Investigations into Normal numbers and Experimental Mathematics

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Definition

A Normal number is an irrational number in which every combination of digits occurs as frequently as any other combination.

That is, when

1. $a_1 a_2 \cdots a_k$ is any combination of k digits, and
2. $N(t)$ is the number of times this combination occurs among the first t digits in the base b expansion,

then

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{b^k}.$$

Known Normal Numbers

Date	The Number	Author
1933	$0.123456789\ldots$	Champernowne
1946	$0.23571113\ldots$	Copeland and Erdös Constant
1952	$0.f(1)f(2)f(3)\ldots$	Davenport and Erdös
1973	$\sum_{k=1}^{\infty} \frac{1}{b^{c^k} c^k}$	Stoneham
2001	0.11011100101_2	Binary Champerownes, Bailey and Crandall

We investigated the Davenport-Erdös numbers, that when $f \in \mathbb{Q}[x]$ such that $f(x) \geq 0$ for $x > 0$, have the form

$$0.f(1)f(2)f(3)\ldots.$$

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Theorem (?)

Let $f(x) \in \mathbb{Q}[x]$, so that when $x \in \mathbb{N}$, $f(x) \geq 0$. Then the decimal $.f(1)f(2)f(3)\ldots_{10}$ is 10-normal.

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Theorem (Us)

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[?]

Definition (Simply Strongly Normal)

1. Let $\alpha \in \mathbb{R}$ with base-b fractional part $0.a_0a_1a_2\ldots$, and
2. $m_k(n) := \#\{i : a_i = k, i \leq n\}$.

α is simply strongly normal in base b if for each $0 \leq k \leq b-1$

$$\limsup_{n\to\infty} \frac{m_k(n) - n/b}{\sqrt{2n\log\log n}} = \frac{\sqrt{b-1}}{b}, \text{ and}$$

$$\liminf_{n\to\infty} \frac{m_k(n) - n/b}{\sqrt{2n\log\log n}} = -\frac{\sqrt{b-1}}{b}.$$

Moreover...

A number is strongly normal in base b if it is simply strongly normal in each base b^j for $j = 1, 2, 3 \ldots$, and is absolutely strongly normal if it is strongly normal in every base.

It was proved in (**?**)

1. If a number is strongly normal, it is normal.
2. "Almost all" numbers are strongly normal in any base.

Strong Normality

Let $\alpha \in \mathbb{R}$ have base ten expansion $0.a_1 a_2 \ldots$, and take

$$p_k(n) = \frac{m_k(n) - n/b}{\sqrt{2n \log\log n}}$$

(this is equivalent to removing the limits from the definition of Simply Strongly Normal).

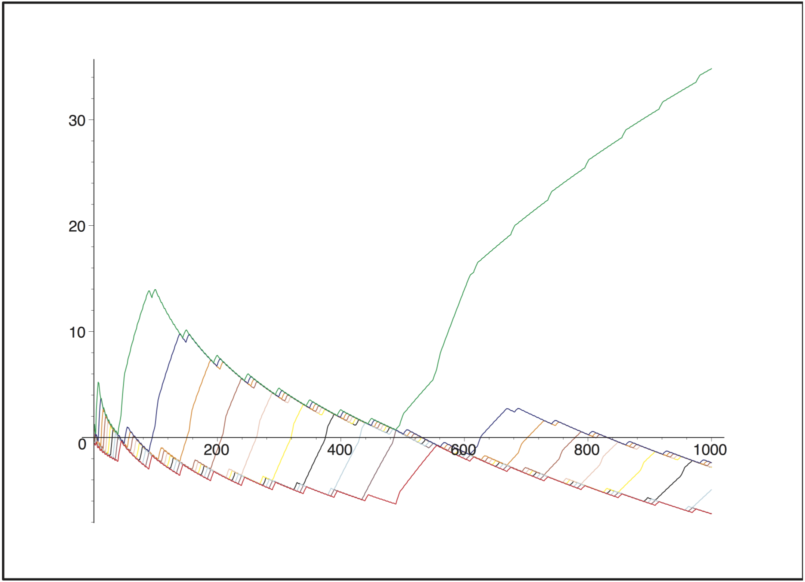
We plot, for various Davenport-Erdös numbers, p against n for all k of α and can observe whether $p_k(n)$ is tending towards $\pm\frac{3}{10}$ or not.

Strong Normality

Colour	k
Red	0
Green	1
Blue	2
Coral	3
Orange	4

Colour	k
Pink	5
Yellow	6
Black	7
Turquoise	8
Maroon	9

$f(x) = x$ for $n = 1, \ldots, 10^6$.



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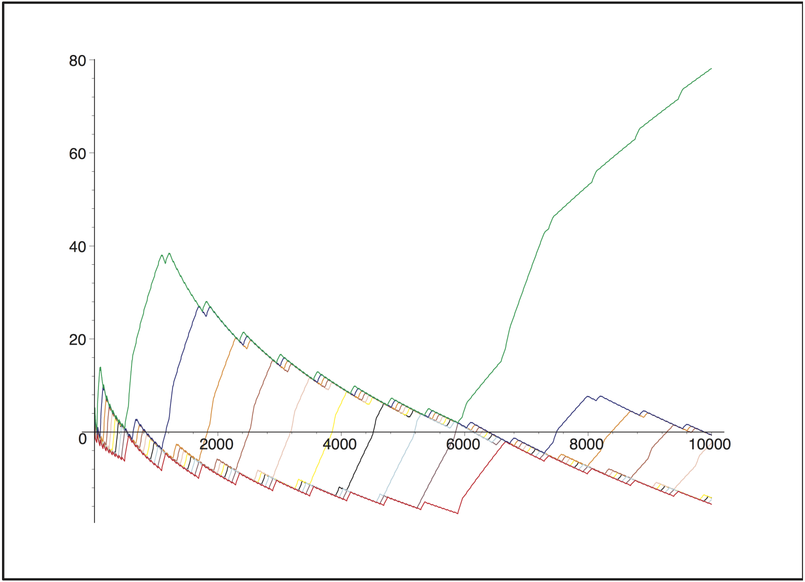

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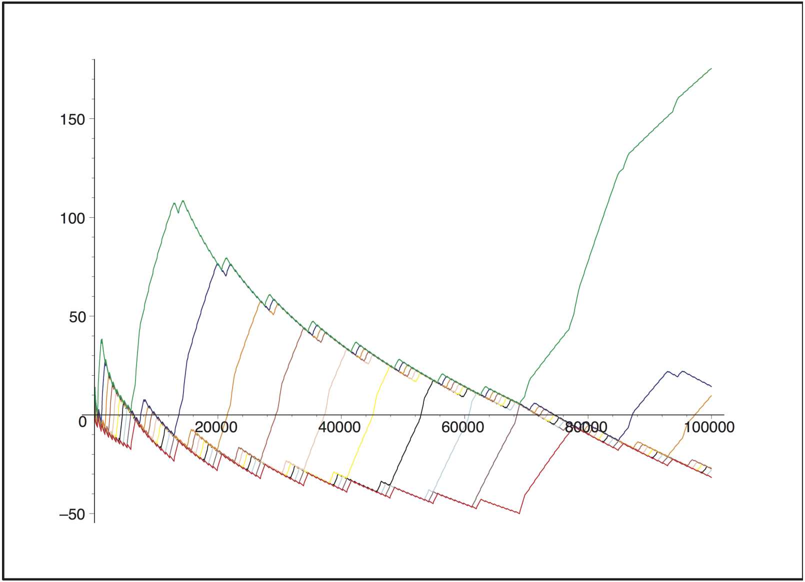

