I Prefer Pi

Corey Sinnamon

Febuary 3, 2015

Big π Day

$$3/14/15$$

3/14/15

Themes

3/14/15

Themes

- History

3/14/15

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- Irrationality and Transcendence

3/14/15

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- $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

3/14/15

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- Series and Products for π

3/14/15

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- History
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- Series and Products for π
- Computation

Machin-like Formulae

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$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

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$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

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There are countless variations on Machin's Formula, e.g.

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{8}$$

$$\frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}$$

$$\frac{\pi}{4} = 12 \arctan \frac{1}{38} + 20 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} + 24 \arctan \frac{1}{268}$$

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Consider the Gregory series for $\frac{1}{x}$,

$$\arctan \frac{1}{x} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots$$

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$$\arctan\frac{1}{x} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots$$

Lehmer observed that the number of terms that must be calculated to acquire n digits of π is approximately $\frac{n}{\log x}$, since

$$x^{-i} < 10^{-n} \iff i > \frac{n}{\log_{10} x}.$$

Machin-like Formulae

Thus, given a Machin-like formula of the form

$$\frac{k\pi}{4} = \sum_{i=1}^{n} a_i \arctan \frac{1}{m_i}$$

Lehmer defined its $measure$ as

$$\sum_{i=1}^{n} \frac{1}{\log m_i}$$

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Using the shorthand $[x] = \arctan \frac{1}{x}$, Lehmer computed the measures of most of the interesting Machin-like formulae of the time.

Machin-like Formulae

(14)	$[1]=[2]+[3]$, (5.4178)	(Hutton, Euler)
(15)	$[1]=[2]+[5]+[8]$, (5.8599)	(Daze)
(16)	$[1]=2[3]+[7]$, (3.2792)	(Clausen)
(17)	$[1]=3[4]+[19.8]$, (2.4322)	
(18)	$[1]=4[5]-[239]$, (1.8511)	(Machin)
(19)	$[1]=4[5]-[70]+[99]$, (2.4737)	(Euler, Rutherford)
(20)	$[1]=5[6]-[31.4375]-[117]$, (2.4364)	
(21)	$[1]=5[7]+2[79/3]$, (1.8873)	(Hutton, Euler)
(22)	$[1]=6[8]+[19.8]-3[268]$, (2.2904)	
(23)	$[1]=8[10]-[239]-4[515]$, (1.2892)	(Klingenstierna)
(24)	$[1]=8[10]+3[18]+2[100]+2[307]-3[515]+2[9901]$, (2.3177)	
(25)	$[1]=8[10]-2[452761/2543]-[1393]$, (1.2624)	
(26)	$[1]=8[10]-[100]-[515]-[371498882/3583]$, (1.0681)	
(27)	$[1]=8[10]-[100]-2[1000]+5[100000]-[719160]-\cdots$, ($<.8414$)	
(28)	$[1]=7[10]+2[50]+4[100]+[682]+4[1000]+3[1303]-4[90109]$, (1.9644)	(Wrench)
(29)	$[1]=7[10]+8[100]+[682]+4[1000]+3[1303]-4[90109]-2[500150]$, (1.5513)	(Wrench)
(30)	$[1]=8[10.1]-[239]+4[52525]$, (1.6280)	
(31)	$[1]=12[15]-[239]-4[433.1]$, (1.6500)	
(32)	$[1]=12[18]+8[57]-5[239]$, (1.7866)	(Gauss)
(33)	$[1]=12[18]+3[70]+5[99]+8[307]$, (2.2418)	(Bennett)
(34)	$[1]=12[18]+8[99]+3[239]+8[307]$, (2.1203)	
(35)	$[1]=16[20.05]-[239]-4[515]+8[1620050]$, (1.7182)	
(36)	$[1]=22[26]-2[2057]-5[3240647/38479]$, (1.5279)	

Machin-like Formulae

Although they were not the lowest-measure relations, Lehmer recommended the following for practical computing,

$$\frac{\pi}{4} = 8 \arctan \frac{1}{10} - \arctan \frac{1}{239} - 4 \arctan \frac{1}{515}$$

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$$

Why?

Lehmer recognized that this measure could be grossly inaccurate when a relation involves some repeated calculations. For example, in 1939 J. P. Ballantine observed that

$$\arctan\frac{1}{18} = 18\left(\frac{1}{325} + \frac{2}{3\cdot 325^2} + \frac{2\cdot 4}{3\cdot 5\cdot 325^3} + \cdots\right)$$

$$\arctan\frac{1}{57} = 57\left(\frac{1}{3250} + \frac{2}{3\cdot 3250^2} + \frac{2\cdot 4}{3\cdot 5\cdot 3250^3} + \cdots\right)$$

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$$\arctan\frac{1}{57} = 57\left(\frac{1}{3250} + \frac{2}{3\cdot 3250^2} + \frac{2\cdot 4}{3\cdot 5\cdot 3250^3} + \cdots\right)$$

This produces the lovely relation

$$\pi = \frac{864}{18}\arctan\frac{1}{18} + \frac{1824}{57}\arctan\frac{1}{57} - 5\arctan\frac{1}{239},$$

which (according to Ballantine) was the fastest known method for computing π to many digits.

