When does an arithmetic progression through an infinite sequence remember the whole sequence?

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HECKE OPERATORS

History: Hecke and Atkin introduced a very useful class of linear operators that act on modular forms.

Here we use some of their linear operators, but we replace the space of modular forms with the space of Hypergeometric series, and we let the Hecke operators U_n act on these sets of functions instead.

Definition.

Given a power series $f(x) = \sum_{k=1}^{\infty} c_k x^k$,

we define the Hecke U_n operator as follows:

$$U_n(f)(x) = \sum_{k=1}^{\infty} c_{nk}\, x^k,$$

for any positive integer n.

We may also think of the operator as U_p acting on an infinite vector (any input signal of complex numbers) as a "filter":

$$(c_0, c_1, c_2, \ldots, c_k, \ldots,)$$

$$\to (c_0, c_p, c_{2p}, \ldots, c_{kp}, \ldots,)$$

Sometimes this operator is called the "forgetful" operator because it forgets about a lot of the coefficients, except for an arithmetic progression's worth of them; that is, only every n'th coefficient is remembered.

A natural question, already raised by Hecke himself, is

"when does the forgetful operator in fact remember everything - i.e. the whole function?"

This question can be phrased in terms of eigenvalues:

For which Hypergeometric functions f is it true that

$$U_n(f) = \lambda f \quad ?$$

Also, classify the eigenvalues λ of all the Hecke operators U_n, for $n \in \mathbb{Z}_{\geq 0}$.

These operators are very useful in studying:

1. Completely multiplicative functions on the integers.

2. Bases for various spaces of modular forms in number theory.

3. Relationships/identities between various theta functions and other modular forms.

4. Useful Bases for rational functions in one variable.

5. Relationships between some Hypergeometric functions.

Formal Power Series

Assume U_n has an eigenfunction f, a formal power series of the form

$$f(x) = x^j \sum_{k=0}^{\infty} a_k x^k,$$

with eigenvalue λ.

If n divides j, then we conclude that $j = 0$ and $\lambda = 1$.

If n does not divide j, then we conclude that $j = 1$.

Definition of a Hypergeometric function

Every hypergeometric function that we consider has, by definition, a canonical Taylor series representation of the form

$$ {}_pF_q(\mathbf{a}, \mathbf{b}; x) := a_0 x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \, (a_2)_k \, \cdots \, (a_p)_k}{(b_1)_k \, (b_2)_k \, \cdots \, (b_q)_k} \frac{x^k}{k!}, $$

where $j \in \mathbb{Z}_{\geq 0}$, and $\mathbf{a} := (a_1, a_2, \cdots, a_p) \in \mathbb{C}^p$ $\mathbf{b} := (b_1, b_2, \cdots, b_q) \in \mathbb{C}^q$ are the *parameters* of ${}_pF_q$.

These parameters satisfy a_i, $b_i \neq 0$ and $-b_i \notin \mathbb{N}$. Here a_0 is any complex constant, and

$$(c)_k := c(c+1)(c+2)\cdots(c+k-1),$$

and where $(c)_0 := 1$. Here $(c)_k$ is called the ascending factorial symbol. For example $(1)_k = k!$, and $(0)_k = 0$.

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Examples of Hypergeometric functions

for any nonzero $a \in \mathbb{C}$. The error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

can also be represented as a hypergeometric function, namely

$$\mathrm{erf}(x) = \frac{2x}{\sqrt{\pi}} {}_1F_1(\frac{1}{2}, \frac{3}{2}; x).$$

The exponential function e^x.

The logarithm $ln(1+x)$.

The Polylog functions $\sum_{k=1}^{\infty} \frac{1}{k^n} x^k$, for each $n \in \mathbb{Z}_{\geq 1}$.

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Example.

Is the Hypergeometric function

$$Polylog_j(x) = \sum_{k=1}^{\infty} \frac{1}{k^j} x^k$$

an eigenfunction of U_n?

Well, let's see -

$$U_n(Polylog_j)(x) = \sum_{k=1}^{\infty} \frac{1}{(kn)^j} x^k$$

$$= \frac{1}{(n)^j} \sum_{k=1}^{\infty} \frac{1}{(k)^j} x^k$$

$$= n^{-j} Polylog_j(x).$$

So yes it is......

The strange thing is that there are NO OTHER Hypergeometric eigenfunctions. (!)

This is the main result of the recent paper "Hecke operators acting on Hypergeometric functions", by V. Moll, S. Robins, and K. Soodhalter, Journal of the Australian Mathematical Society, 2010.

Back to Formal power series

Lemma.

Let $j,\, n \in \mathbb{N}$. Then

$$U_n\left(x^j \sum_{k\geq 0} a_k x^k\right) = \begin{cases} x^{1+\lfloor j/n \rfloor} \sum a_{n(k+1-\{j/n\})} x^k \\ \quad \text{if } n \text{ does not divide } j \\ \\ x^{\lfloor j/n \rfloor} \sum a_{kn} x^k \\ \quad \text{if } n \text{ divides } j, \end{cases}$$

where the sums are over $k \geq 0$.