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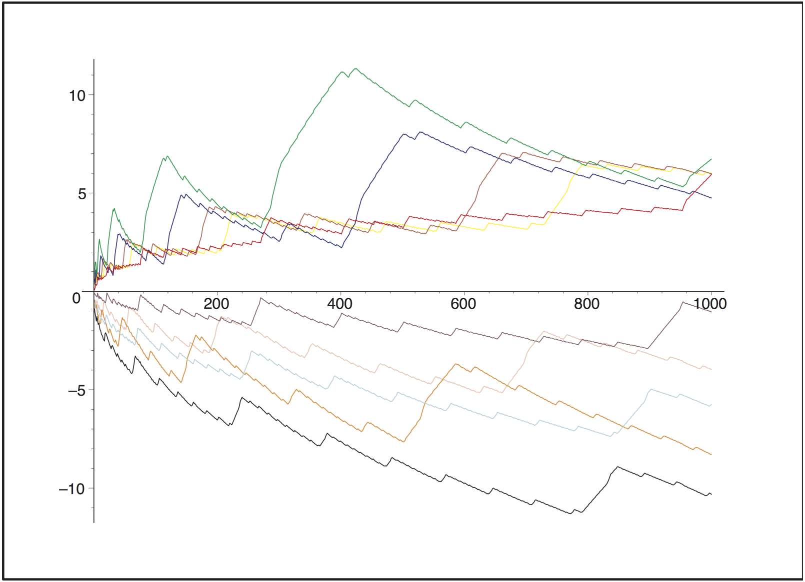

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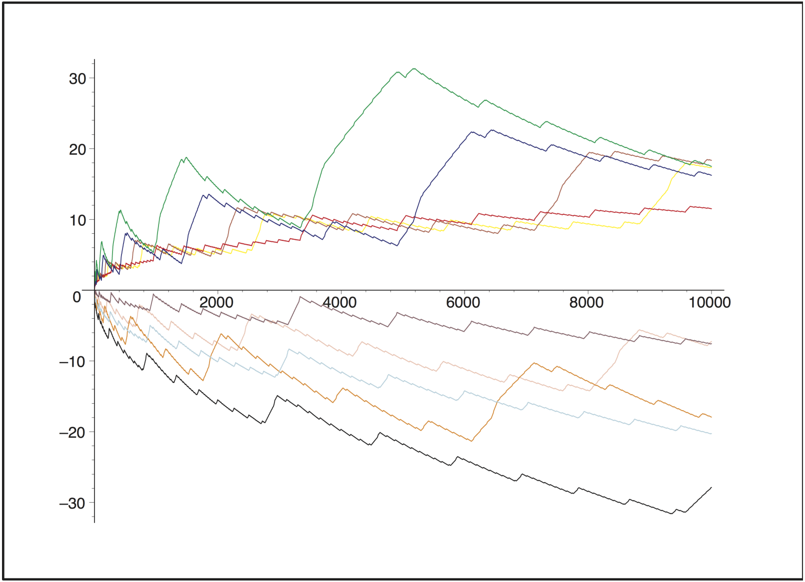

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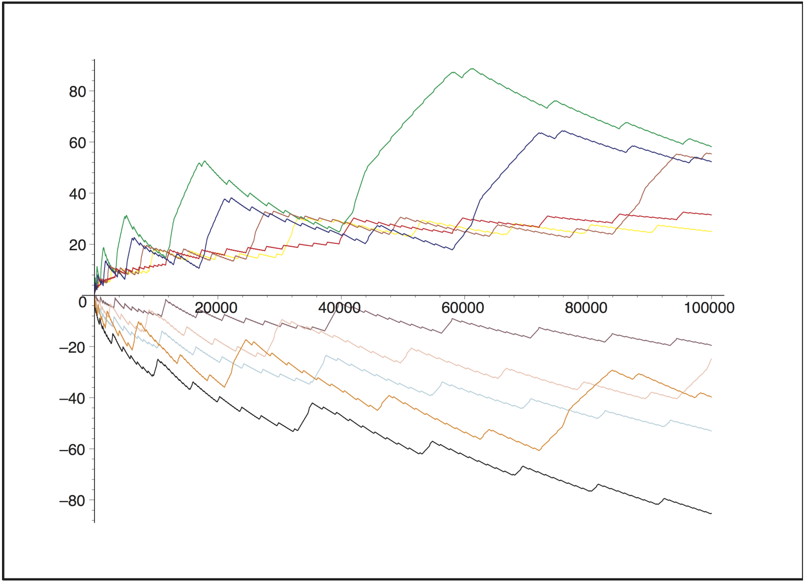

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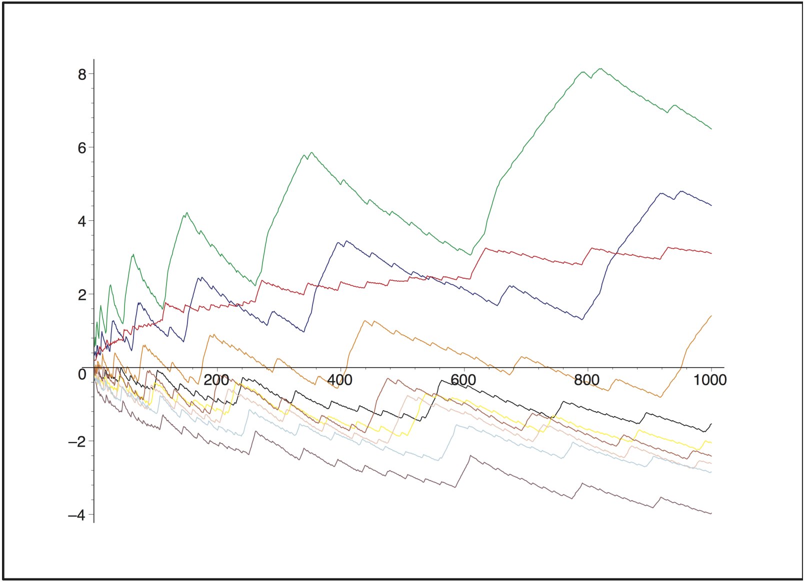