Monroe Calculator

MONROE
REGISTERED TRADE MARK
HIGH SPEED ADDING CALCULATOR

Infinite Products for πe and $\frac{\pi}{e}$

Z. A. Melzak (1961)

Infinite Products for πe and $\frac{\pi}{e}$

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$$\lim_{n \to \infty} V(C_n)/V(S_n) = \sqrt{\frac{2}{\pi e}}$$

Infinite Products for πe and $\frac{\pi}{e}$

Two simple products,

$$\frac{\pi}{2e} = \prod_{n=1}^{\infty}(1 + \frac{2}{n})^{(-1)^{n+1}n}$$

$$\frac{6}{\pi e} = \prod_{n=2}^{\infty}(1 + \frac{2}{n})^{(-1)^{n}n}$$

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But then

$$\frac{\pi}{6e} = \prod_{n=2}^{\infty} (1 + \frac{2}{n})^{(-1)^{n+1} n} = \frac{\pi e}{6}$$

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What's going on here?

Infinite Products for πe and $\frac{\pi}{e}$

$$\prod_{n=2}^{N}(1+\frac{2}{n})^{(-1)^{n+1}n}$$

Infinite Products for πe and $\frac{\pi}{e}$

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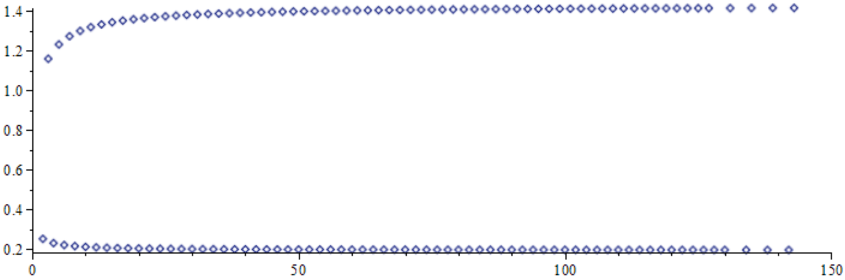

$$\frac{\pi}{6e} \approx 0.1926212250 \qquad \frac{\pi e}{6} \approx 1.423289037$$

Infinite Products for πe and $\frac{\pi}{e}$

Two not-as-simple products,

$$\frac{\pi}{2e} = \lim_{N\to\infty} \prod_{n=1}^{2N} (1 + \frac{2}{n})^{(-1)^{n+1}n}$$

$$\frac{6}{\pi e} = \lim_{N\to\infty} \prod_{n=2}^{2N+1} (1 + \frac{2}{n})^{(-1)^{n}n}$$

A Spigot Algorithm for π

Rabinowitz and Wagon (1995)

A Spigot Algorithm for π

Rabinowitz and Wagon (1995)

Presented an algorithm to compute digits of π that

- "drips" digits of π one by one and does not use them afterwards,
- is easy to implement,
- uses only integer arithmetic

A Spigot Algorithm for π

Column Head $\frac{a_i}{b_i} = \frac{i}{2i+1}$ Column Number $c_i = 2$.

	Digits of π		$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{4}{9}$	$\frac{5}{11}$	$\frac{6}{13}$	$\frac{7}{15}$	$\frac{8}{17}$	$\frac{9}{19}$	$\frac{10}{21}$	$\frac{11}{23}$	$\frac{12}{25}$
Initialize		2	2	2	2	2	2	2	2	2	2	2	2	2
× 10		20	20	20	20	20	20	20	20	20	20	20	20	20
Carry	**3**	+10	+12	+12	+12	+10	+12	+7	+8	+9	+0	+0	+0	—
		30	32	32	32	30	32	27	28	29	20	20	20	20
Remainders		0	2	2	4	3	10	1	13	12	1	20	20	20
× 10		0	20	20	40	30	100	10	130	120	10	200	200	200
Carry	**1**	+13	+20	+33	+40	+65	+48	+98	+88	+72	+150	+132	+96	—
		13	40	53	80	95	148	108	218	192	160	332	296	200
Remainders		3	1	3	3	5	5	4	8	5	8	17	20	0

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	Digits of π		1/3	2/5	3/7	4/9	5/11	6/13	7/15	8/17	9/19	10/21	11/23	12/25
Initialize		2	2	2	2	2	2	2	2	2	2	2	2	2
× 10		20	20	20	20	20	20	20	20	20	20	20	20	20
Carry	3	+10	+12	+12	+12	+10	+12	+7	+8	+9	+0	+0	+0	—
		30	32	32	32	30	32	27	28	29	20	20	20	20
Remainders		0	2	2	4	3	10	1	13	12	1	20	20	20
× 10		0	20	20	40	30	100	10	130	120	10	200	200	200
Carry	1	+13	+20	+33	+40	+65	+48	+98	+88	+72	+150	+132	+96	—
		13	40	53	80	95	148	108	218	192	160	332	296	200
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Initialize		2	2	2	2	2	2	2	2	2	2	2	2	2
× 10		20	20	20	20	20	20	20	20	20	20	20	20	20
Carry	3	+10	+12	+12	+12	+10	+12	+7	+8	+9	+0	+0	+0	—
		30	32	32	32	30	32	27	28	29	20	20	20	20
Remainders		0	2	2	4	3	10	1	13	12	1	20	20	20
× 10		0	20	20	40	30	100	10	130	120	10	200	200	200
Carry	1	+13	+20	+33	+40	+65	+48	+98	+88	+72	+150	+132	+96	—
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- Multiply each c_i by 10.

