Polyhedral cones, their theta functions, and what they say about polytopes

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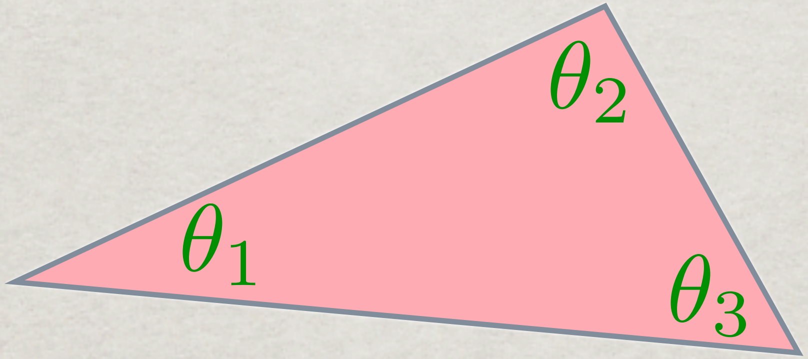
Our grade school triangle experience:



$$\theta_1 + \theta_2 + \theta_3 = \frac{1}{2}$$

The sum of all three angles at the vertices of a triangle is equal to 1/2.

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How does this elementary result extend to higher dimensions?

1. What is a polygon in higher dimensions?

2. What is an angle in higher dimensions?

3. How do these relations among angles extend to higher dimensions?

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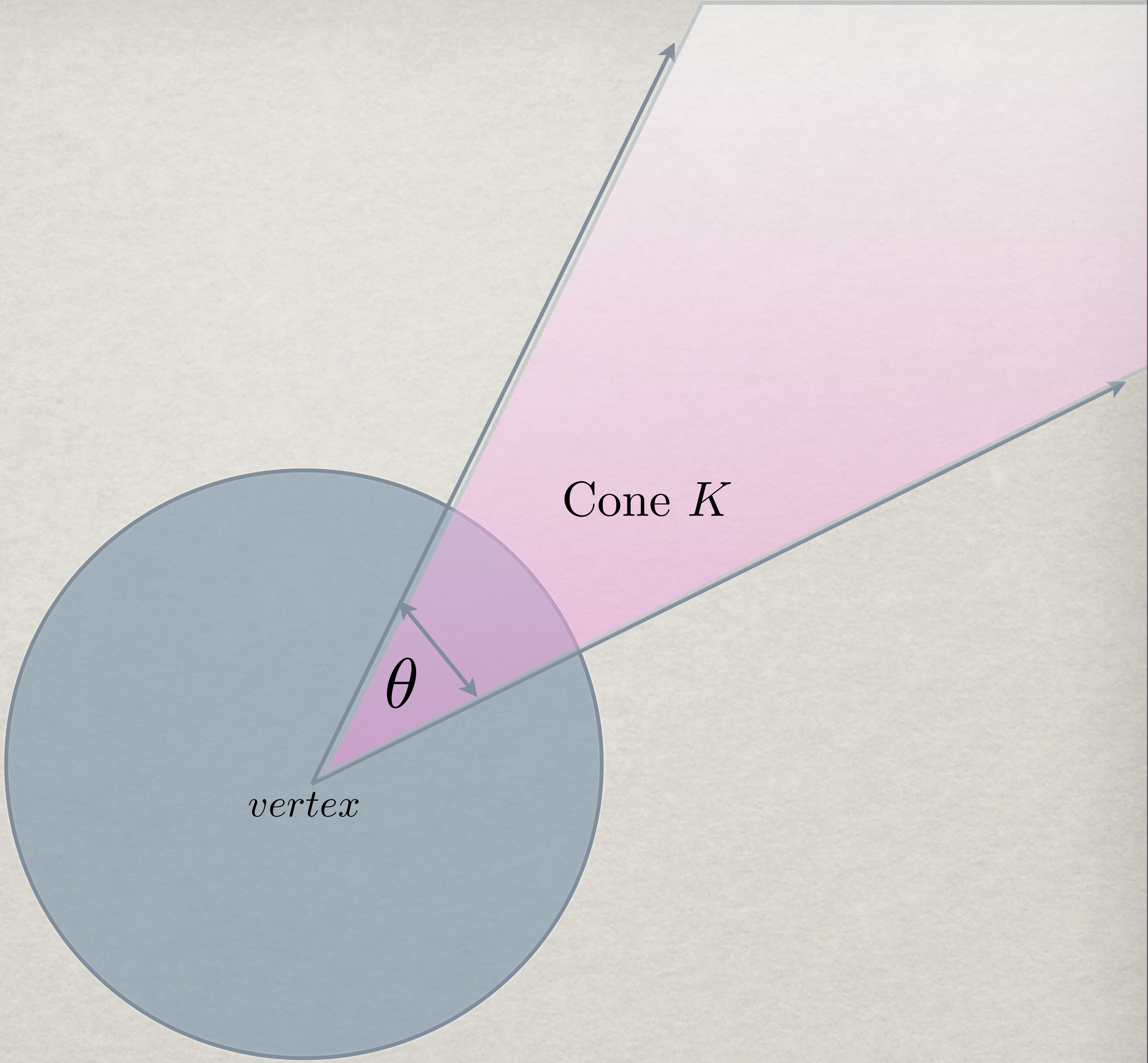

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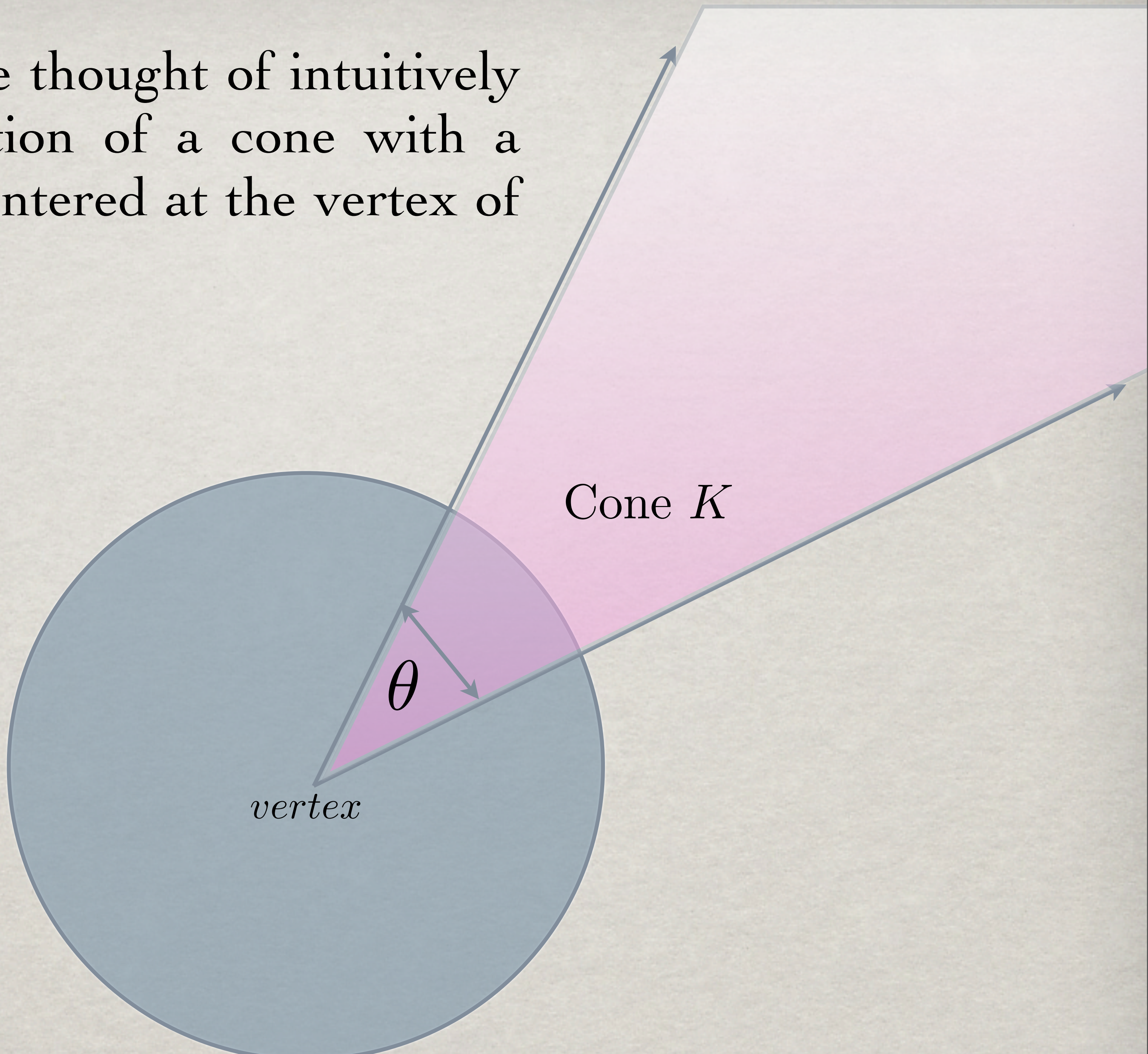
An angle can be thought of intuitively as the intersection of a cone with a small sphere, centered at the vertex of the cone.