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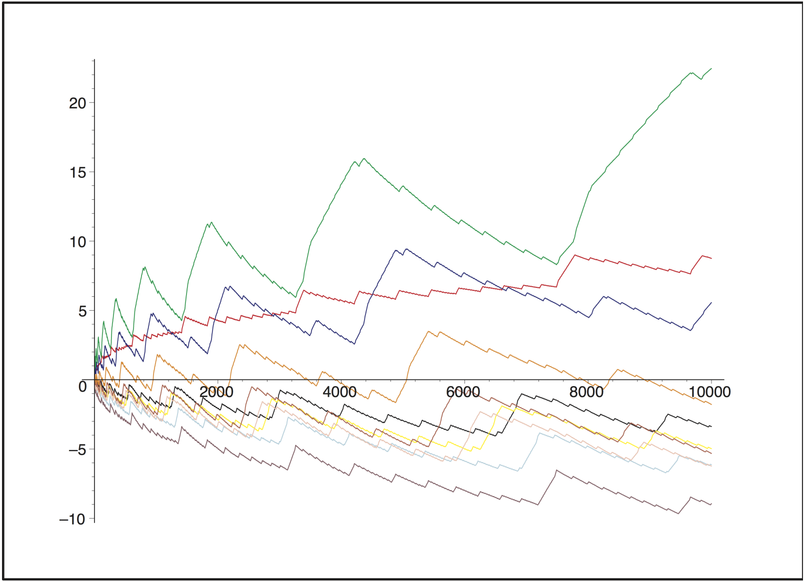

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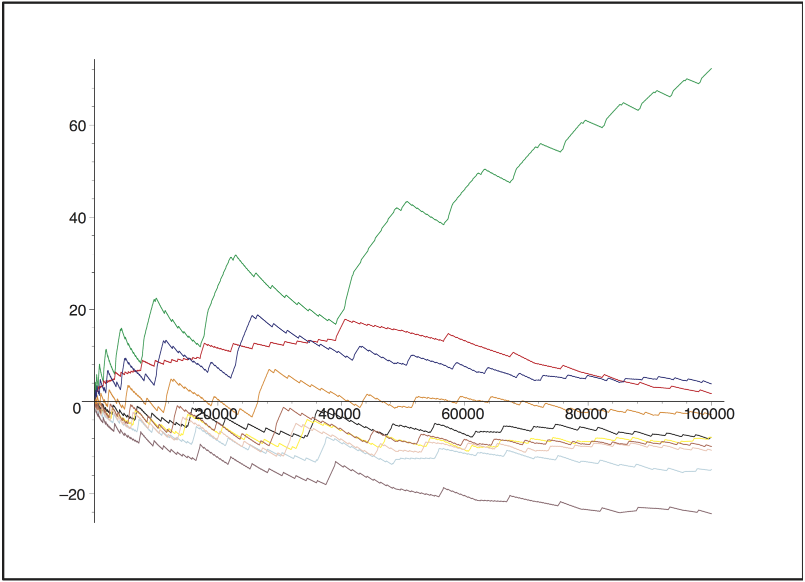
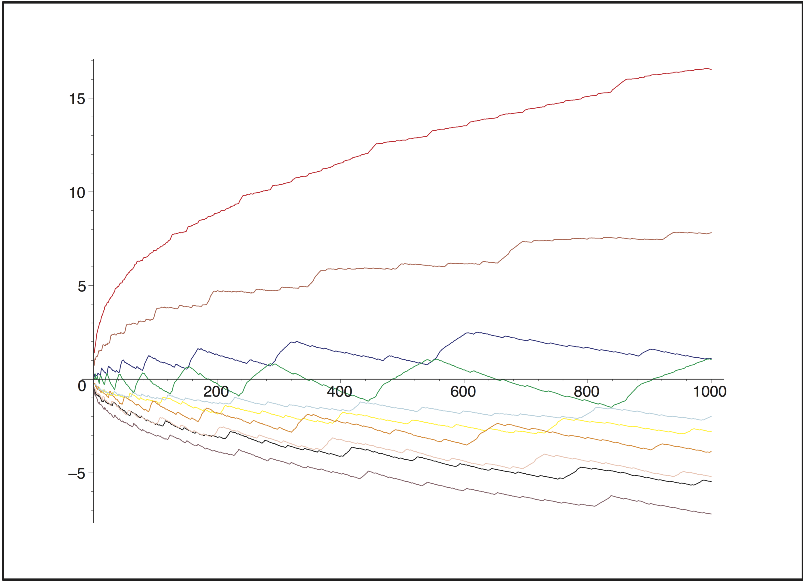


$f(x) = 2x^4 + 2x^2$ for $n = 1, \ldots, 10^6$.



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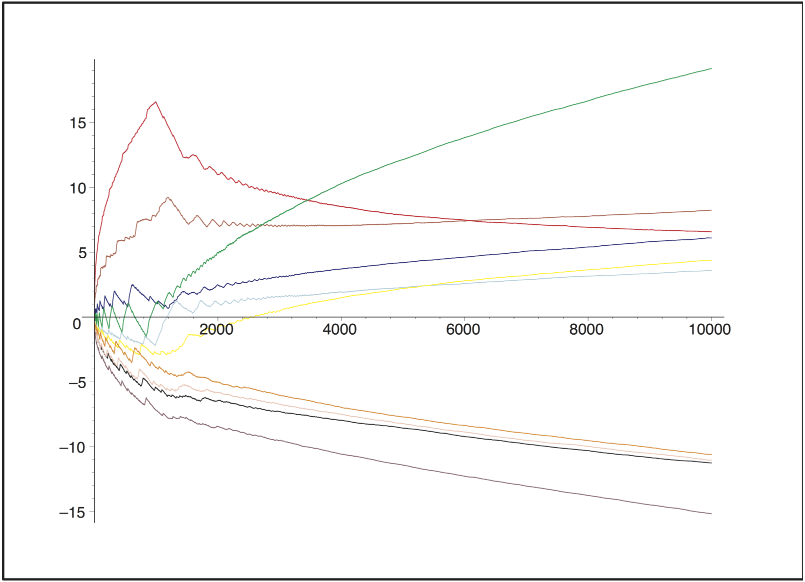