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Initialize		2	2	2	2	2	2	2	2	2	2	2	2	2
× 10		20	20	20	20	20	20	20	20	20	20	20	20	20
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Algorithm:

- Multiply each c_i by 10.
- Proceeding from the rightmost to the leftmost column, find positive integers q and r such that $q \cdot b_i + r = c_i$ with $0 < r < b_i$, then assign $c_i \leftarrow r$ and carry $q \cdot a_i$ left.

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Initialize		2	2	2	2	2	2	2	2	2	2	2	2	2
× 10		20	20	20	20	20	20	20	20	20	20	20	20	20
Carry	3	+10	+12	+12	+12	+10	+12	+7	+8	+9	+0	+0	+0	—
		30	32	32	32	30	32	27	28	29	20	20	20	20
Remainders		0	2	2	4	3	10	1	13	12	1	20	20	20
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- On the leftmost column, output the first digit of the column number c_0. This is the next digit of π!

A Spigot Algorithm for π

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Initialize		2	2	2	2	2	2	2	2	2	2	2	2	2
× 10		20	20	20	20	20	20	20	20	20	20	20	20	20
Carry	3	+10	+12	+12	+12	+10	+12	+7	+8	+9	+0	+0	+0	
		30	32	32	32	30	32	27	28	29	20	20	20	20
Remainders		0	2	2	4	3	10	1	13	12	1	20	20	20
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Carry	1	+13	+20	+33	+40	+65	+48	+98	+88	+72	+150	+132	+96	
		13	40	53	80	95	148	108	218	192	160	332	296	200
Remainders		3	1	3	3	5	5	4	8	5	8	17	20	0
× 10		30	10	30	30	50	50	40	80	50	80	170	200	0
Carry	4	+11	+24	+30	+40	+40	+42	+63	+64	+90	+120	+88	+0	
		41	34	60	70	90	92	103	144	140	200	258	200	0
Remainders		1	1	0	0	0	4	12	9	4	10	6	16	0
× 10		10	10	0	0	0	40	120	90	40	100	60	160	0
Carry	1	+4	+2	+9	+24	+55	+84	+63	+48	+72	+60	+66	+0	
		14	12	9	24	55	124	183	138	112	160	126	160	0

A Spigot Algorithm for π

Why does this work?

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The algorithm is based on the following representation for $\frac{\pi}{2}$,

$$\frac{\pi}{2} = \sum_{i=0}^{\infty} \frac{i!}{(2i+1)!!} = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots$$

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Observe that we can rewrite this as

$$\frac{\pi}{2} = 1 + \frac{1}{3}\left(1 + \frac{2}{5}\left(1 + \frac{3}{7}\left(1 + \cdots\right)\right)\right)$$

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Observe that we can rewrite this as

$$\frac{\pi}{2} = 1 + \frac{1}{3}\left(1 + \frac{2}{5}\left(1 + \frac{3}{7}\left(1 + \cdots\right)\right)\right)$$

or equivalently,

$$\pi = 2 + \frac{1}{3}\left(2 + \frac{2}{5}\left(2 + \frac{3}{7}\left(2 + \cdots\right)\right)\right)$$

A Spigot Algorithm for π

In some ways, this *mixed-radix* form is similar to having a closed form for the digits of π in some base b, except the base changes with every term.

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$$\pi = a_0 + \frac{1}{b}\left(a_1 + \frac{1}{b^2}\left(a_2 + \frac{1}{b^3}\left(a_3 + \cdots\right)\right)\right)$$

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$$\pi = a_0.a_1a_2a_3\ldots_b$$

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$$\pi = a_0.a_1a_2a_3\ldots_b$$

$$\pi = 2 + \frac{1}{3}\left(2 + \frac{2}{5}\left(2 + \frac{3}{7}\left(2 + \cdots\right)\right)\right)$$

$$\pi = 2.222\ldots_{[1,\frac{1}{3},\frac{2}{5},\ldots]}$$

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$$\pi = 2.222\ldots_{[1,\frac{1}{3},\frac{2}{5},\ldots]}$$

In this sense, the algorithm is simply changing the base of the representation.

Unbounded Spigot Algorithms for the Digits of π

Jeremy Gibbons (2006)

Developed an algorithm using the same principle, but much cleaner in the details. Particularly, it is unbounded - no need to decide on the number of digits before starting.

Assorted Series

Assorted Series

Wallis

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Assorted Series

Wallis

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Ramanujan

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}$$

Assorted Series

Wallis

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Ramanujan

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}$$

D. H. Bailey, P. Borwein, S. Plouffe

$$\pi = \sum_{k=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right]$$