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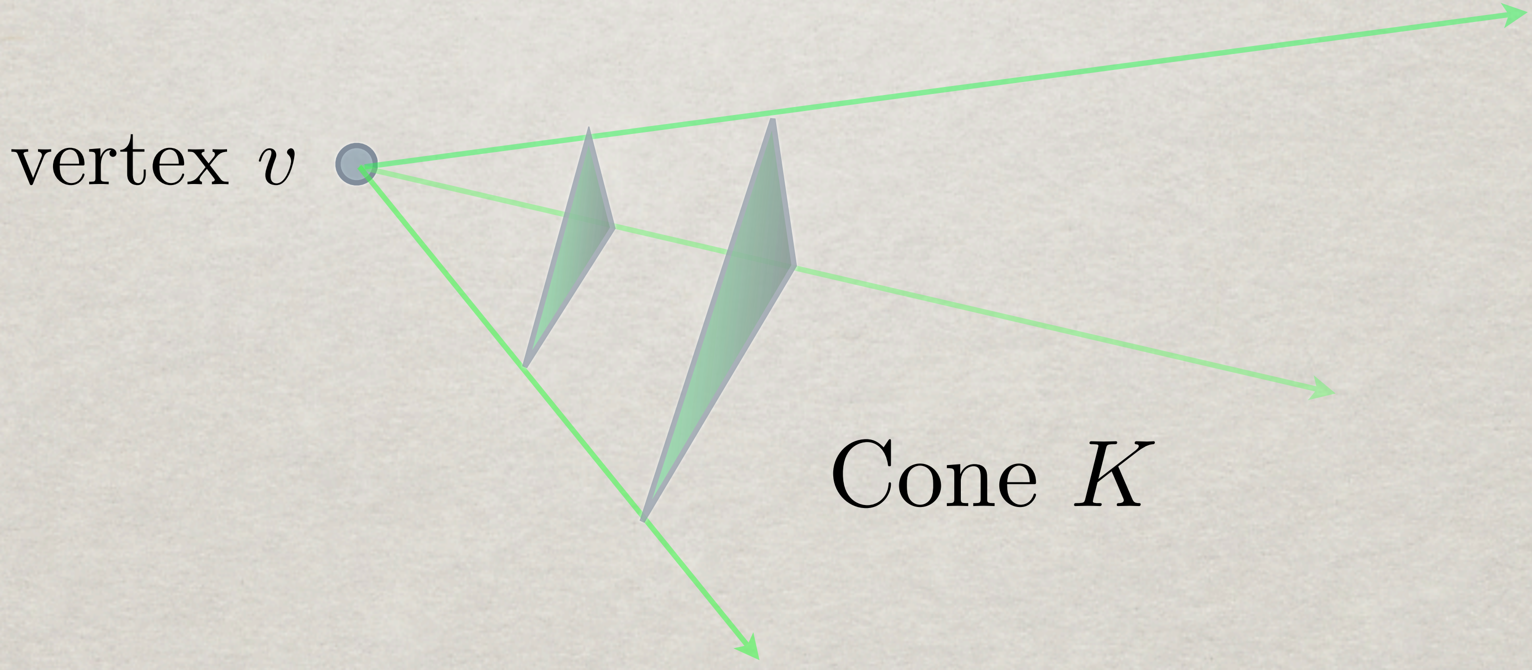
A cone $K \subset \mathbb{R}^d$ is the nonnegative real span of a finite number of vectors in $\mathbb{R}^d$.

That is, a cone is defined by

$$K = \{\lambda_1 w_1 + \cdots + \lambda_d w_d | \text{ all } \lambda_j \geq 0\},$$

where we assume that the edge vectors $w_1, \ldots, w_d$ are linearly independent in $\mathbb{R}^d$.

Example: a 3-dimensional cone.



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How do we describe a higher dimensional angle analytically, though?

A nice analytic description is given by:

A two dim'l angle $= \int_K e^{-\pi(x^2+y^2)} dxdy$

The solid angle at the vertex of a cone K is

$$\omega_K = \int_K e^{-\pi||x||^2} dx.$$

A solid angle in dimension d is equivalently:

1. The proportion of a sphere, centered at the vertex of a cone, which intersects the cone.

2. The probability that a randomly chosen point in Euclidean space, chosen from a fixed sphere centered at the vertex of K, will lie inside K.

3. A solid angle $= \int_K e^{-\pi ||x||^2} dx$

4. The volume of a spherical polytope.

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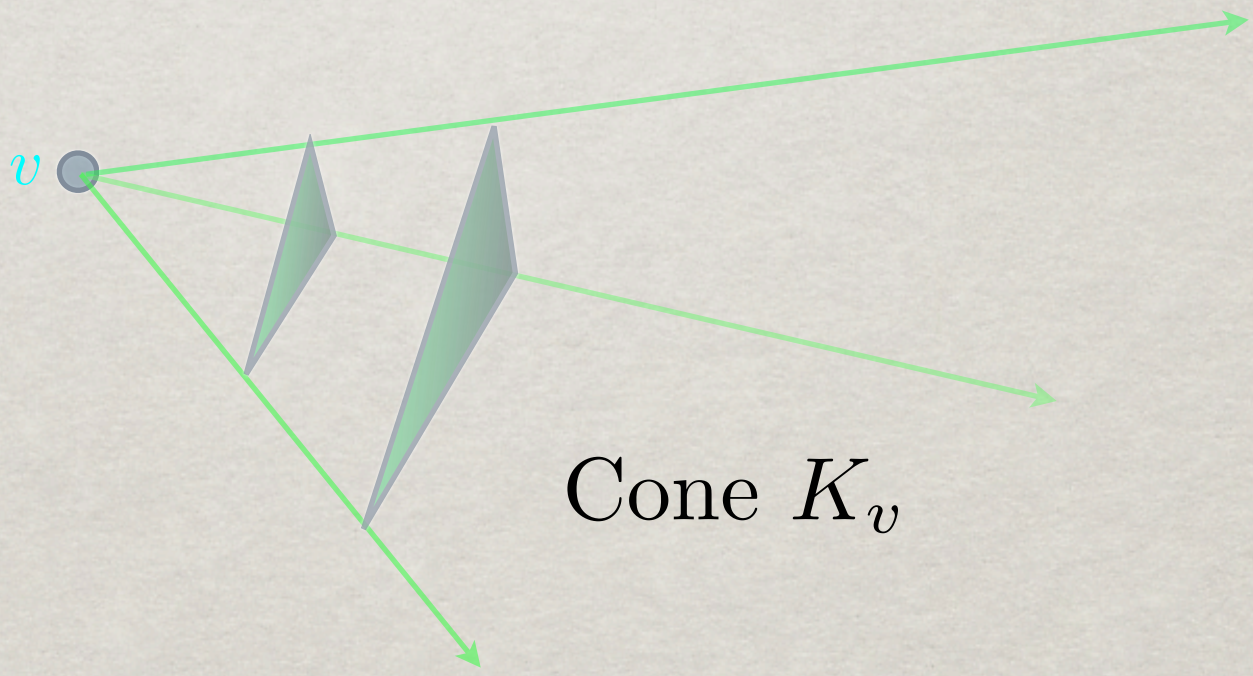
Example: defining the solid angle at a vertex of a 3-dimensional cone.