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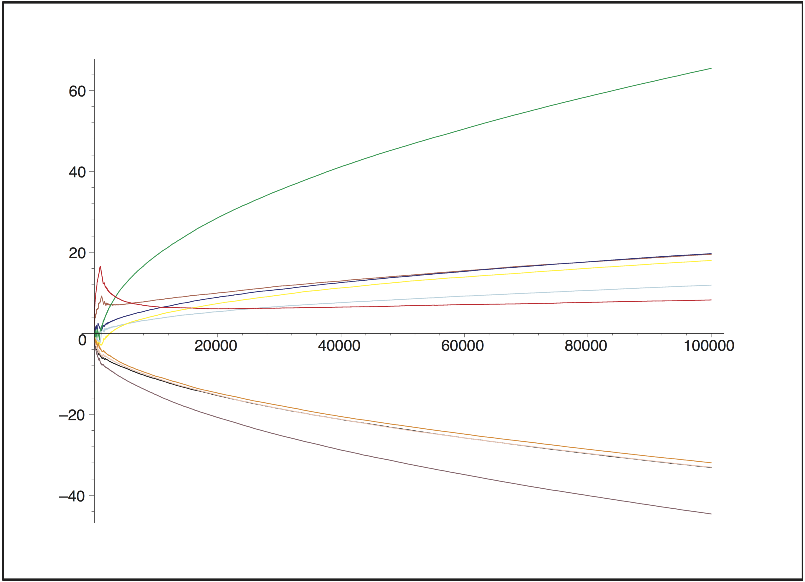
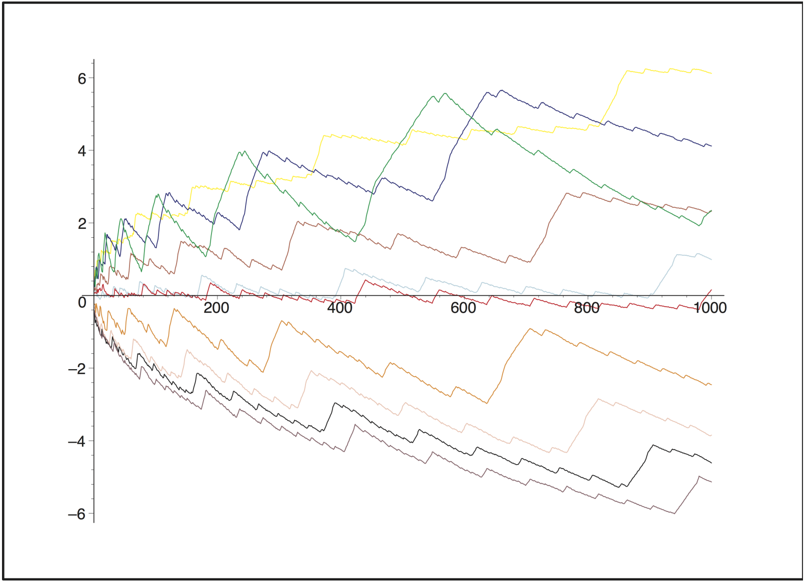


$f(x) = 3x^3 - 2x^2 + x$ for $n = 1, \ldots, 10^6$.



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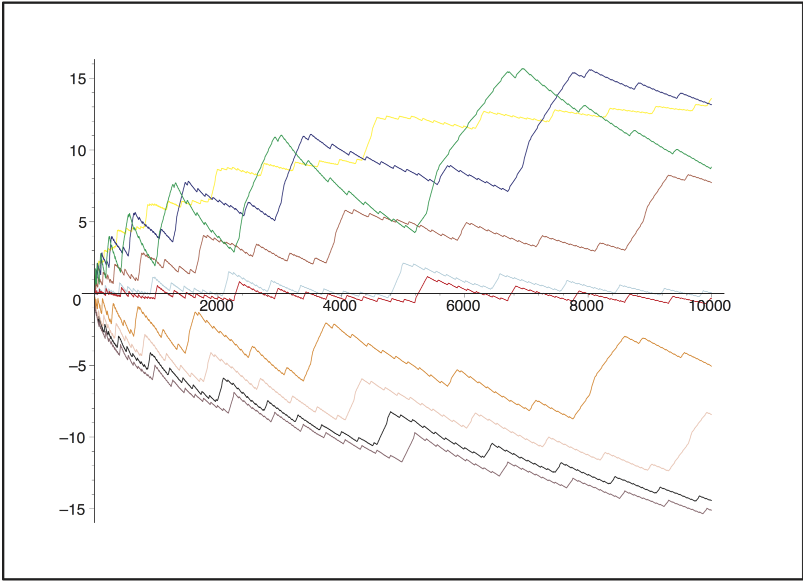

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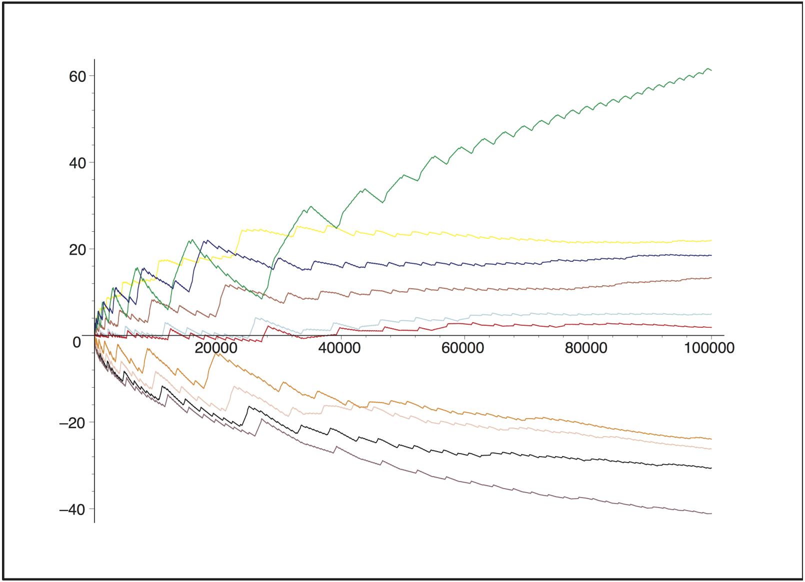

Strong Normality

Conjecture

Let $f(x) \in \mathbb{Q}[x]$, then the decimal

$$.f(1)f(2)f(3)\ldots_{10}$$

is <u>not</u> strongly normal when $x \in \mathbb{N}$ and $f(x) \geq 0$.

Strong Normality

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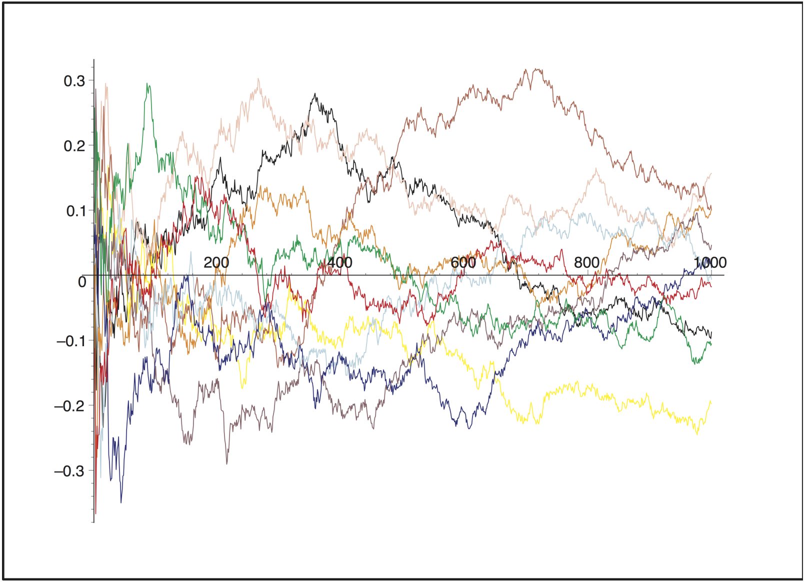
We shall see that with these constants $p_k(n)$ is generally bound between $\pm\frac{3}{10}$ as $n \to \infty$.

