Finite Horizon Investment Risk Management

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Kelly's formula

Let $p = Prob(H)$ and $q = 1 - p = Prob(T)$ and let f be the bet as % of bankroll. Then expected gain per play in log scale is

$$l(f) = p\ln(1+f) + q\ln(1-f).$$

Solving $l'(f) = 0$ we have

Kelly's formula

The best betting size

$$\kappa = p - q.$$

Kelly's formula (picture)

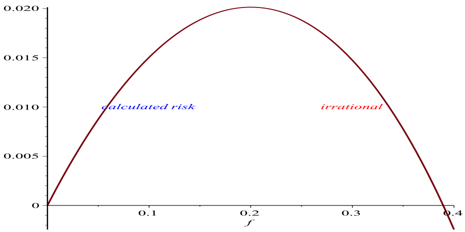

Figure : Log return curve

Edward O. Thorp

- Professor and hedge fund manager
- author of 1962 classic "Beat the Dealer" is still the standard reference of Blackjack player,
- in which he applied Kelly's formula to provide a guide to Blackjack betting size.

Edward O. Thorp

- He then generalized it to handle investment allocation with Kassouf in "Beat the market (1967)",
- which was dubbed 'fortunes formula' by Pounderstone in his NY Times best seller of the same title.
- The idea of statistic arbitrage in "Beat the market" also stimulated Black, Scholes and Merton to derive the Black-Scholes formula.

Limitations of the fortunes formula

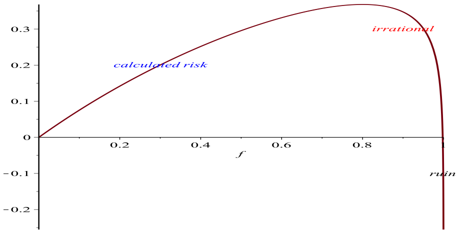

Figure : Log return curve of 9:1 coin flip

In practice

Practitioners know that one cannot use the full Kelly bet size. But there is no careful discussion on how to do it.

"if you bet half the Kelly amount, you get about three-quarters of the return with half the volatility. So it is much more comfortable to trade. I believe that betting half Kelly is psychologically much better." –Ed Thorp

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What is missing?

- Accurate only when the gambler playing forever.
- Risk is not adequately addressed.

Betting size as proxy of risk

- Maximum drawdown is largest relative percentage loss,
- a very important risk measure but hard to estimate.
- However, drawdown is approximately proportional to the bet size f.

Betting size as proxy of risk

Assuming sequence of consecutive returns (mostly losses) of $l_1, l_2, \ldots, l_m$ cause the maximum drawdown.
Then the max drawdown with betting size f is

$$\begin{aligned} & (1+fl_1)(1+fl_2)\ldots(1+fl_m)-1 \\ = \ & f\sum_{i=1}^{m} l_i + f^2 \sum_{1\le i<j\le m} l_i l_j + f^3 \sum_{1\le i<j<k\le m}^{m} l_i l_j l_k + \ldots. \end{aligned} \tag{1}$$

Usually $f, l_1, \ldots, l_m << 1$ and, therefore, drawdown $\sim \sum_{i=1}^{m} l_i f$.

Return curve in finite horizon

When we play Q games the total return is

$$r_Q(f) = \exp(Ql(f)) - 1.$$

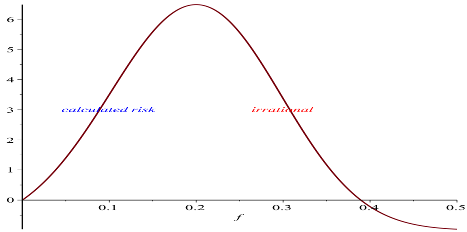


Figure : Return curve

Return / bet size optimal point

We can maximize $r_Q(f)/f$ as a proxy for the return/ drawdown ratio. Geometrically

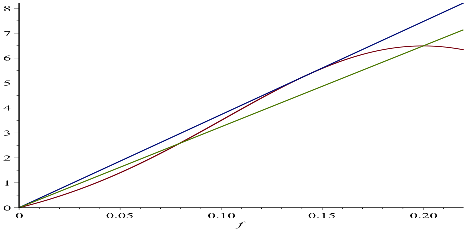


Figure : Return/ f maximum

Return / bet size optimal point

We can maximize $r_Q(f)/f$ as a proxy for the return/ drawdown ratio. Analytically: solve

$$\left(\frac{r_Q(f)}{f}\right)' = \frac{r_Q'(f)f - r_Q(f)}{f^2} = 0.$$

Equivalent to

$$r_Q'(f) - \frac{r_Q(f)}{f} = 0.$$

Solution depends on Q and we denote it ζ_Q.