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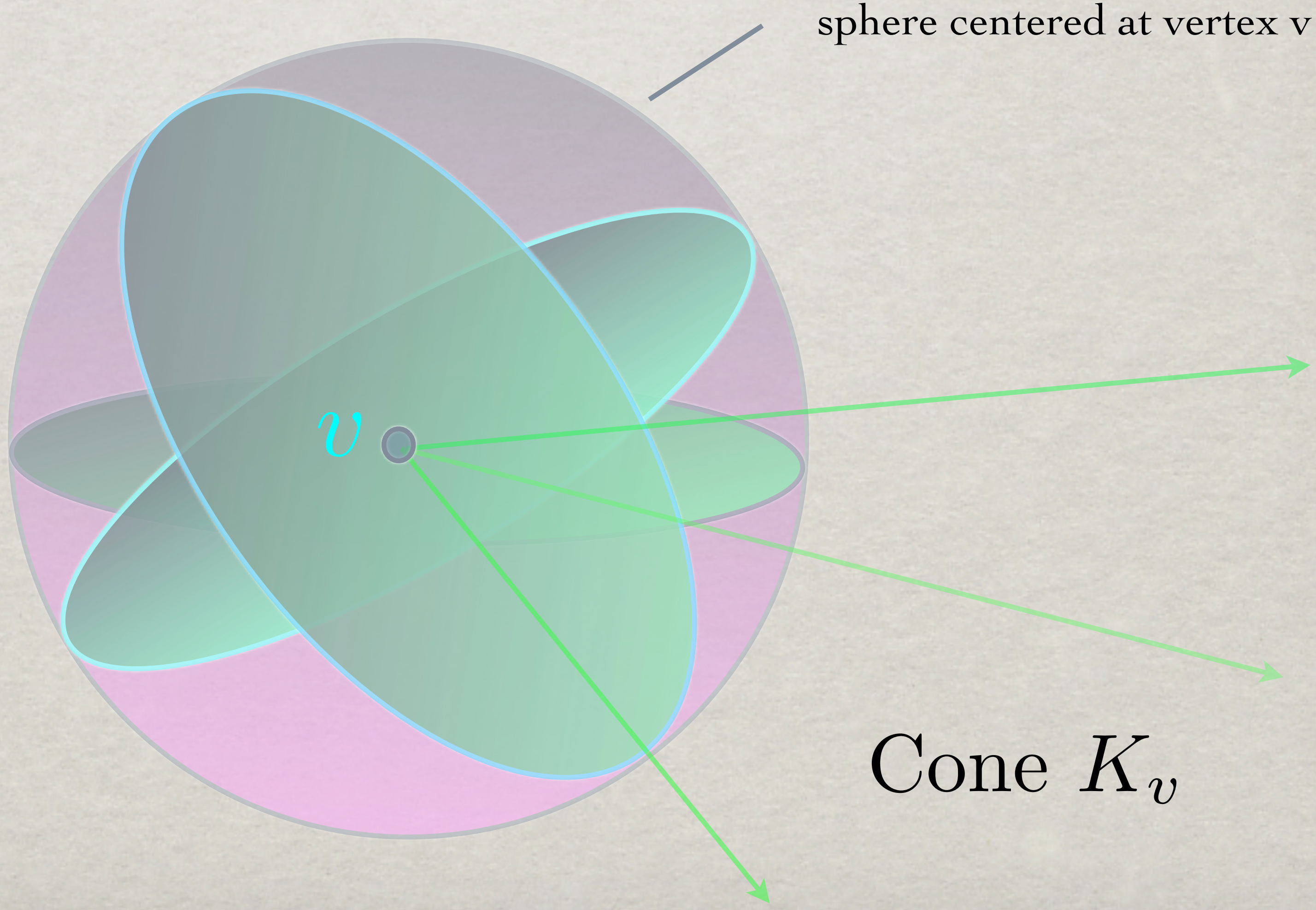
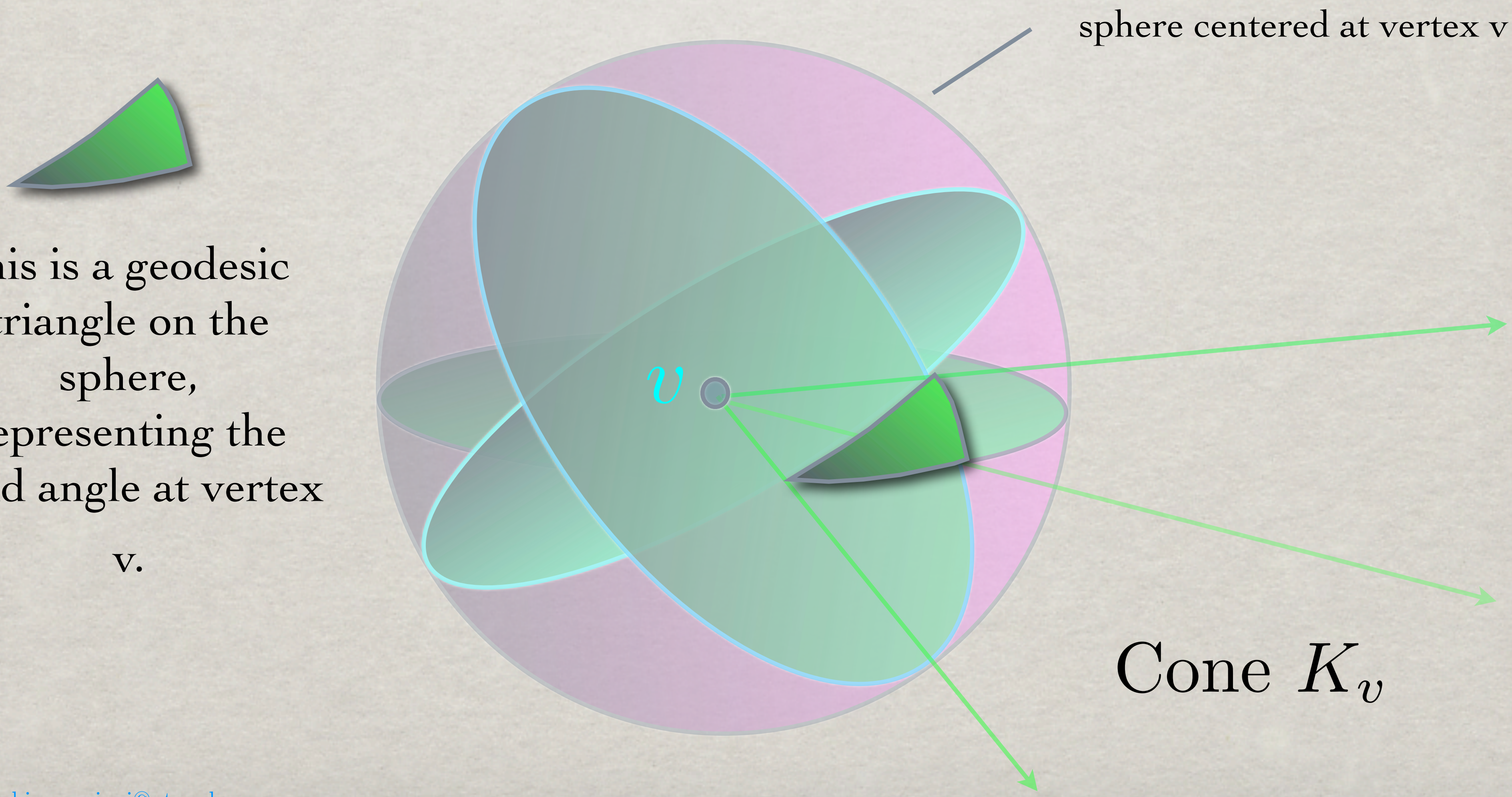
Example: defining the solid angle at a vertex of a 3-dimensional cone.



Example: defining the solid angle at a vertex of a 3-dimensional cone.



this is a geodesic triangle on the sphere, representing the solid angle at vertex v.

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The moral: a solid angle in higher dimensions is really the volume of a spherical polytope.

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We need to extend the notion of a solid angle at a vertex of a cone to the more general notion of a solid angle at ANY point, relative to any polytope P.

Given any convex polytope P, we define the solid angle

$$\omega_P(x) = \lim_{\epsilon \longrightarrow 0} \frac{vol(S_\epsilon(x) \cap P)}{vol(S_\epsilon)}$$

where S_ϵ is a sphere of radius ϵ, centered at x.

This can also be thought of as the proportion of a small sphere, centered at x, that lies inside P.

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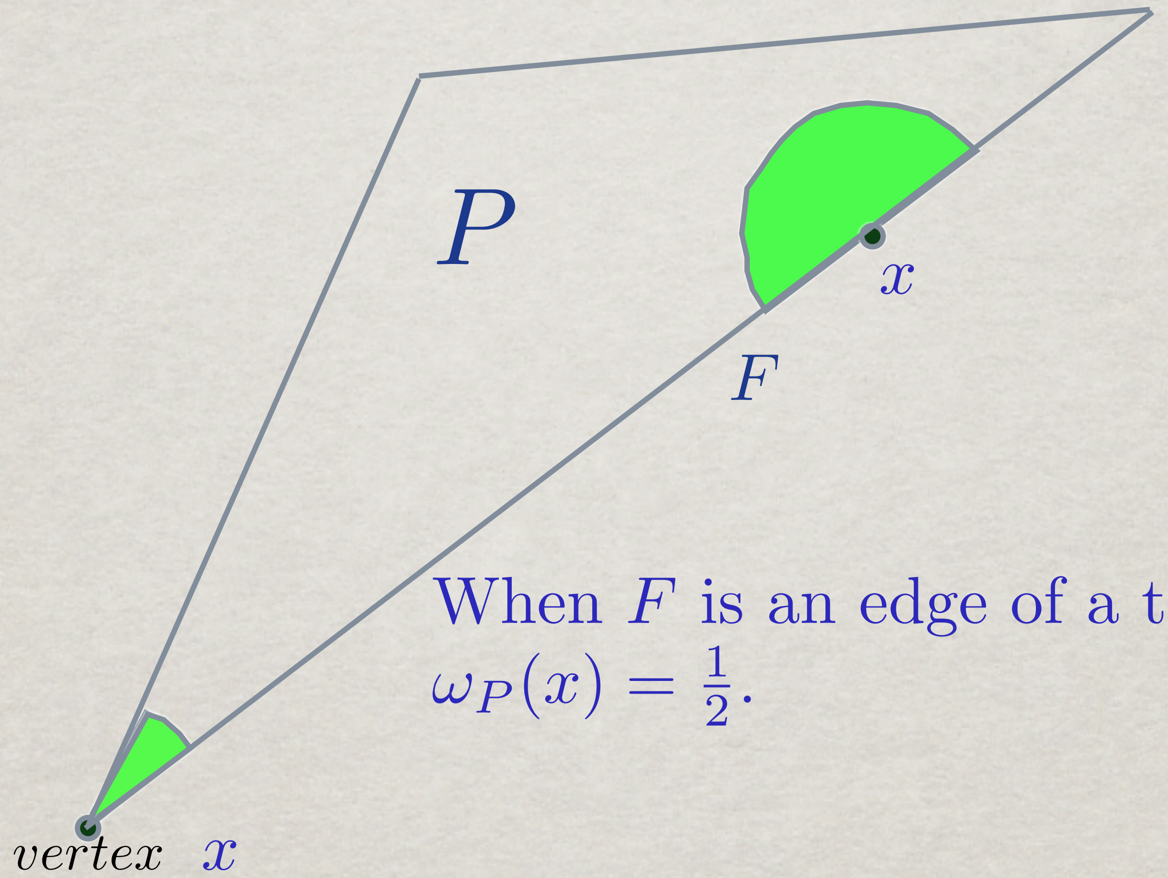
Example. Let P be a triangle.



When F is an edge of a triangle, we get $\omega_P(x) = \frac{1}{2}$.

When F is a vertex of a triangle, we get $\omega_P(x) = \theta$.

