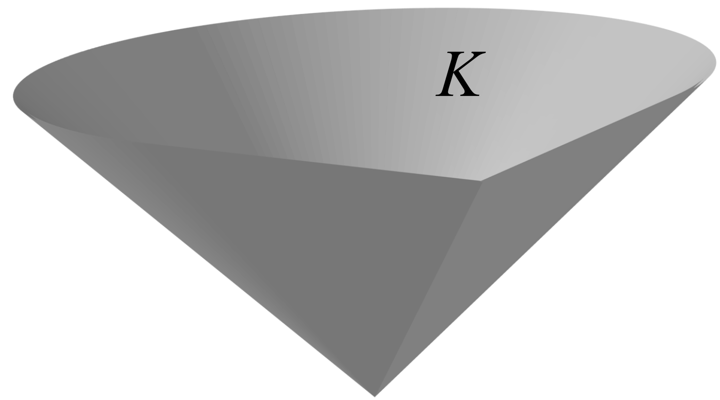

(frame 1 of 7)

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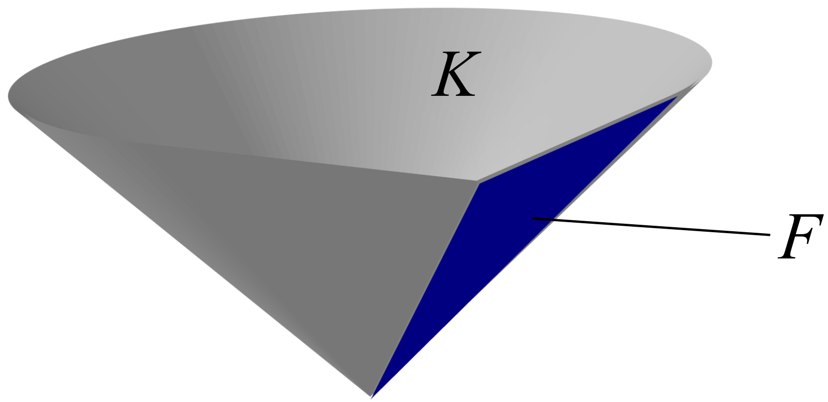

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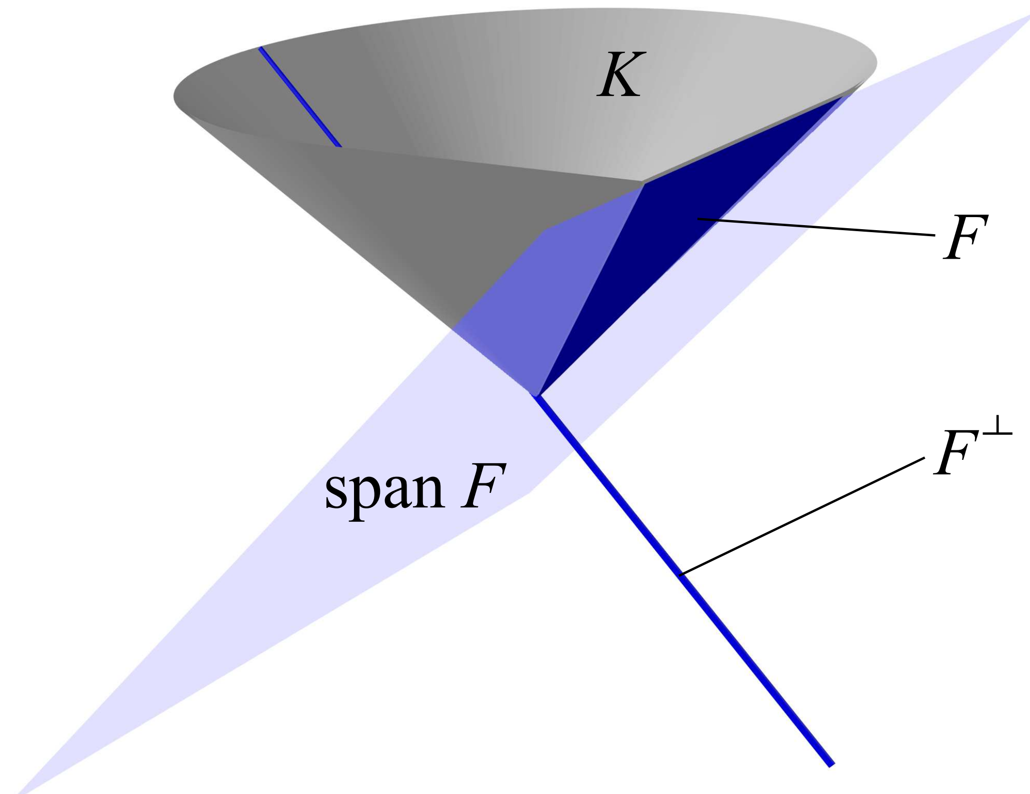