π for $n = 1, \ldots, 10^6$



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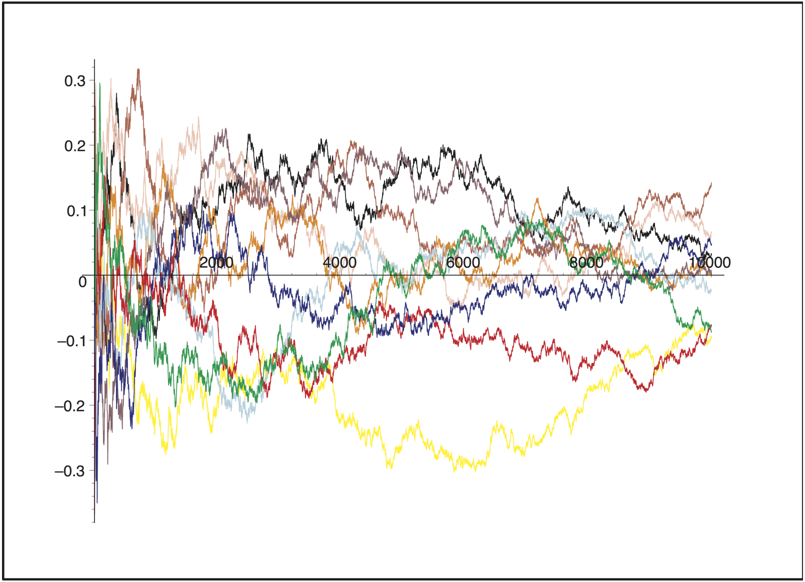

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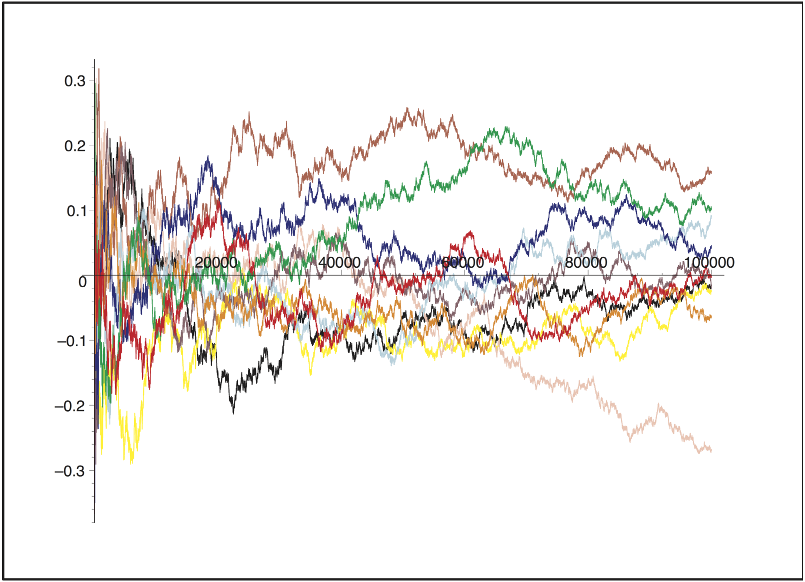
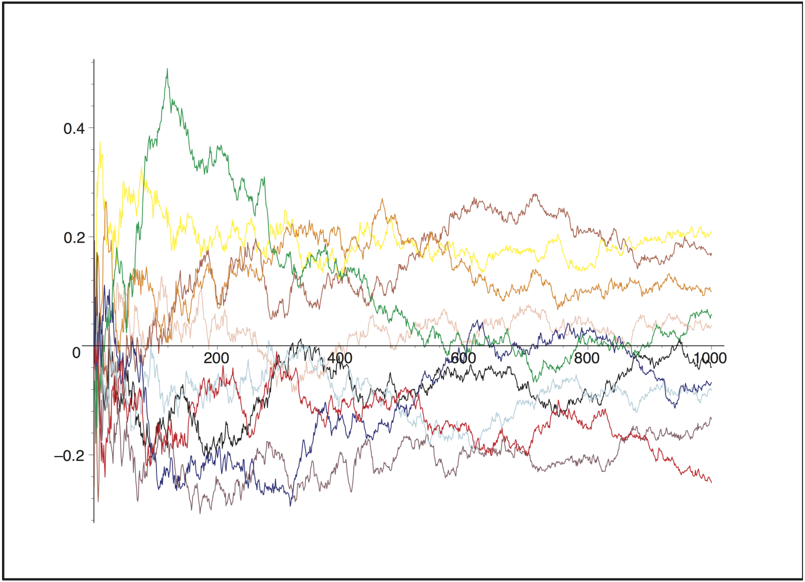


e for $n = 1, \ldots, 10^6$



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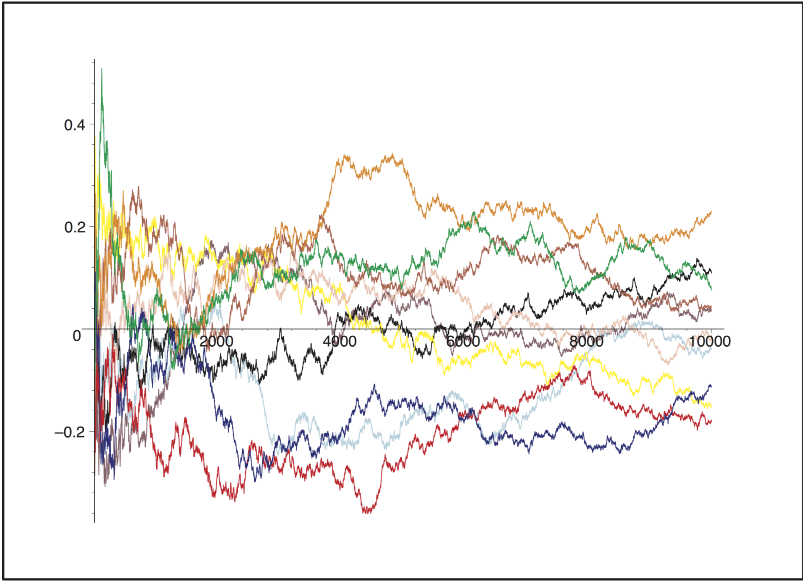