Formal power series

Theorem.

Assume U_n has an eigenfunction of the form

$$f(x) = x^j \sum_{k=0}^{\infty} a_k x^k,$$

with eigenvalue λ. If n divides j, then we conclude that $j = 0$ and $\lambda = 1$. If n does not divide j, then we conclude that $j = 1$.

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Action of U_n on Hypergeometric series

Theorem.

(a)

If n divides j, we have

$$U_n\left(x^j\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_{q+1})_k}x^k\right)=x^{j/n}\sum_{k=0}^{\infty}n^{nk(p-q-1)}\frac{(c_1)_k(c_2)_k\cdots(c_{np})_k}{(d_1)_k(d_2)_k\cdots(d_{n(q+1)-1})_k}\frac{x^k}{k!}$$

where we define the parameters

$$\text{(5.3)}\qquad \begin{aligned} c_{in+l} &= \frac{a_{i+1}+l-1}{n}, &&\text{for } 0\le i\le p-1,\ 1\le l\le n\\ d_{in+l} &= \frac{b_{i+1}+l-1}{n}, &&\text{for } 0\le i\le q,\ 1\le l\le n. \end{aligned}$$

(b)

If n does not divide j, we have

$$U_n\left(x^j\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_{q+1})_k}x^k\right)=x^{1+\lfloor j/n\rfloor}\sum_{k=0}^{\infty}n^{nk(p-q-1)}\frac{(c_1)_k\cdots(c_{pn})_k}{(d_1)_k\cdots(d_{(q+1)n-1})_k}\frac{x^k}{k!}$$

where we now define the parameters

$$(5.4)\qquad \begin{aligned} c_{in+l} &= \frac{a_{i+1}+r+l}{n}, &&\text{for } 0\le i\le p-1,\ 1\le l\le n\\ d_{in+l} &= \frac{b_{i+1}+r+l}{n}, &&\text{for } 0\le i\le q,\ 1\le l\le n \end{aligned}$$

and $r = n(1-\{j/n\})-1.$

Theorem.

The parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ satisfy

$$\sum_{i=1}^{np} c_i - \sum_{i=1}^{n(q+1)-1} d_i = \sum_{i=1}^{p} a_i - \sum_{i=1}^{q} b_i + \frac{(n-1)}{2}(p-q-1).$$

Corollary.

Suppose that we have $p = q + 1$.

Then U_n preserves the quantity $\sum a_i - \sum b_i$.

Lemma.
(the case of f having a nonzero constant term)

Assume that n divides j and that we have a solution to the eigenvalue equation $U_n(f) = \lambda f$ for a Hypergeometric function f.

Then we have the combinatorial restriction

$$\bigcup_{i=1}^{p} \left\{ a_i, \frac{b_i}{n}, \frac{b_{i+1}}{n}, \ldots, \frac{b_{i+n-1}}{n} \right\}$$

$$= \bigcup_{i=1}^{p} \left\{ b_i, \frac{a_i}{n}, \frac{a_{i+1}}{n}, \ldots, \frac{a_{i+n-1}}{n} \right\}$$

It turns out that this combinatorial condition implies that

$$\sum_{i=1}^{p} a_i^k = \sum_{i=1}^{p} b_i^k,$$

for all $k \geq 1$.

That is, all of the MOMENTS are the same.

By a standard statistical argument involving the moment generating function of the a_i's and the moment generating function of the b_i's, we arrive at the conclusion that the sets are equal.

Theorem.

Suppose there exists an eigenfunction of U_n of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_{q+1})_k} x^k$$

corresponding to the eigenvalue λ.
Then $\lambda = 1$, $p = q + 1$ and $a_i = b_i$ for all i.

Therefore we conclude that $f(x) = \frac{1}{1-x}$.

Theorem.

Let let $f(x) = \sum_{k=1}^{\infty} c_k x^k$ be a hypergeometric function.
(Note that f has no constant term)
Then f is an eigenfunction of the Hecke operator U_n
for some n if and only if

$$f(x) = C \sum_{k=1}^{\infty} k^a x^k,$$

with $a \in \mathbb{Z}$ and $C \in \mathbb{C}$. In other words, f is a polylogarithm,
or $f = \left(x \frac{d}{dx}\right)^a \left(\frac{1}{1-x}\right)$.

Theorem. When the base space is the set of Hypergeometric functions, we have

$$\mathrm{Spec}(U_n) = \{n^a | a \in \mathbb{Z}\}.$$

Which Hypergeometric functions are simultaneously eigenfunctions of U_n for all n?

Here is the complete description of all the Hecke simultaneous eigenfunctions.

Theorem.