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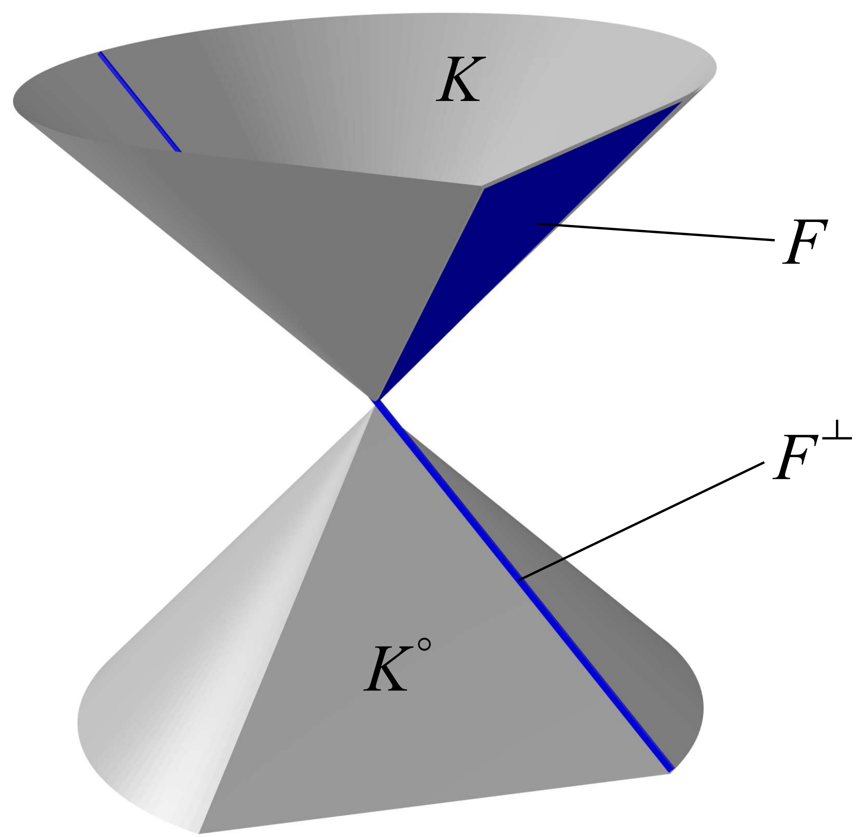

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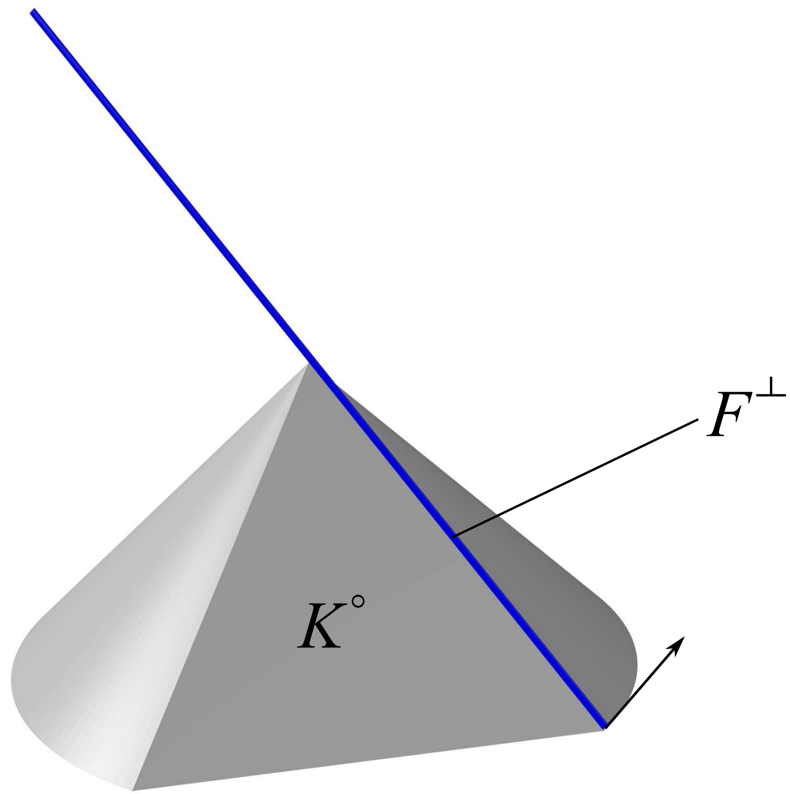