Inflection point

Another important point is the inflection point

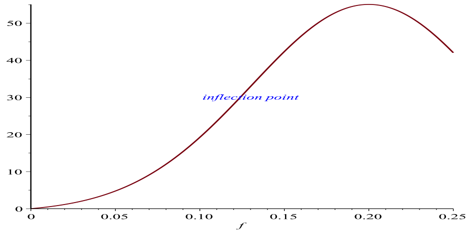


Figure : Inflection point

Importance: critical point for the marginal increase of return with respect to f.

Find inflection point

Solve equation

$$0 = r''_Q(f) = Q\exp(Ql(f))[Q(l'(f))^2 + l''(f)]$$

or equivalently

$$Q(l'(f))^2 + l''(f) = 0.$$

The inflection point also depends on Q and we denote it by ν_Q.

Relationship

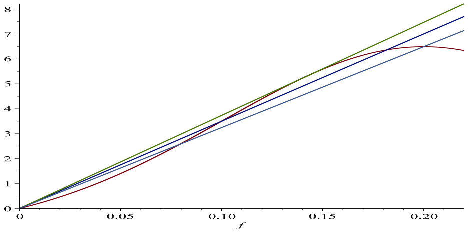


Figure : Return/size ratios as slopes of the top line at ζ_Q, middle line at ν_Q and bottom line at κ

$$\nu_Q < \zeta_Q < \kappa$$

Blackjack simulation: Main Rules

- Use six decks.
- Dealer stop at soft 17.
- Player may split once and double on split.

Blackjack simulation: Basic strategy

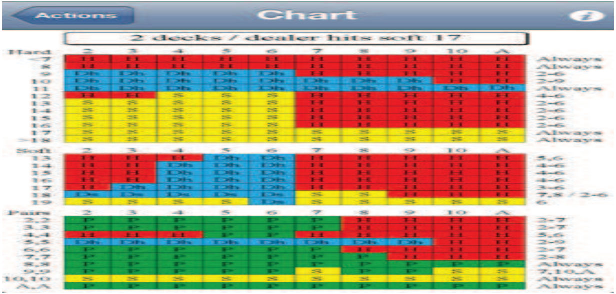


Figure : Basic Strategy

Blackjack simulation: Revere counting system

Lawrence Revere: Playing Blackjack as a Business

- Ace through Ten: -2 +1 +2 +2 +2 +2 +1 0 0 -2
- True Count Calculation: divide by full decks.
- Play only when true counts > 2.

Probability of different scenarios

a_n	Frequency $\sim p_n$
-4.000000	0.000206
-3.000000	0.001638
-2.000000	0.045842
-1.000000	0.425114
0.000000	0.090411
1.000000	0.319943
1.500000	0.051173
2.000000	0.063130
3.000000	0.002102
4.000000	0.000441

Table 1. Frequencies from ten million hands

Based on scenario probability

Q	ν	ζ	κ
15000	nonexist	nonexist	0.02539
20000	0.00093	0.0014	0.02539
25000	0.00351	0.0052	0.02539
30000	0.00542	0.0080	0.02539

Table 2. Optimal points at various horizons

Direct simulation

f	r_Q	$Drawdown$	Marginal r_Q	r_Q/f	$r_Q/Drawdown$
0.004	0.26	0.152	0.066	64.09	1.685
0.005	0.32	0.187	0.066	64.44	1.720
0.006	0.39	0.221	0.066	64.66	1.752
0.007	0.45	0.254	0.065	64.77	1.782
0.008	0.52	0.286	0.065	64.76	1.810
...	...	...	...	...	...
0.014	0.87	0.456	0.052	62.13	1.910
0.015	0.92	0.480	0.049	61.27	1.913
0.016	0.96	0.504	0.046	60.29	1.913
0.017	1.01	0.527	0.042	59.19	1.909

Table 3. Direct simulations

Direct simulation

0.023	1.16	0.647	0.013	50.35	1.789
0.024	1.17	0.664	0.007	48.56	1.754
0.025	1.17	0.681	0.002	46.69	1.714
0.026	1.16	0.697	-0.004	44.76	1.670

Table 3. Direct simulations (continued)

Model for multiple risky assets

- Investing in M assets/strategies represented by a random vector $X = (X_1, \ldots, X_M)$,
- with N different outcomes $\{b^1, \ldots, b^N\}$ where $b^n = (b_1^n, \ldots, b_M^n)$;
- for Q holding periods and suppose that $Prob(X = b^n) = p_n$.
- Define $w_m = \min\{b_m^1, \ldots, b_m^N\}$ and scale $Y = (-X_1/w_1, \ldots, -X_M/w_M)$;
- the scaled outcome is $a^n = (-b_1^n/w_1, \ldots, -b_M^n/w_M)$ with $Prob(Y = a^n) = p_n$.