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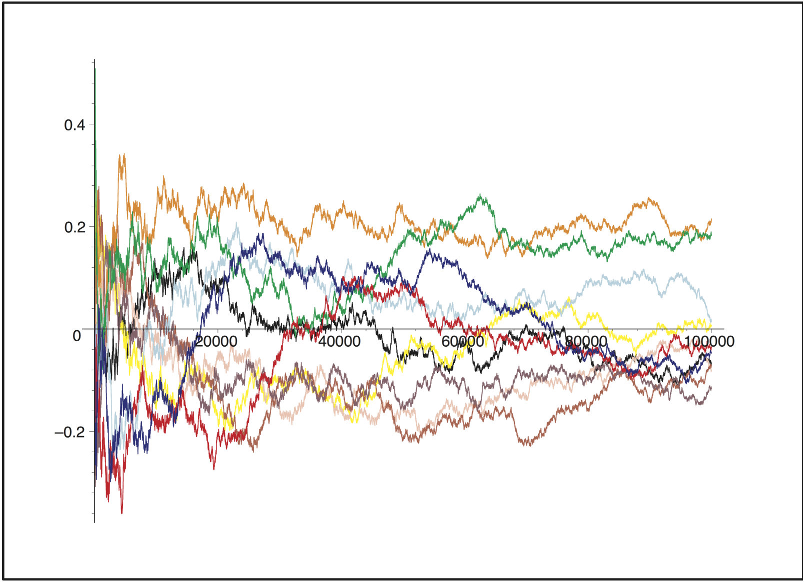
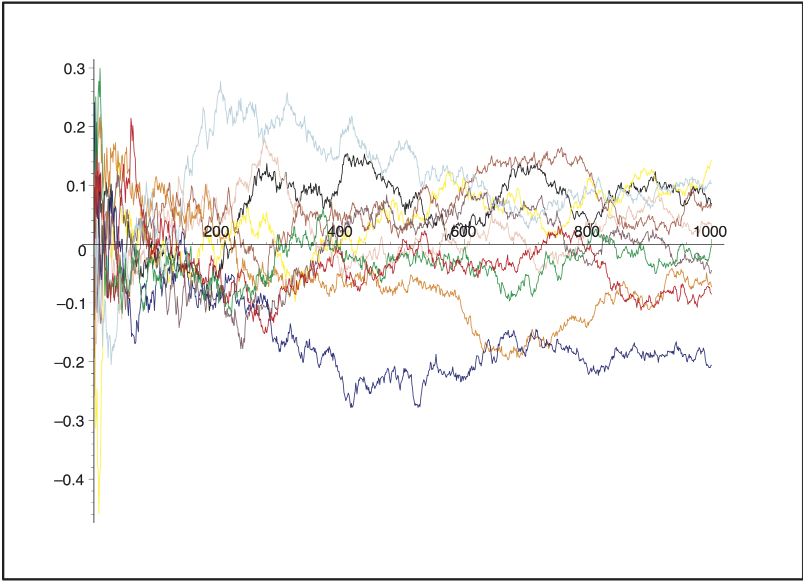


$\varphi = \frac{1+\sqrt{5}}{2}$ for $n = 1, \ldots, 10^6$



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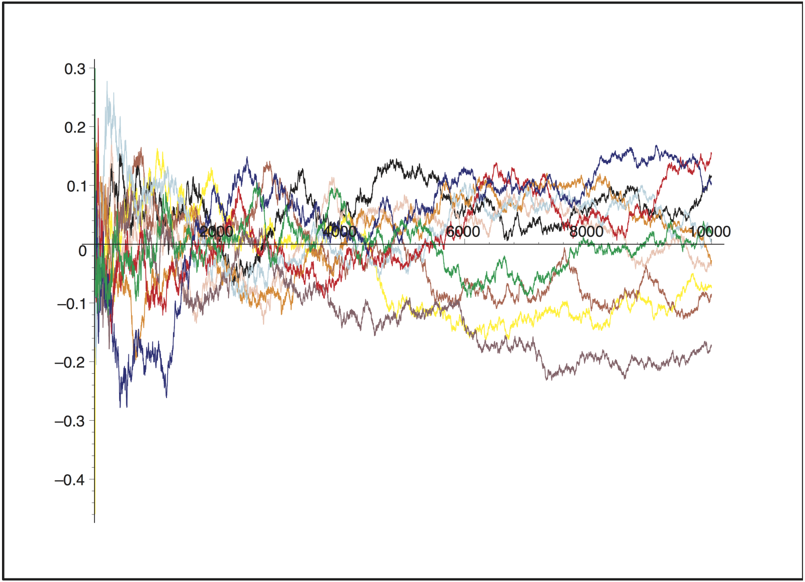

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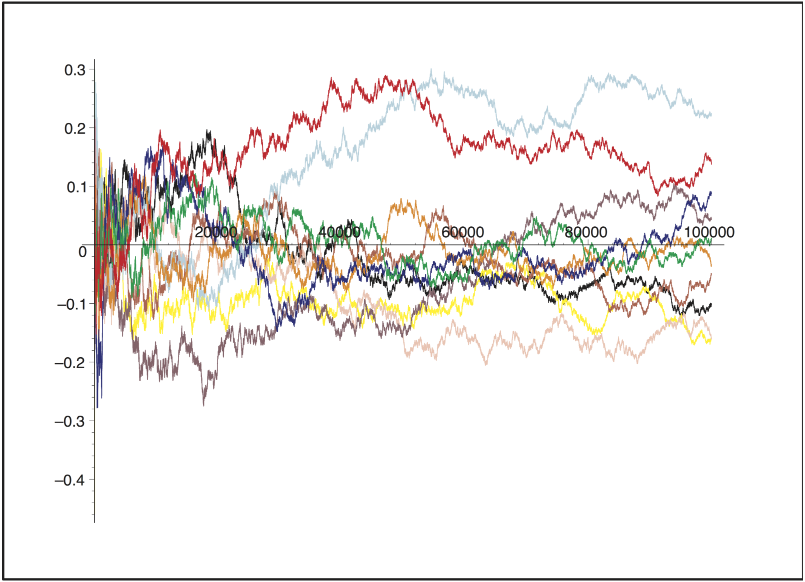
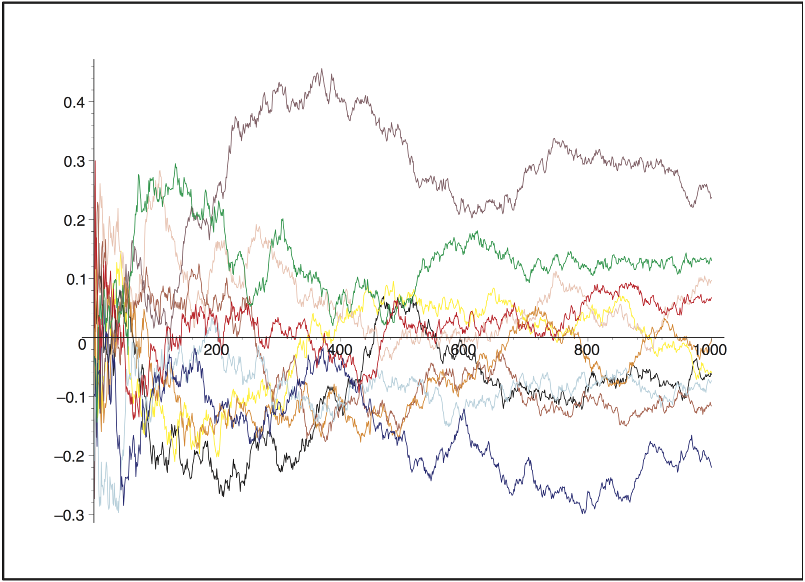