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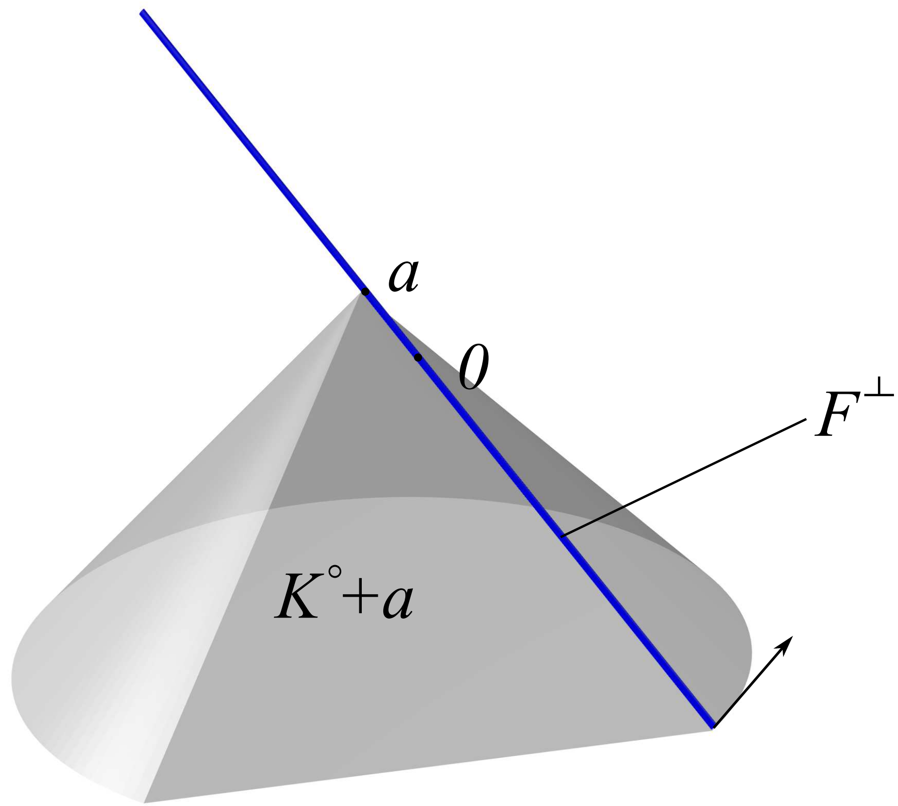

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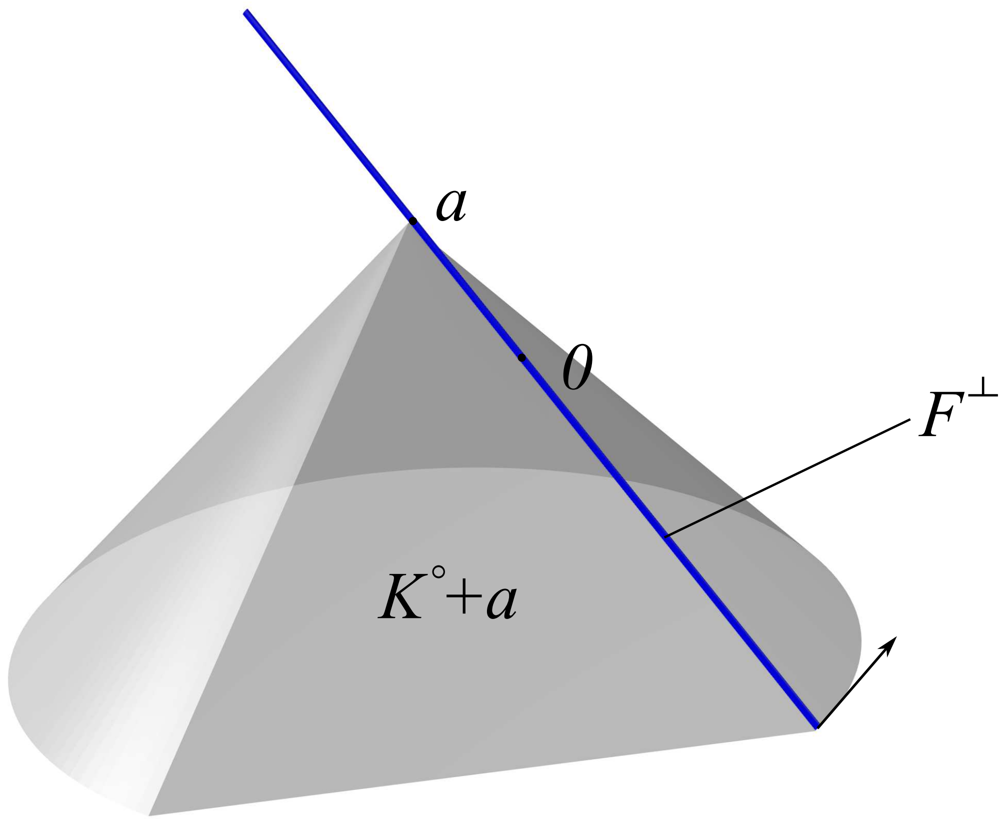