Leverage space

- Each allocation is represented by $f = (f_1, \ldots, f_M) \in [0,1]^M$,
- where f_m represents shares in the mth asset, $[0,1]^M$ the leverage space.
- Define the log return function

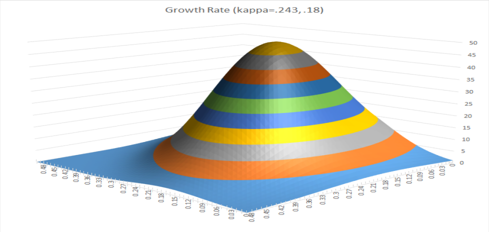
$$l_Y(f) := \sum_{n=1}^{N} p_n \ln(1 + f \cdot a_n) \tag{2}$$

- Then Q-period return is $r_Q(f) = \exp(Q l_Y(f)) - 1$.
- $r_Q(f)$ attains a unique maximum κ (Kelly optimal) determined by

$$\nabla l_Y(f) = 0.$$

Total return surface

Log return function $l_Y(f)$ is convex but $r_Q(f)$ may not be. The following is the return of playing two coins for $Q = 50$ times: Coin 1 is .50/.50 that pays 2:1, and Coin 2 is .60/.40 that pays 1:1.



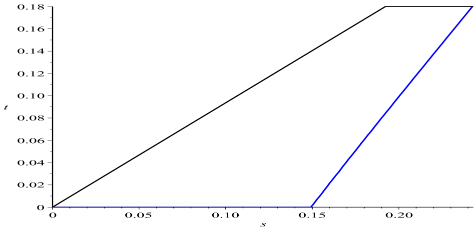
Return/ Risk Paths

One usually allocate in between 0 (too conservative) and Kelly optimal κ (too aggressive). A return / risk path $f : [a, b] \to R^M$ is defined by the following properties

1. f is piecewise C^2.
2. $f(a) = 0$ and $f(b) = \kappa$.
3. $t \mapsto r_Q(f(t))$ is increasing on $[a, b]$.
4. There is a risk measure m such that $t \mapsto m(f(t))$ is increasing.

Following the Return/ Risk Paths

In theory one can use the one asset method on such a return/risk path. However, such path are not unique, even if we insist on path that optimize return / risk on every level of return. The following are two corresponding to the two coin flipping game. Blue path assuming drawdown completely correlate and **Black path** completely independent.



Difficulty in Following the Return/ Risk Paths

- Computing for each risk measure the optimal path is rather costly.
- In general, it is more efficient in following a few heuristically determined paths among infinitely many possible;
- Even so use the one-dimensional method to each of such path is cumbersome.

So we turn to determine the manifolds of inflection points and return / risk maximum points.

Manifold of inflection points

This manifold can be determined by using the Sylvester's criterion for negative definite matrix on the hessian of $r_Q(f)$.
The limitation is that computation is too costly when M is large.
A practical (conservative) approximation is

$$\left\{ f \in [0,1]^M : \max\left[\frac{\partial^2 l_Y(f)}{\partial f_n^2} + Q\left(\frac{\partial^2 l_Y(f)}{\partial f_n^2}\right)^2, n = 1, \ldots, M \right] = 0 \right\}.$$

Manifold of return / risk maximum points

It turns out this manifold has a clean characterization:

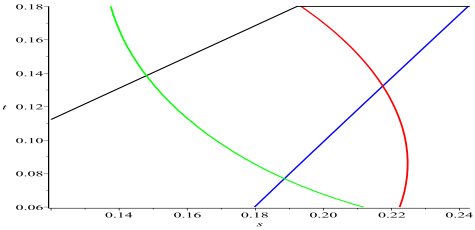
Return / risk maximum points

Let $m(f)$ be a risk measure homogeneous in f. Then the set of allocations f that maximizes $r_Q(f)/m(f)$ is represented by

$$\{f : \langle \nabla r_Q(f), f \rangle = r_Q(f)\}.$$

Example

This is another look of the two coin flip example: as before Blue path assuming drawdowns completely correlate and **Black path completely independent**. We added Green curve– manifold of inflection points and Red curve – manifold of return /risk maximization points.



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THANK YOU

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THANK YOU