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DEFINITION. THE **SOLID ANGLE OF A FACE F** OF A POLYTOPE P IS GIVEN BY THE PROPORTION OF A SMALL SPHERE, CENTERED AT ANY POINT

$x \in$ **THE RELATIVE INTERIOR OF F.**

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Jorgen Gram

1850-1916
Born in Denmark

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Yes, the same Gram, but a different theorem.

THEOREM. (J. GRAM, CIRCA 1860)

(THE GRAM RELATIONS)

Given a convex polytope P, we have the following linear equality for the solid angles of its faces:

$$\sum_{F \subset P} (-1)^{dim F} \omega_F = 0.$$

Note: all sums include the face $F = P$.

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Why is the Gram theorem really a d-dimensional extension of our 2-dimensional elementary school theorem?

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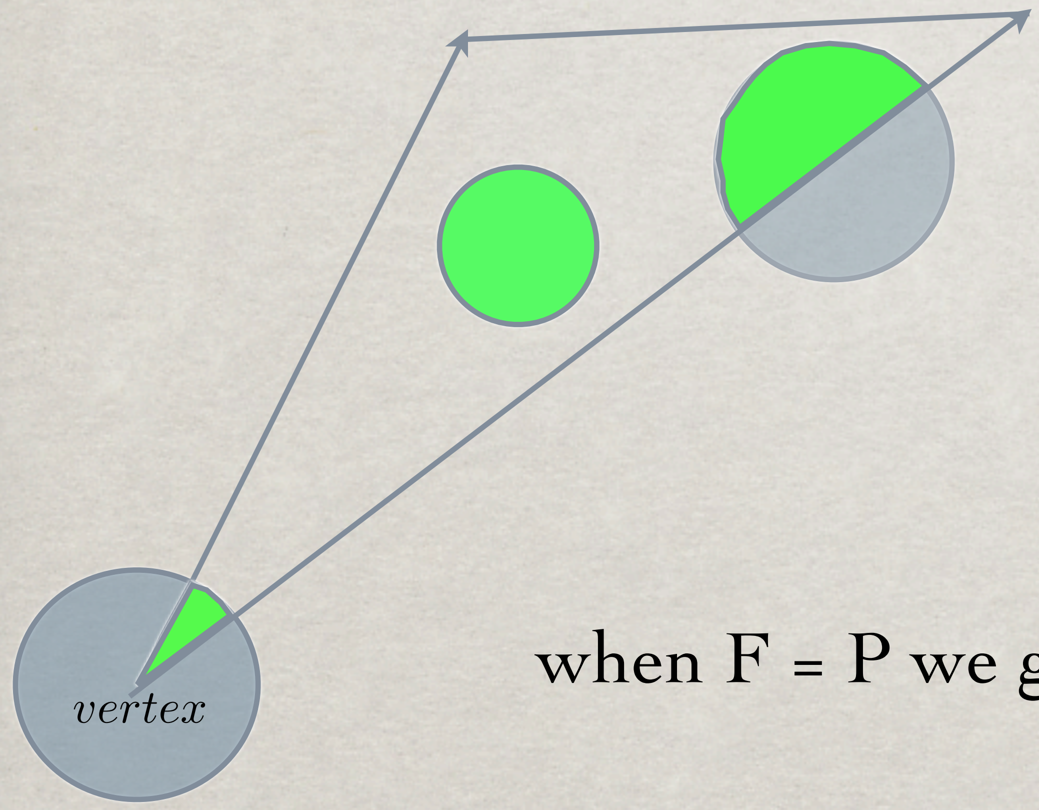
For a triangle P, the Gram relations give

$$0 = (-1)^0(\omega_{v_1} + \omega_{v_2} + \omega_{v_3}) + (-1)^1(1/2 + 1/2 + 1/2) + (-1)^2(1)$$

$$= \theta_1 + \theta_2 + \theta_3 - 1/2.$$



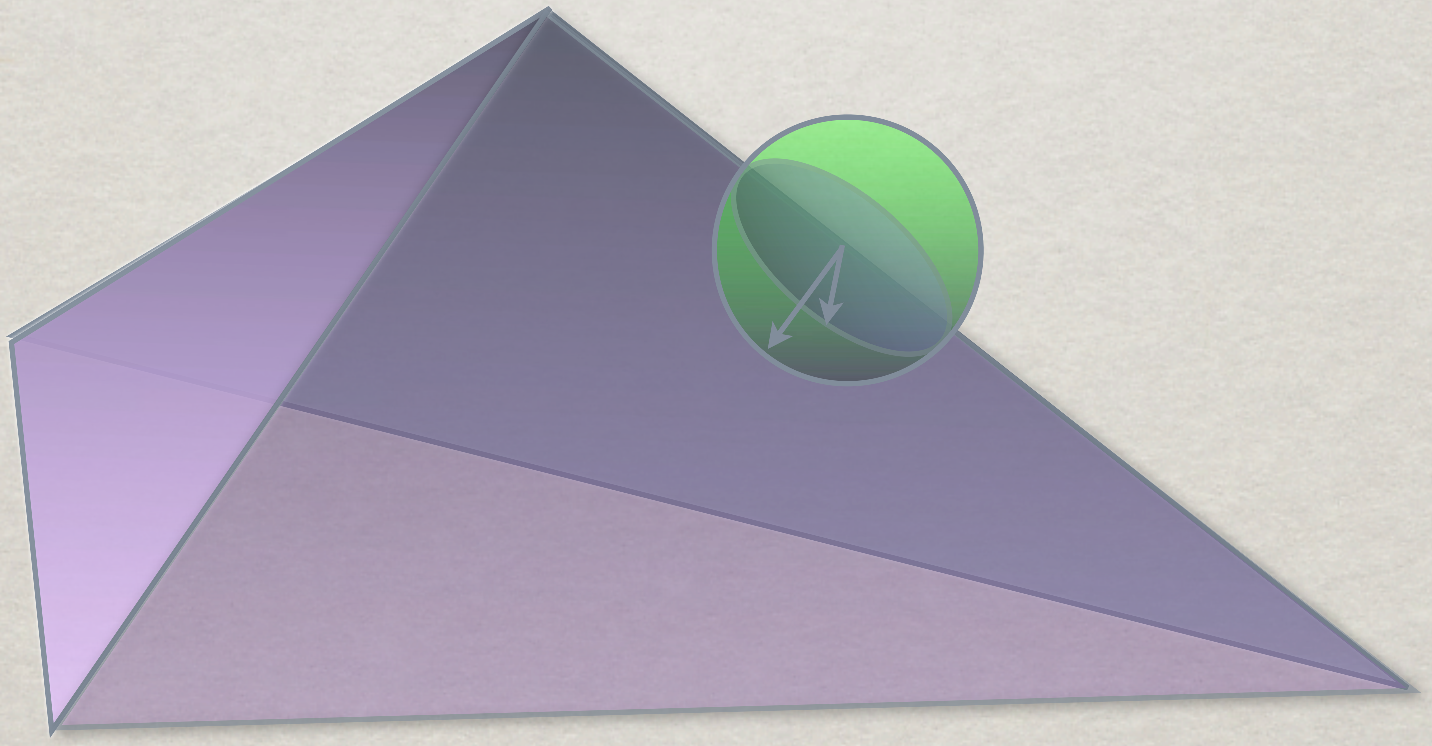
When F is an edge, we get $\omega_F = 1/2$

when F = P we get $\omega_P = 1.$

When F is a vertex, we get $\omega_F = \theta.$

Example of a solid angle of a face F - an edge in this case - relative to a polytope P.

For a 3-simplex P, we have the following picture for the solid angles of one of its edges:



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To help us analyze solid angles, We have introduced the following Conic theta function for a any cone K:

Definition.

$$\theta_K(\tau) = \sum_{m \in \mathbb{Z}^d \cap K} e^{\pi i \tau ||m||^2},$$

where τ is in the upper complex half plane.

Why do we define the conic theta function in this way?

One strong motivation comes from simply discretizing the integral that defines a solid angle!

WE OBSERVE THAT THERE IS A SIMPLE BUT VERY USEFUL ANALYTIC LINK BETWEEN SOLID ANGLES AND THESE CONIC THETA FUNCTIONS, GIVEN BY:

Lemma. For any cone K with vertex v,

$$\theta_K(i\epsilon) \approx \epsilon^{-d/2}\omega_K(v),$$

asymptotically as $\epsilon \to 0$.

Another motivator / teaser : There are many identities among these conic theta functions. For example:

Theorem. (R) For any convex polytope P,

$$\theta_P(i\epsilon) = \sum_{F \subset P} (-1)^{dim F} \theta_{K_F}(i\epsilon)$$

where K_F is the tangent cone to the face F.

What are Tangent cones ?