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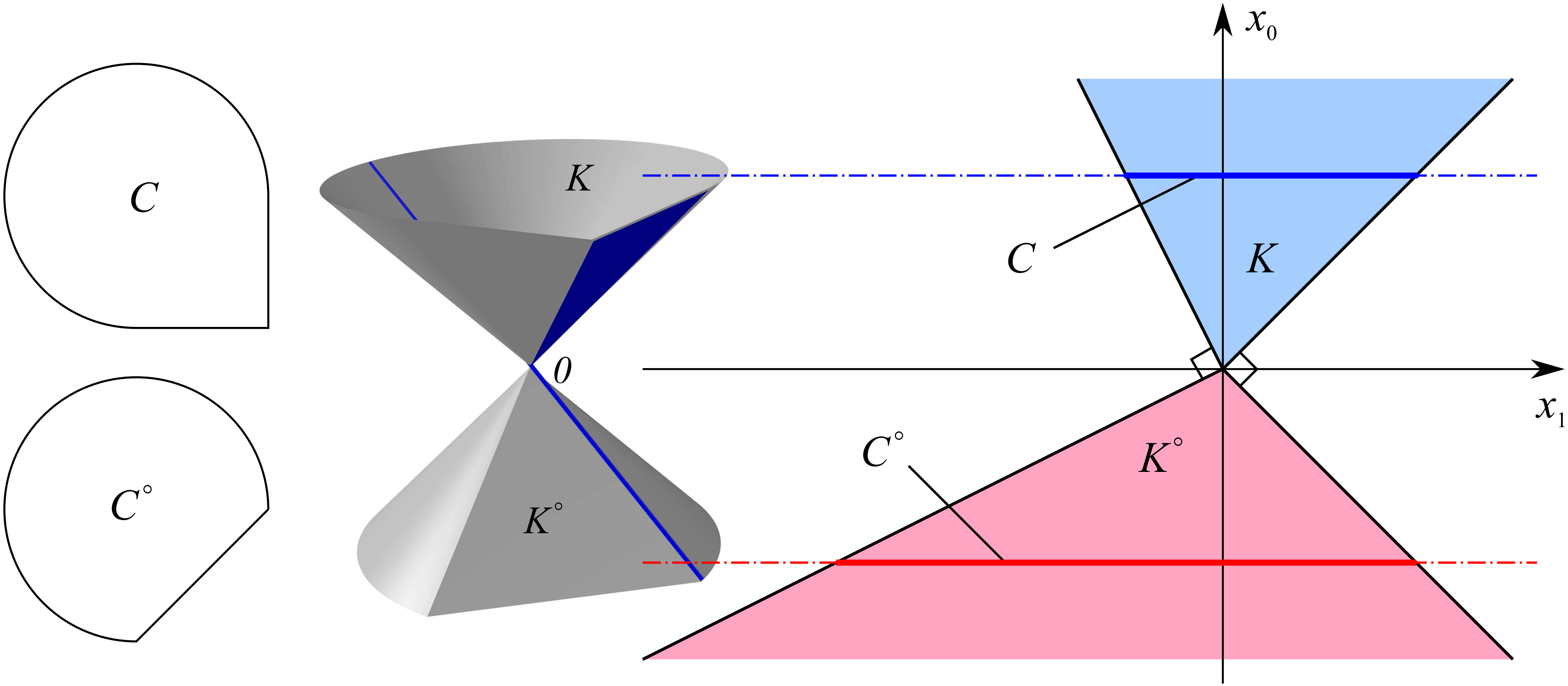
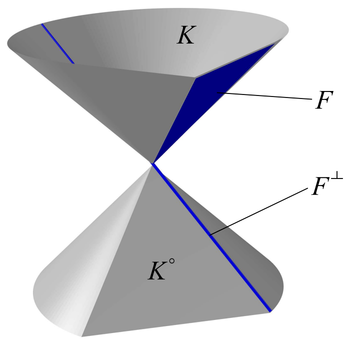
Polars and slices