$\log(2)$ for $n = 1, \ldots, 10^6$



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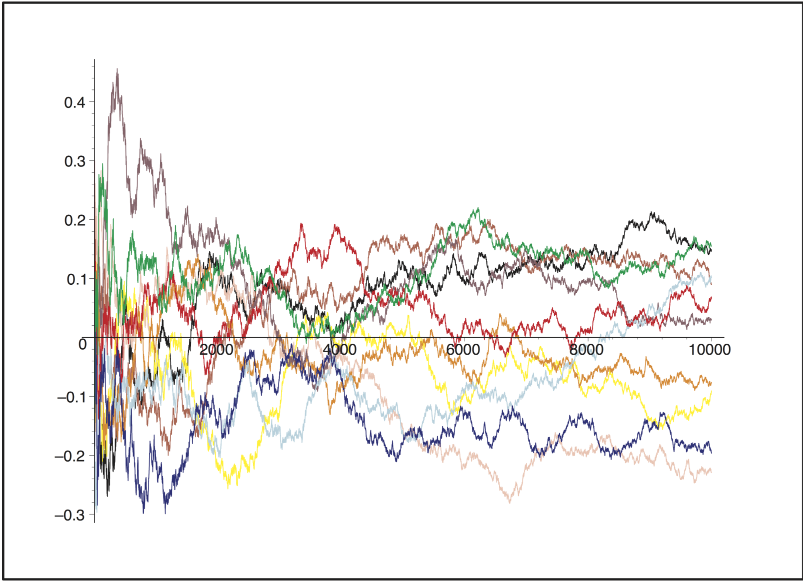

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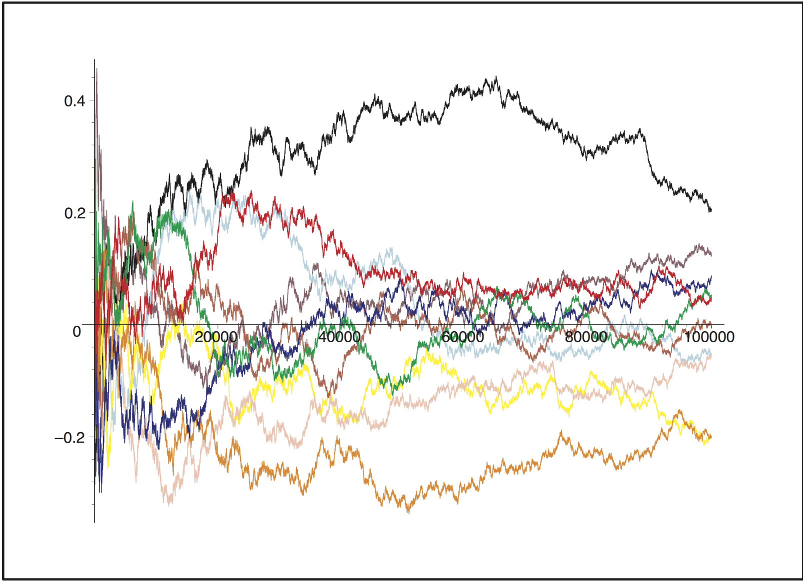
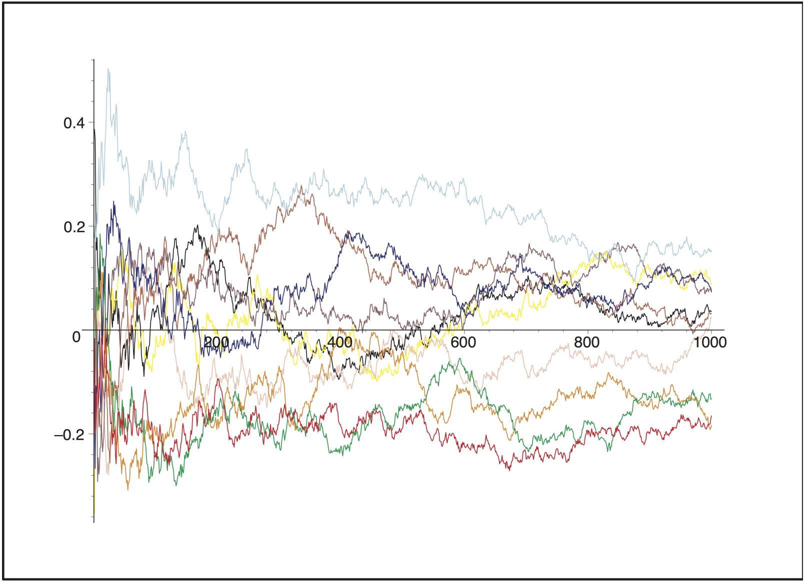


Catalan's Constant for $n = 1, \ldots, 10^6$



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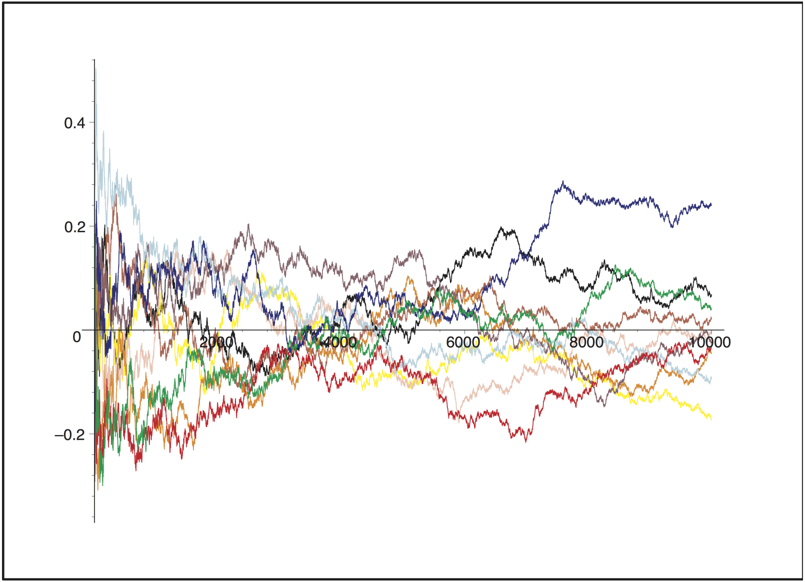