A quick tutorial

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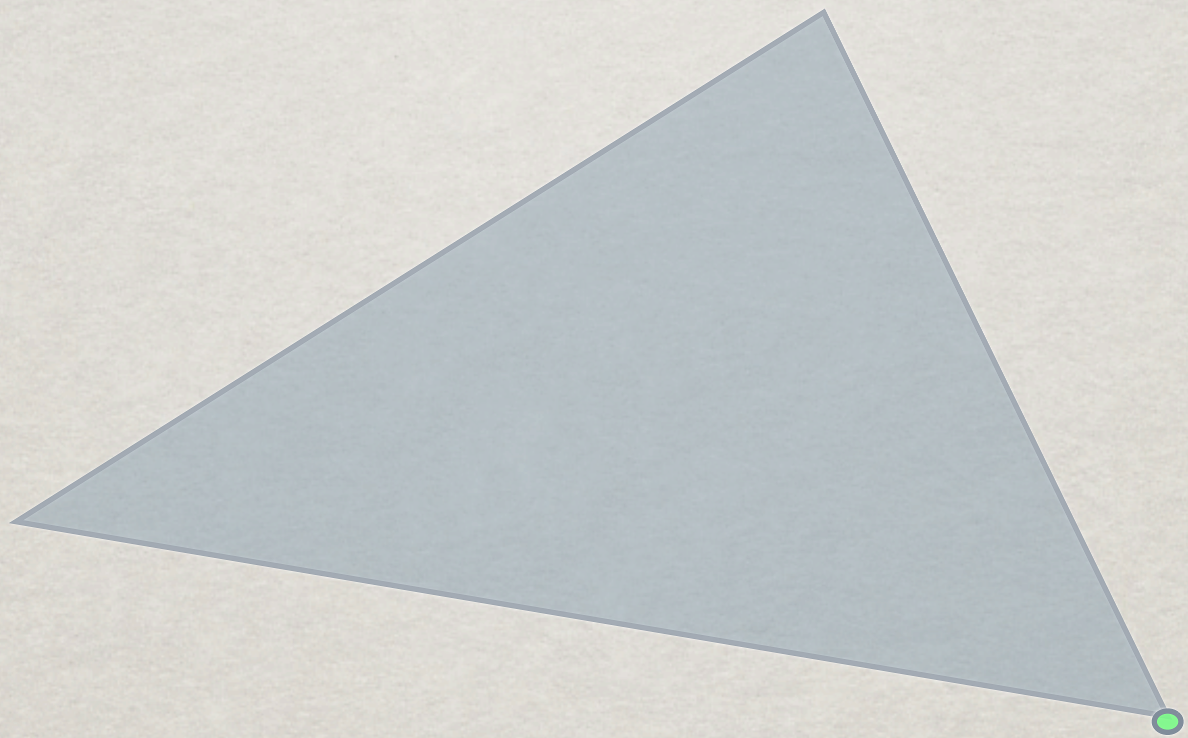
Example: If the face F is a vertex, what does the tangent cone at the vertex look like?



Face $= v$, a vertex

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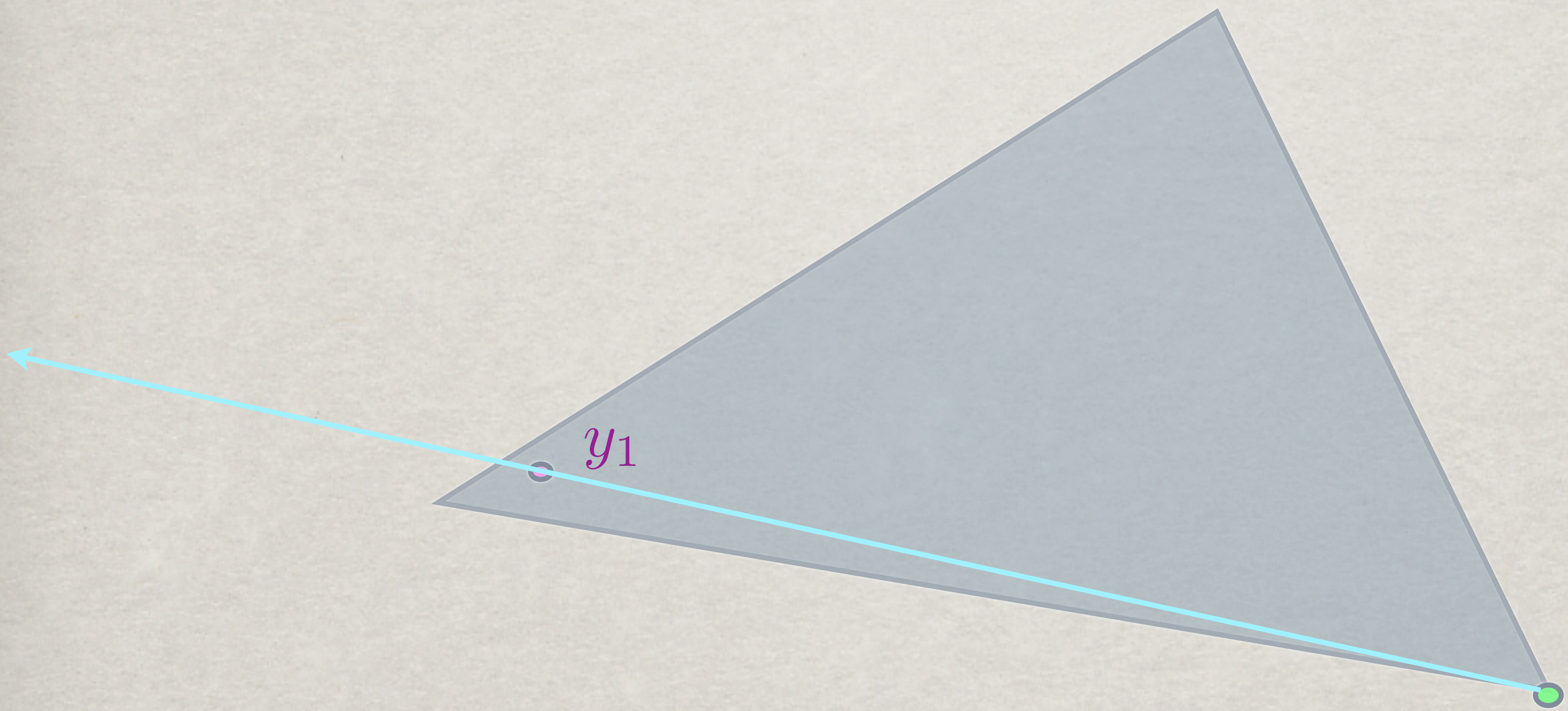
Example: If the face F is a vertex, what does the tangent cone at the vertex look like?



Face $= v$, a vertex

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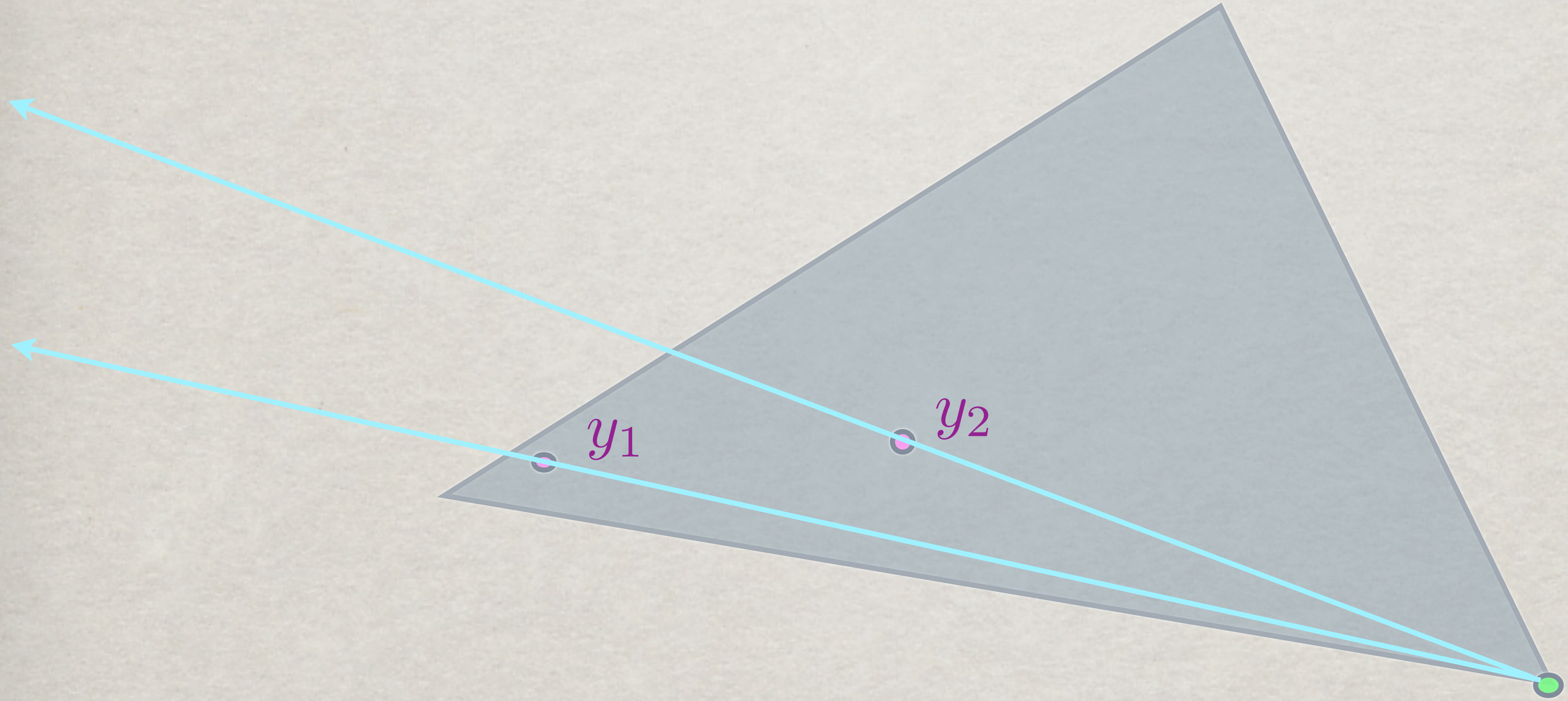