Let let $f(x) = \sum_{k=1}^{\infty} c_k x^k$ be a hypergeometric function. (Note that f has no constant term)
Then f is a simultaneous eigenfunction for the set of all Hecke operators $\{U_n\}_{n=1}^{\infty}$ with respective eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ if and only if

$$f(x) = C \sum_{k=1}^{\infty} k^a x^k,$$

with $a \in \mathbb{Z}$ and $C \in \mathbb{C}$. In other words, f is a polylogarithm, or $f = \left(x \frac{d}{dx}\right)^a \left(\frac{1}{1-x}\right)$.

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In other words, if f is an eigenfunction of any particular Hecke operator U_n, then it is automatically an eigenfunction of all of the Hecke operators.

This fact lies in sharp contrast with the space of rational functions, and also with the space of modular forms on any arithmetic subgroup of $SL_2(Z)$.

Application to completely multiplicative functions

Theorem.

Given any Hypergeometric coefficient (with parameters as above)

$$C(n) = \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}},$$

we conclude that $C(n)$ is a completely multiplicative function of n if and only if it is of the form

$C(n) = K\ n^a$ for some $a \in \mathbb{Z}$ and $K \in \mathbb{C}$.

Rational functions

We now replace the underlying space by the vector space of rational functions.

Theorem.

Let $F(x) = \sum_{k=1}^{\infty} c_k x^k$ be a rational function of x.

Suppose that F is an eigenfunction of the linear operator U_p. Then:

$$c_k = k^{M-1} \sum_{j=1}^{p^{ord(p,L)-1}} e^{2\pi i \frac{kp^j}{L}},$$

for some integers M and L.

Thus, it follows that all poles must have the same multiplicity M, and must all lie on the unit circle at the L'th roots of unity.

Which rational functions are simultaneously eigenfunctions of U_n for all n?

Theorem. Let

$$F(x) = \sum_{k=0}^{\infty} c_k x^k$$

be a *simultaneous* rational eigenfunction of all of the operators U_p, for $p = 2, 3, 4, \ldots$

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Then the coefficients are completely multiplicative, and must have the form:

$$c_k = \chi(k)\, k^{M-1}$$

where M is the common multiplicity of all of the poles of F, and where χ is a "Dirichlet character" on the integers.

Open problems

What are the eigenfunctions of U_n that form a sum of two Hypergeometric functions ?

1. In other words, when is the function ${}_aF_b + {}_cF_d$ an eigenfunction of U_p ?

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2. What happens if we replace the space of rational functions (or hypergeometric functions) by a more general space yet? For example, consider the space of functions of z which are entire for $|z| < 1$, and whose only possible poles are restricted to be roots of unity. What are the eigenfunctions and eigenvalues of U_p here?

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3. What is the growth of the Taylor series coefficients of a rational eigenfunction of U_p ?

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3. What is the growth of the Taylor series coefficients of a rational eigenfunction of U_p ?

4. What happens if we replace the underlying space by the space of rational functions over a finite field?

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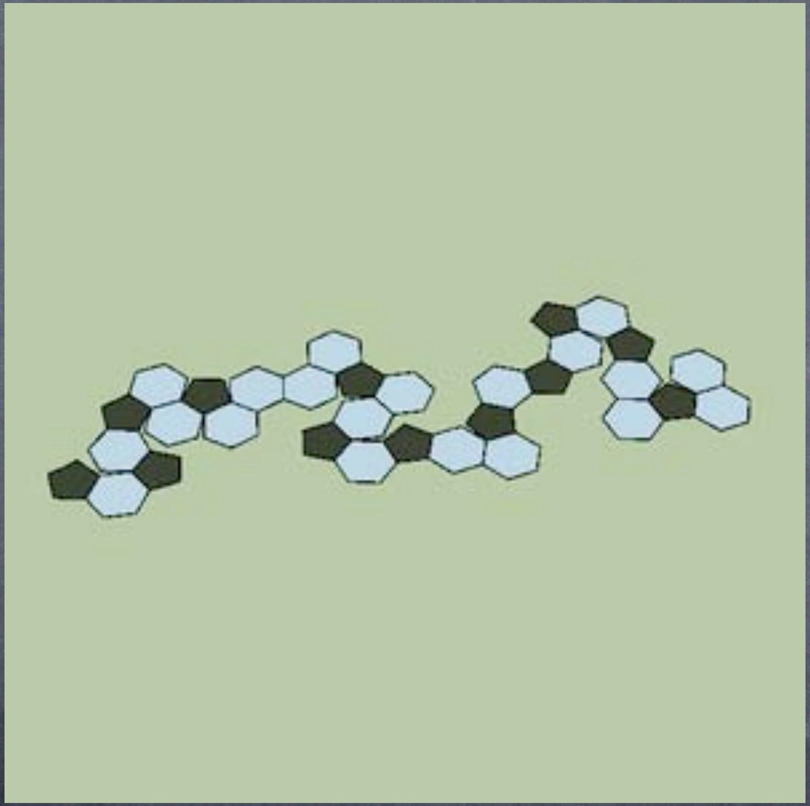
Thank you



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