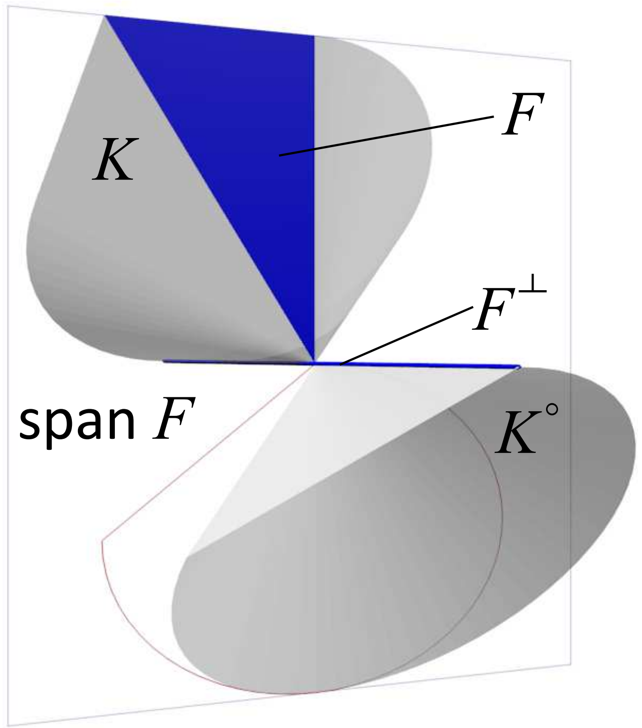


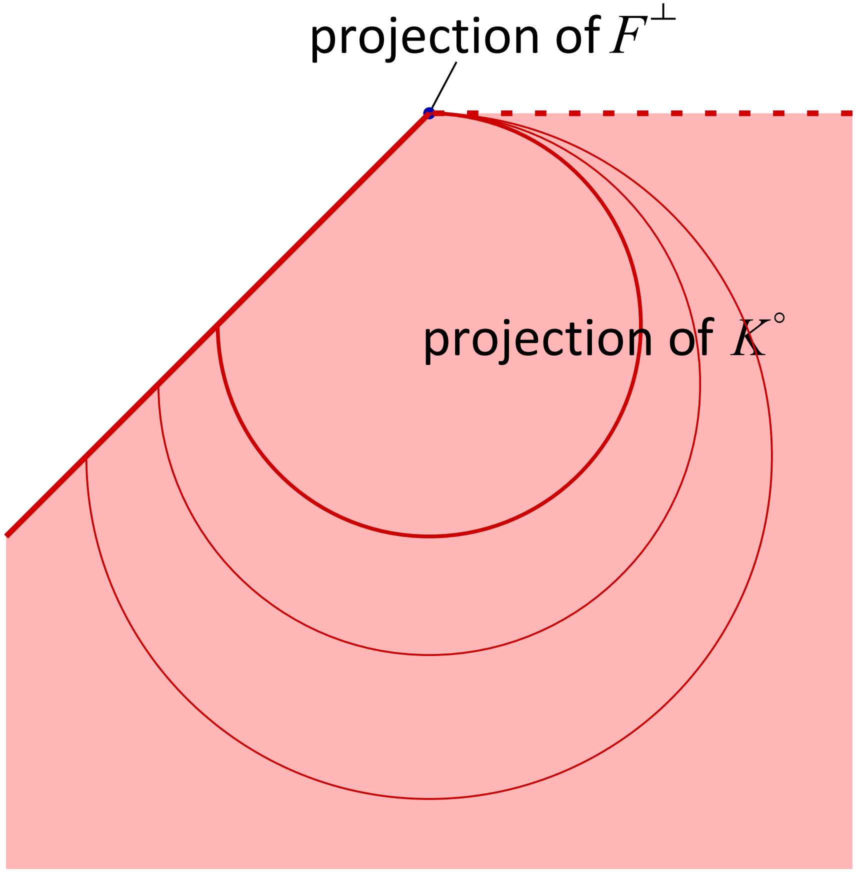
$$K^\circ = \operatorname{cone}(\{-1\} \times C^\circ)$$
$$C^\circ = \{y \mid \sup_{x \in C} \langle x, y \rangle \leq 1\}$$

Lack of niceness: intuition

Observe that if $\Pi_{\mathrm{span}\, F} K^\circ$ is not closed, neither is $F^\perp + K^\circ$.



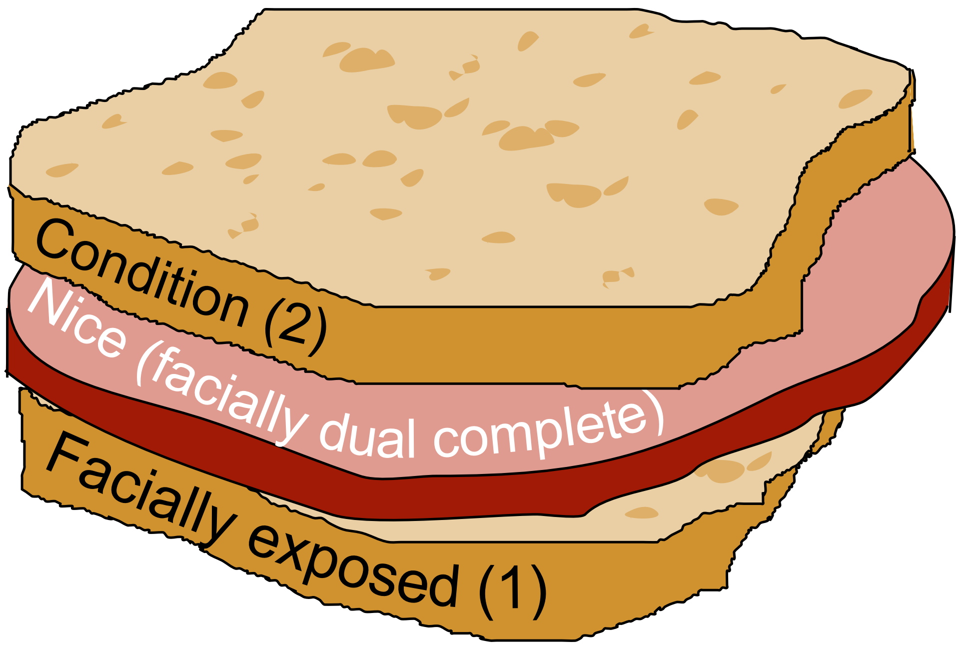




The sandwich theorem

Here K is a closed convex cone in $\mathbb{R}^n$, and $F \trianglelefteq K$ means that F is a face of K.

Pataki sandwich theorem The following statements hold.
(1) If K is nice, then it is facially exposed (sufficient in $\mathbb{R}^3$).
(2) If K is facially exposed, and for all $F \trianglelefteq K$ all properly minimal faces of F^* are exposed, then K is nice (facially dual complete).