Face $= v$, a vertex

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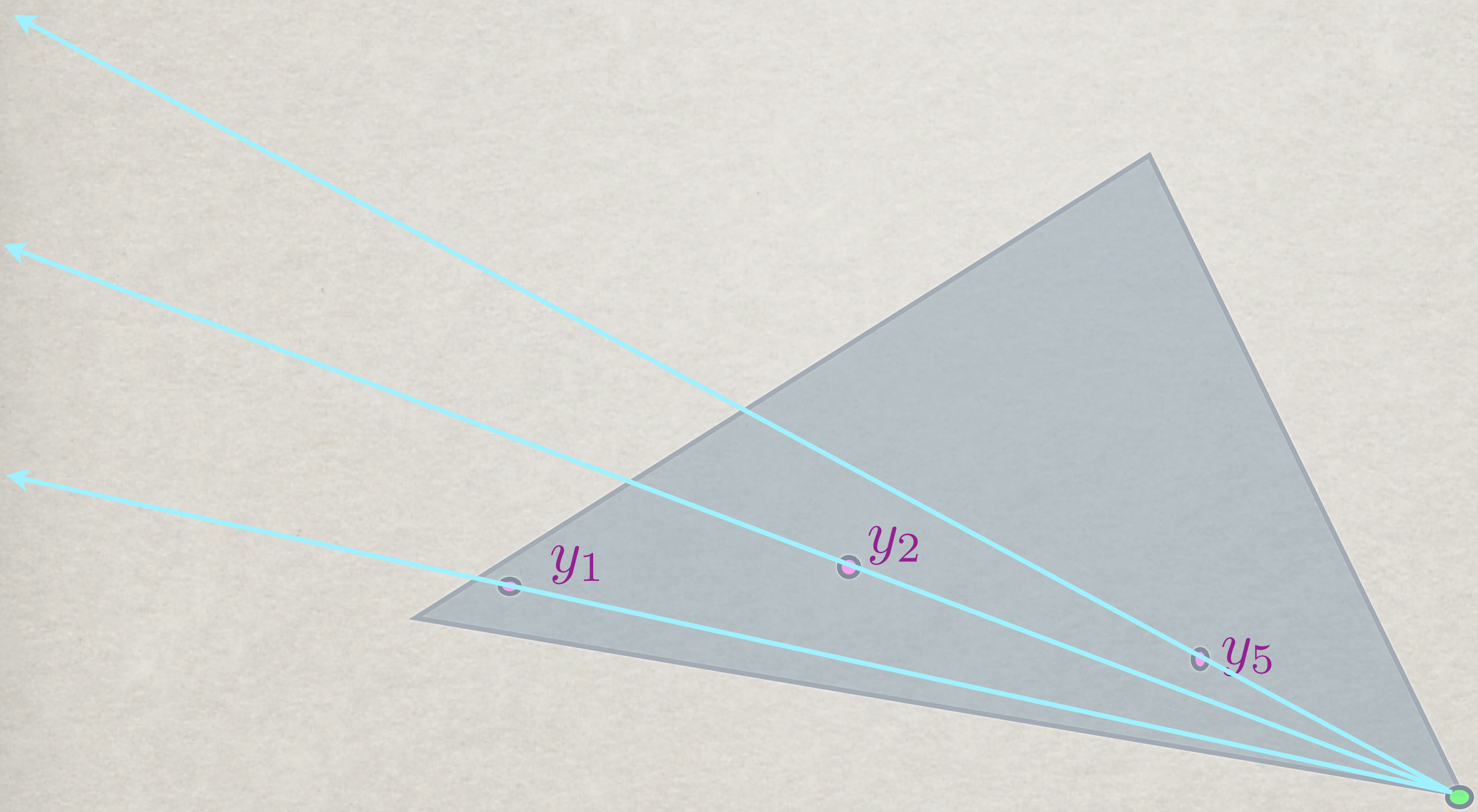
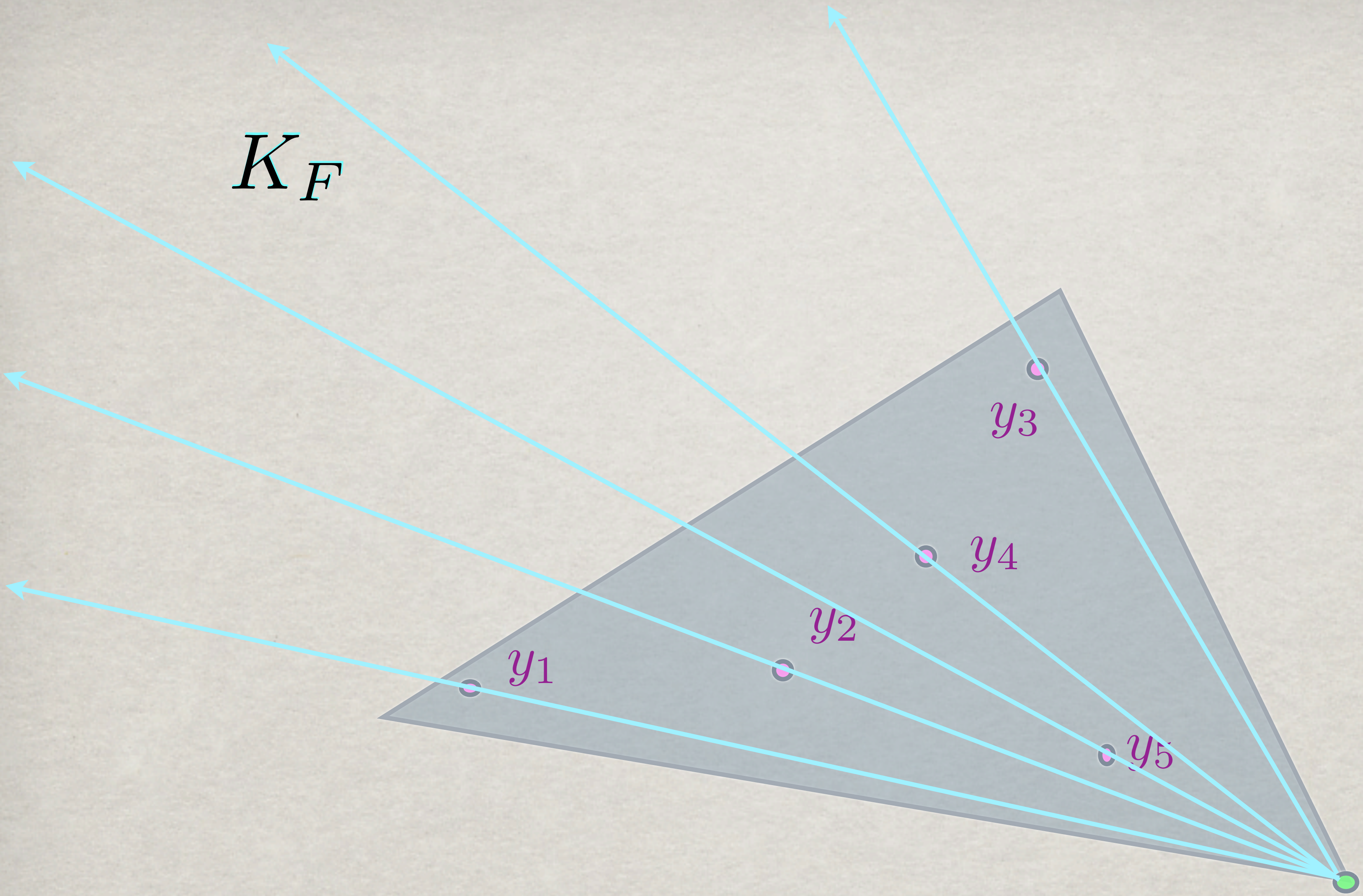
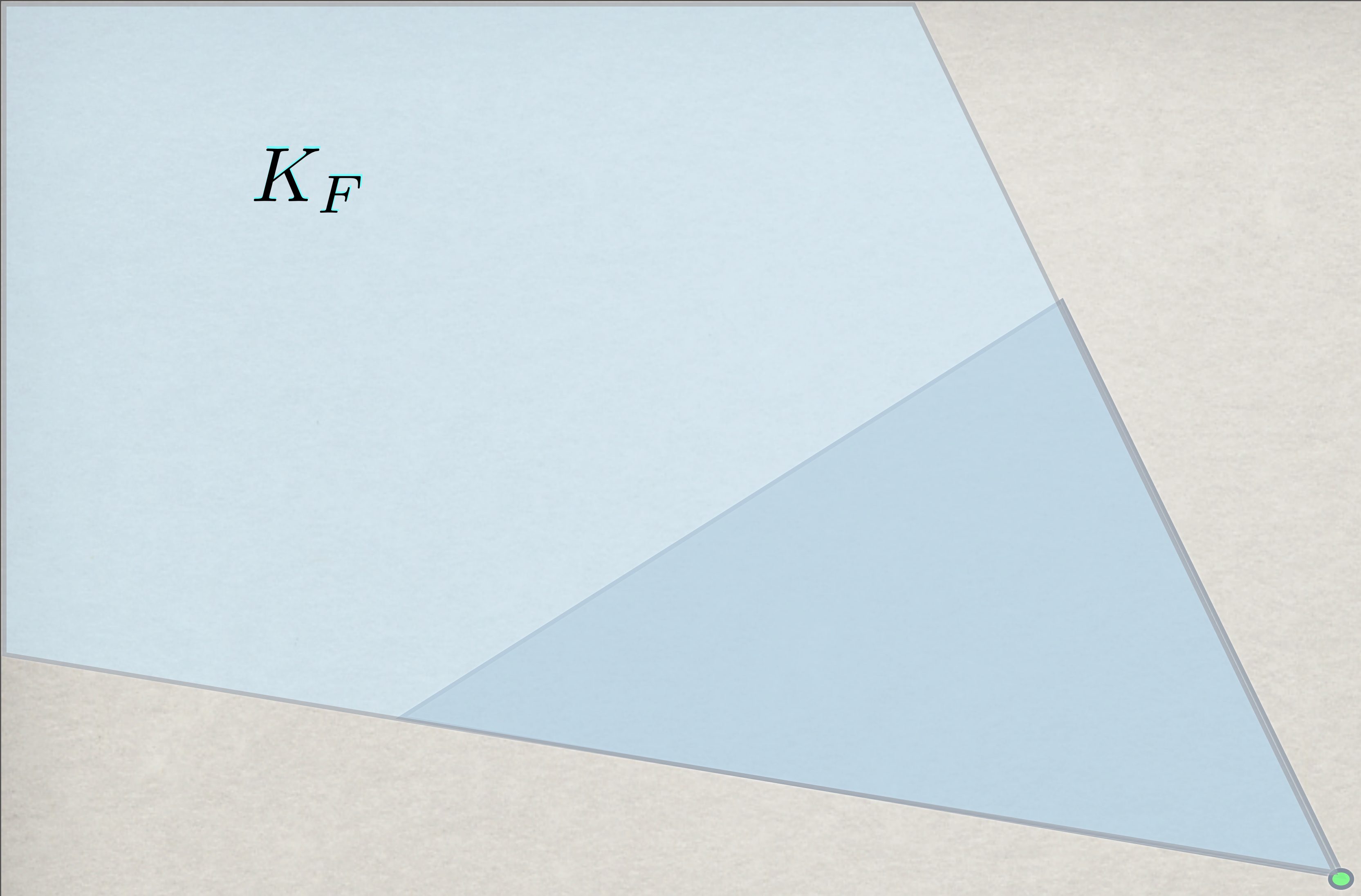