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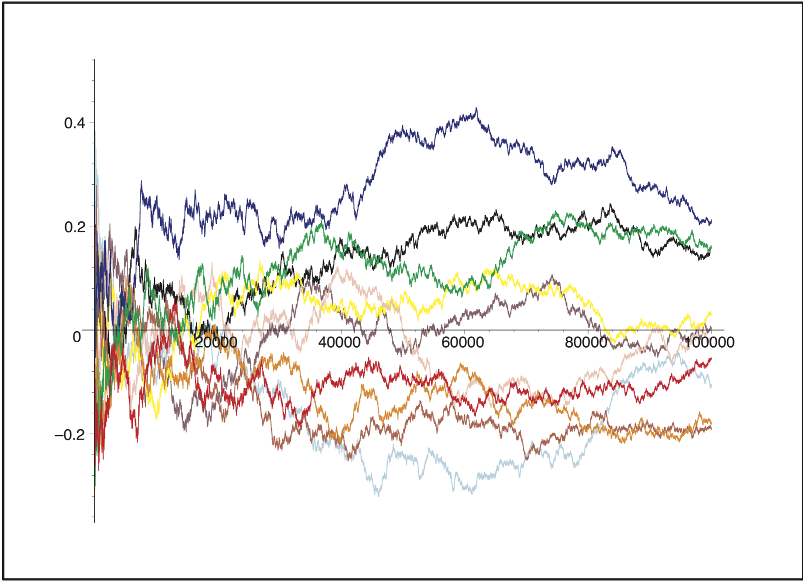
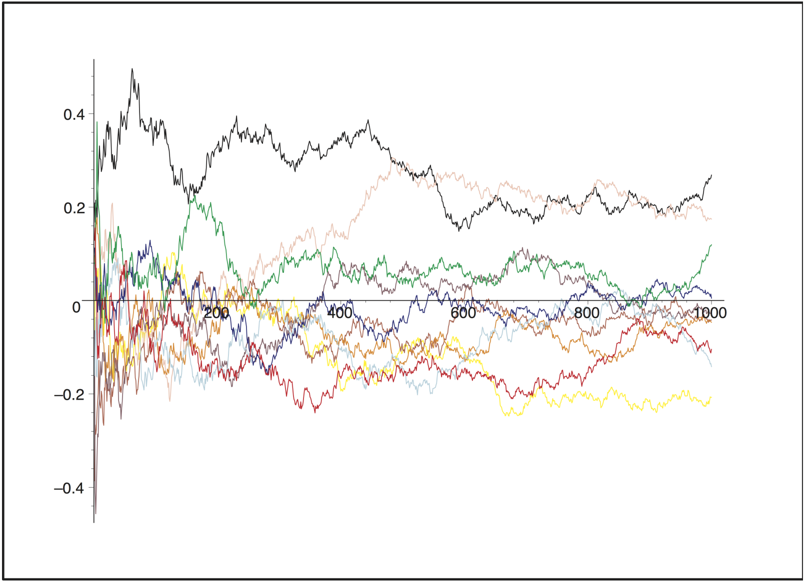


$\zeta(3)$ for $n = 1, \ldots, 10^6$



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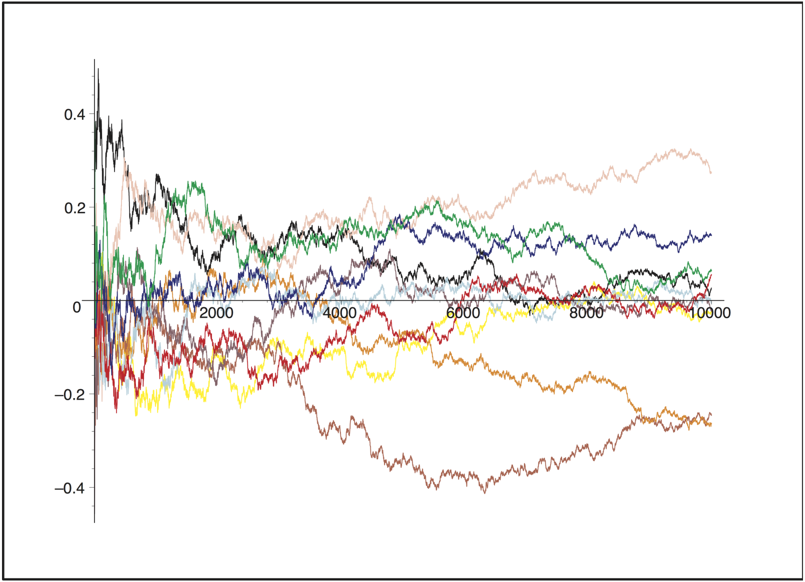

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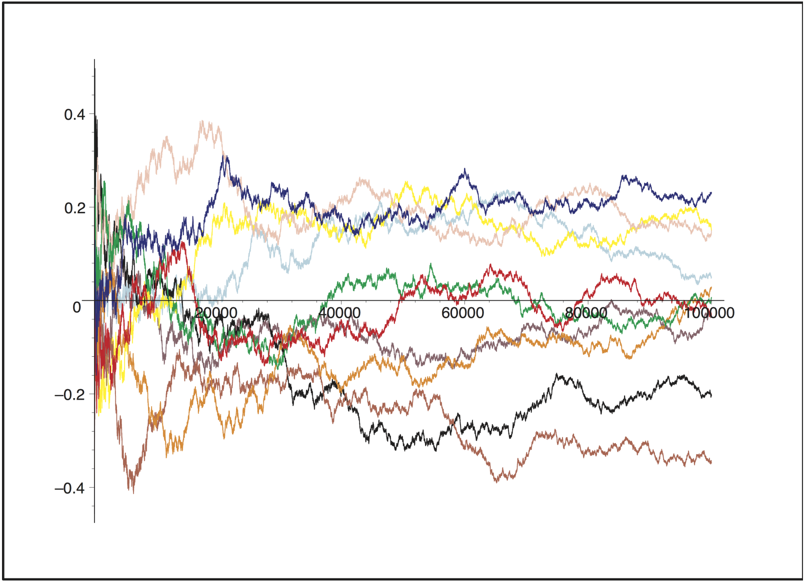

Strong Normality

Conjecture

π, $\zeta(3)$, e, $log(2)$, φ and Catalan's Constants constant are simply strongly normal in base 10.

Future Work

1. Plots for more digits, $10^{10}+$.
2. Plots in other bases apart from 10.
3. Comparing $p_k(n)$ with other functions.
4. A proof for the above conjectures.
5. A proof of the irrationality of $\pi + e$ (Probably above my calibre at this point).
6. Investigations into big data.

Acknowledgements

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- Mr Corey Sinnamon,
- Mr Ghislain McKay,
- Mr Tony Jackson, and
- and all the wonderful people in the Mathematics department.

Thank you!