Remark: Sandwich diagrams will be used instead of the old-fashioned Venn diagrams

Theorem 3 in [Gabor Pataki, "On the connection of facially exposed and nice cones", arxiv:1202.4043 (JMAA 400 (2013))]

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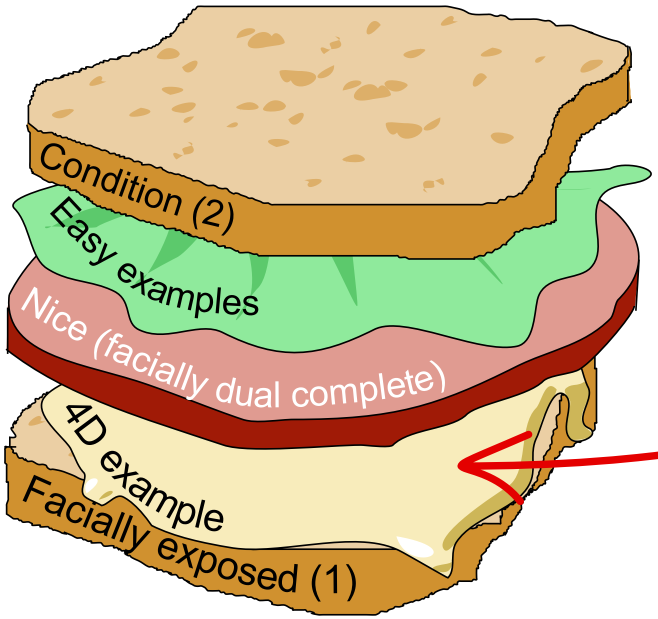

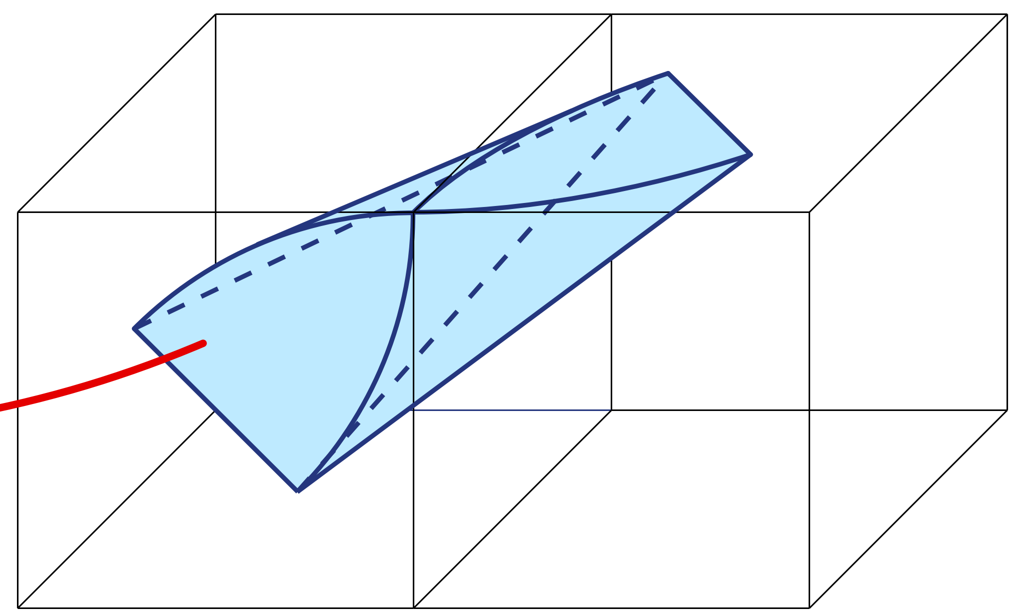

– a slice of the 4D example

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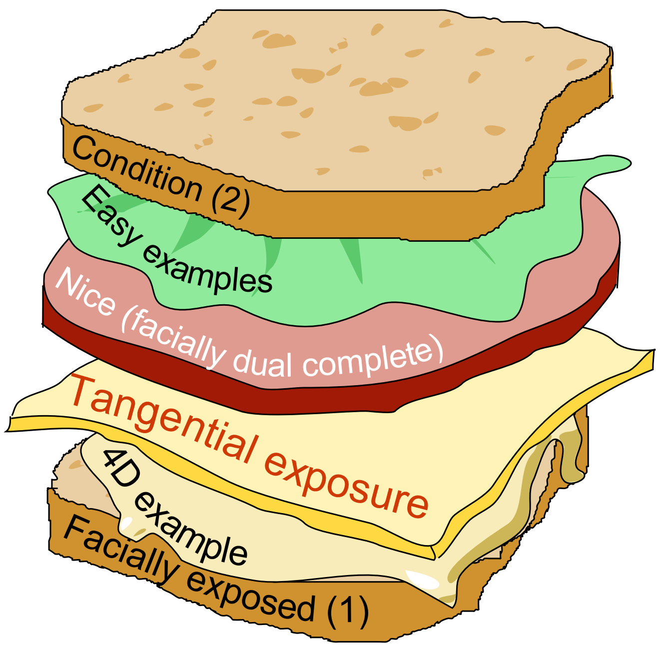

'Tangential exposure' If a cone K is nice (facially dual complete), then for every face $F \triangleleft K$ and every $x \in F$, we have $T(K, x) \cap \operatorname{span} F = T(F, x)$.