Face $= v$, a vertex



Face $= v$, a vertex



Face $= v$, a vertex

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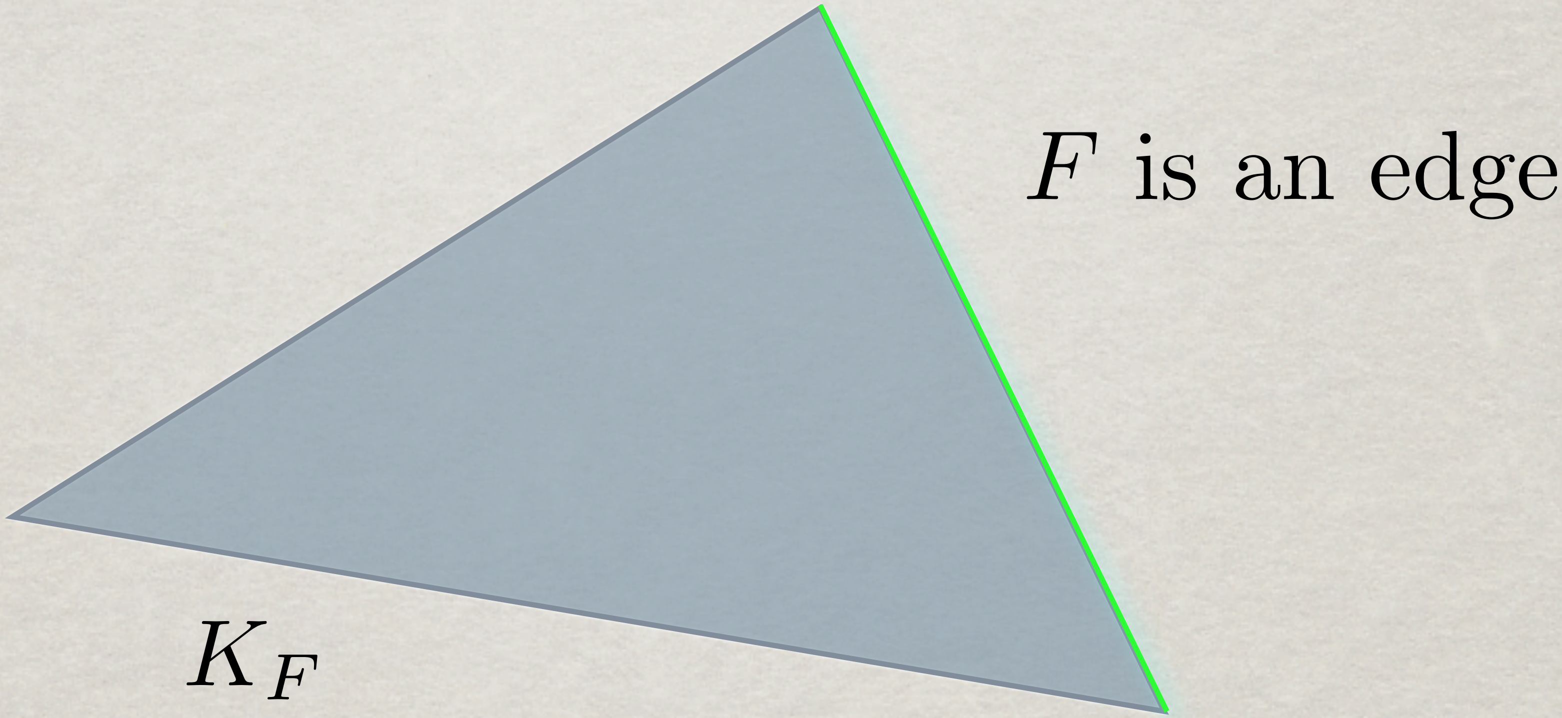
Definition. The tangent cone K_F of a face $F \subset P$ is defined by

$$K_F = \{x + \lambda(y - x) \,|\, x \in F, y \in P, \text{ and } \lambda \geq 0\}.$$

Intuitively, the tangent cone of F is the union of all rays that have a base point in F and point 'towards P'.

We note that the tangent cone of F contains the affine span of the face F.



Example. when the face is a 1-dimensional edge of a polygon, its tangent cone is a half-plane.

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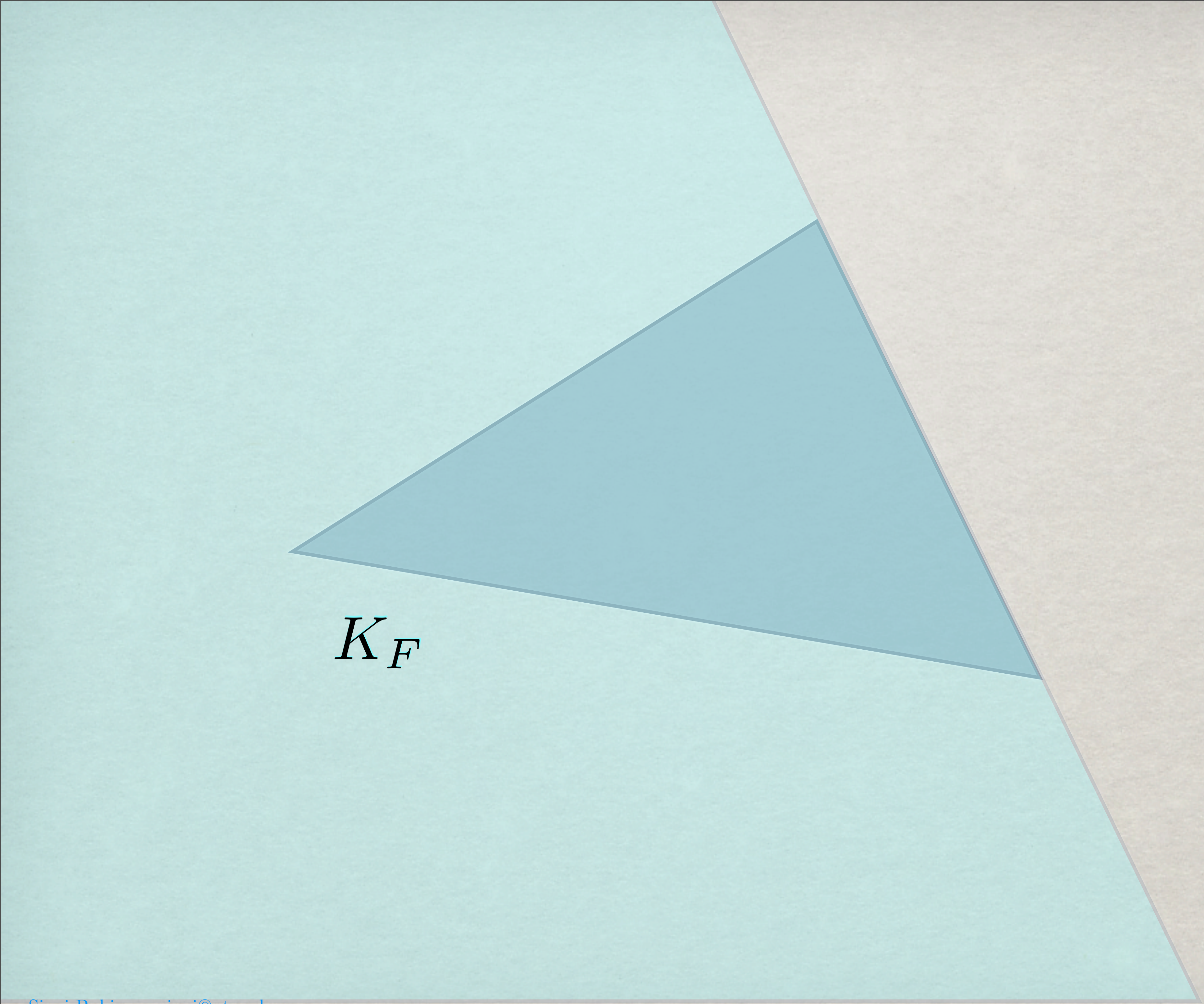
K_F

There is a wonderful and very useful identity, known as the "Brianchon-Gram" identity.

It has an Euler charactistic flavor, and it relates all of the tangent cones of any convex closed polytope as follows:

Theorem. (Brianchon-Gram)

$$1_P = \sum_{F \subset P} (-1)^{\dim F} 1_{K_F}$$

where 1_{K_F} is the indicator function of the tangent cone to F.

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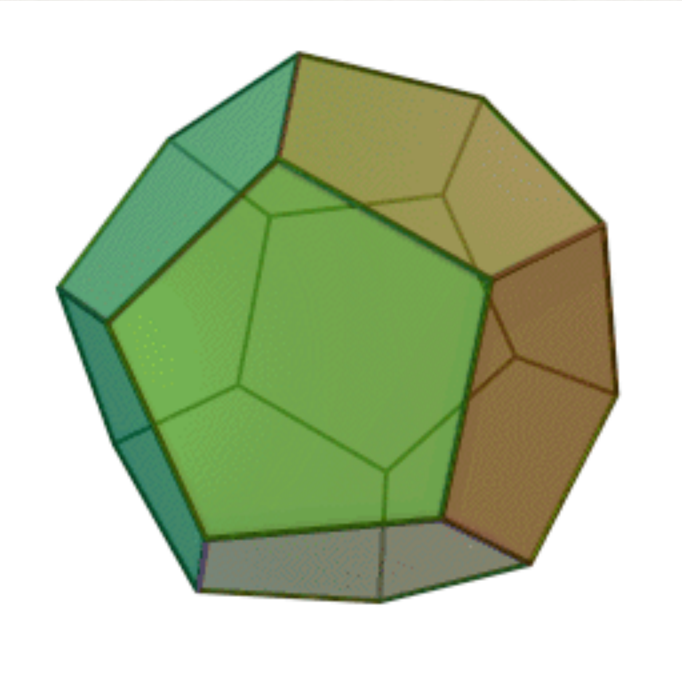
Remarks. The Brianchon identity for indicator functions allows us to transfer the computation of a function f over a polytope P to the **local** computation of **f over each tangent cone of P.**

Definition: A d-dimensional polytope enjoying the property that each of its vertices shares an edge with exactly d other vertices is called a simple polytope.

Example of a simple polytope:
The dodecahedron



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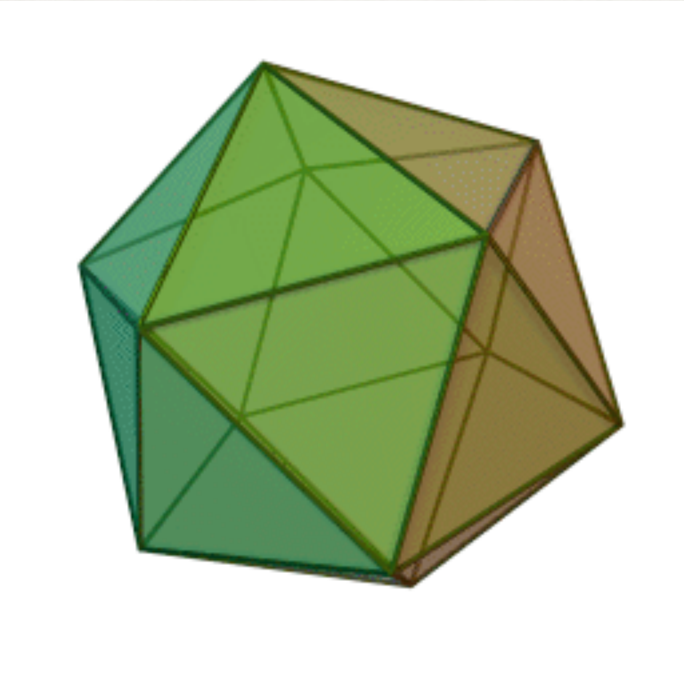
Example of a non-simple polytope: The icosahedron



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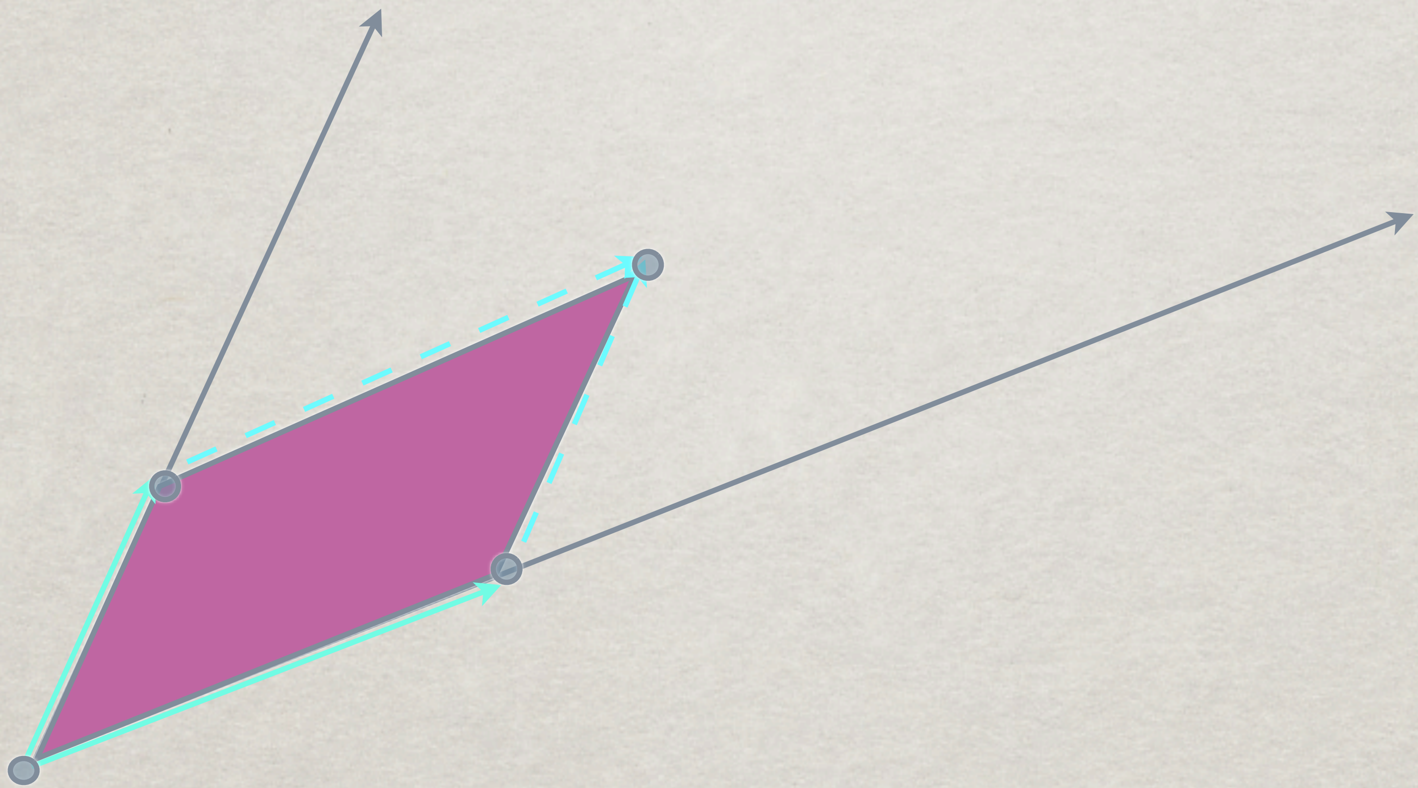
Definition. The **Fundamental Domain** of a cone K is defined by a parallelepiped.

$$\Pi := \{\lambda_1\omega_1 + \cdots + \lambda_d\omega_d \,|\, \text{ all } \, 0 \leq \lambda_j \leq 1\},$$

whereas by comparison, the cone is defined by

$$K := \{\lambda_1\omega_1 + \cdots + \lambda_d\omega_d \,|\, \text{ all } \, 0 \leq \lambda_j\}.$$

Example. In the plane, we have:



The fundamental domain of a two dimensional cone K.

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One ultimate goal:

to find a nice, computably efficient description of each solid angle ω_F as a function of the given data (the rational vertices) of the rational polytope.

Definition. We define the Gauss Sum of a cone K by

$$S_K(p,q) := \sum_{n \in q\Pi \cap \mathbb{Z}^d} e^{2\pi i \frac{p}{q} ||n||^2},$$

where p, q are any two positive integers, and where Π is a fundamental domain for the cone K.

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We arrive at a non-linear extension of the classical Gram relations, using polyhedral Gauss sums:

Theorem. (R)

For any convex, simple rational polytope P,

$$\sum_{F \subset P} (-1)^{\dim F} \omega_F \left\{ \frac{S_{K_F}(p,q)}{\det K_F} \right\} = 0.$$

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Thank You

Reference: www.mathematicaguidebooks.org/soccer/