Here T is the tangent cone:

$$T(K, x) := \operatorname*{Lim\,sup}_{t \to +\infty} t(K - x).$$

Tangentially exposed sets

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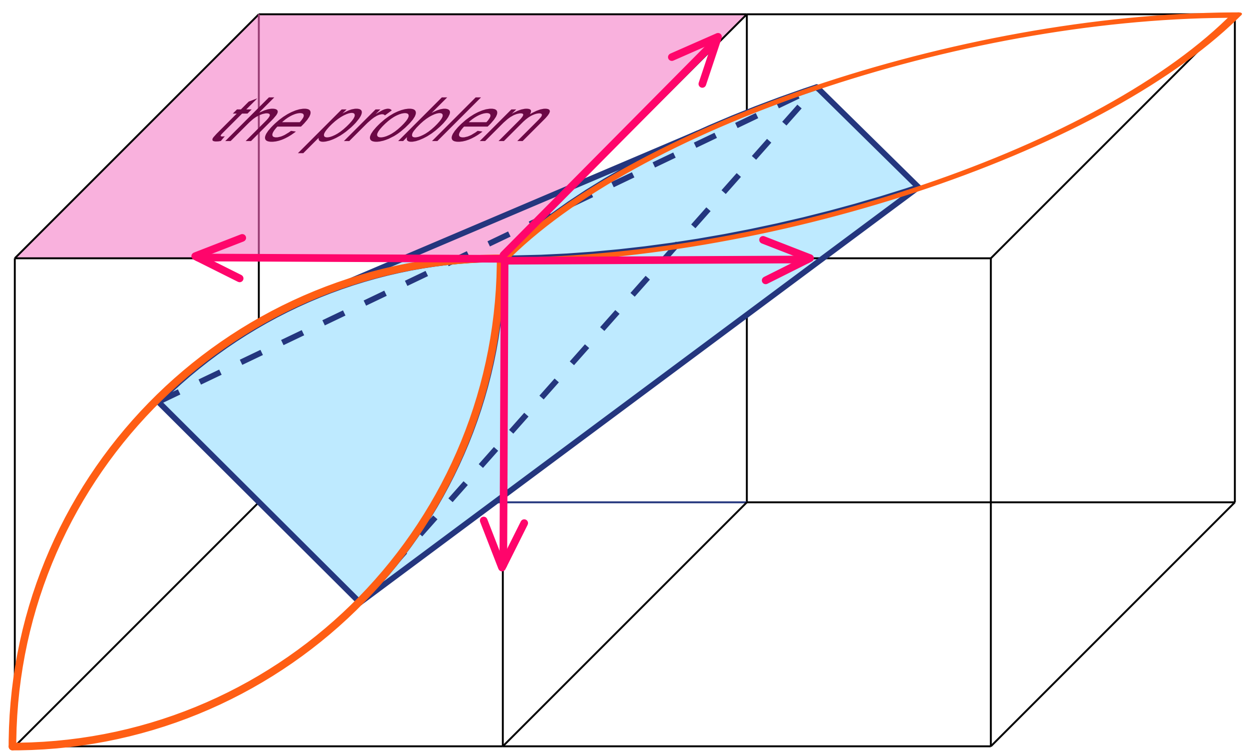
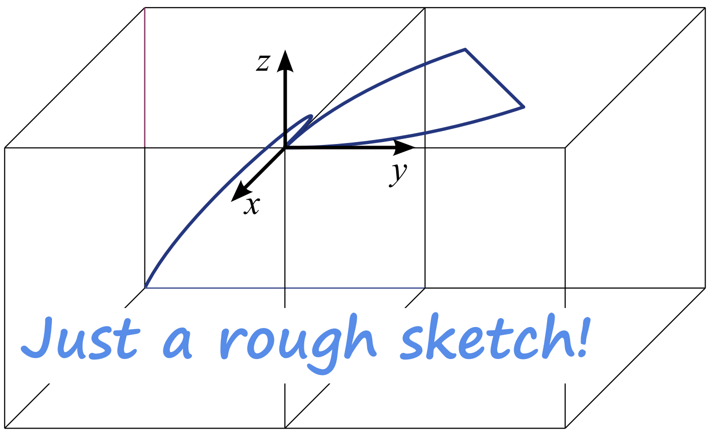
The 4D cone is not tangentially exposed:



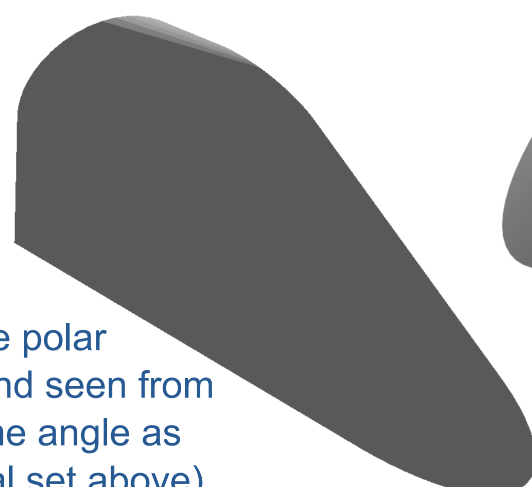
A new example



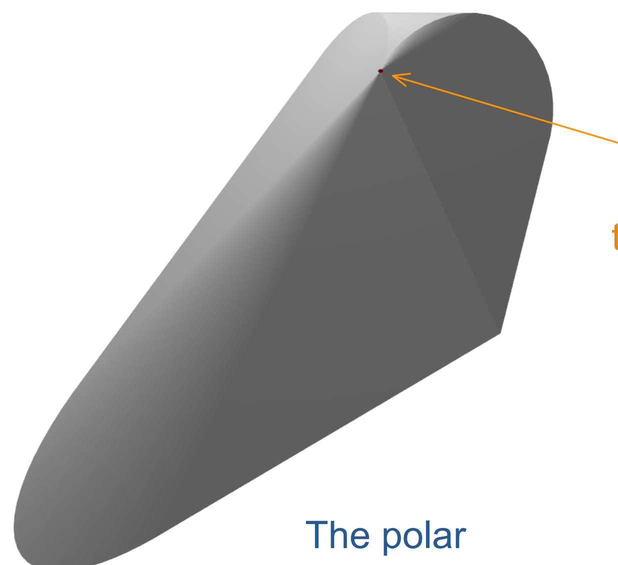
We modify the example to satisfy the new property, but so that it is still not nice.

Instead of the leftmost segment add a curve $\gamma(t) = (-t, -t^2, -t^3)$

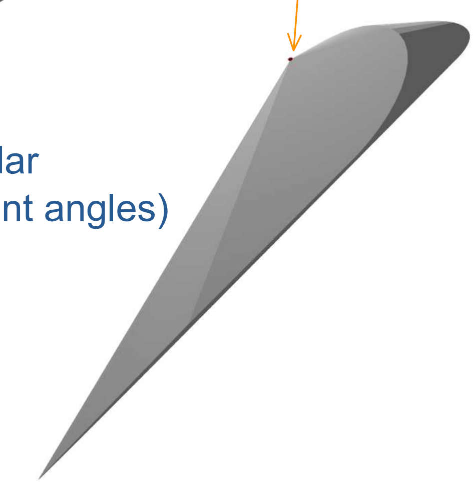
The primal set (shifted and rotated)



The polar
(scaled and seen from
the same angle as
the primal set above)

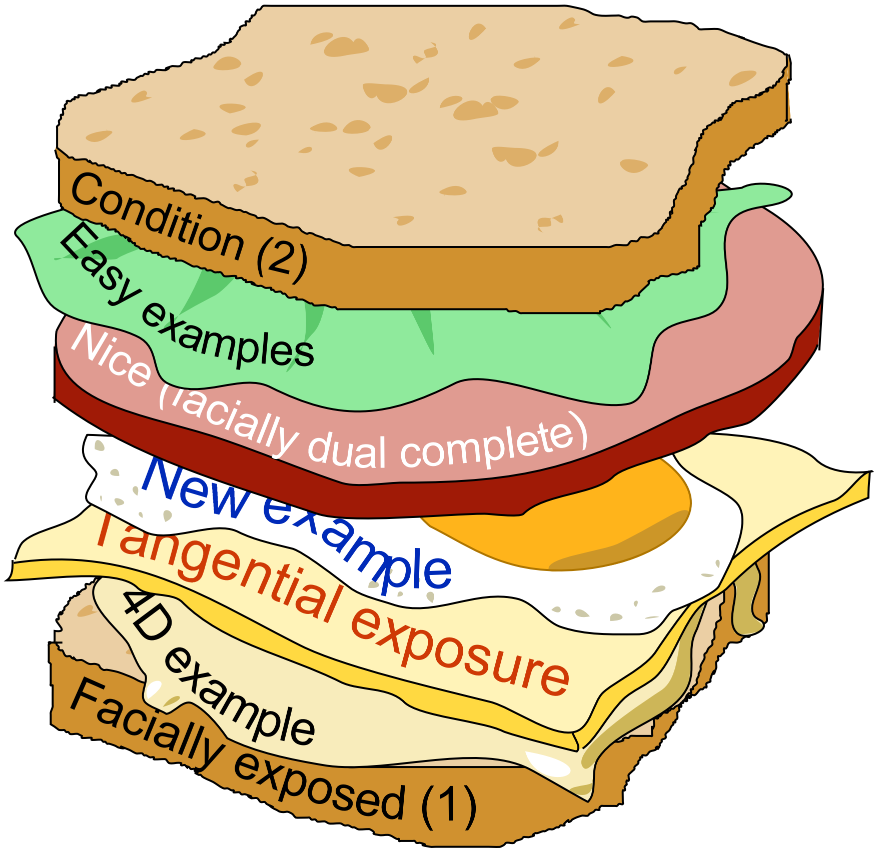


The dual face
that corresponds
to the 2D top face
of the primal set



The polar
(seen at different angles)

State of the sandwich



We have introduced a new condition of 'tangential exposure', which is sharper than facial exposedness, but yet is not equivalent to niceness. Can this be pushed further? The goal is to find an essentially **primal** characterisation of niceness, that is easy to check.