Short Walks in Higher Dimensions

Ghislain McKay

Febuary 3, 2015

What is a Random Walk?

A path formed by a succession of n steps (of unit length) in random directions.

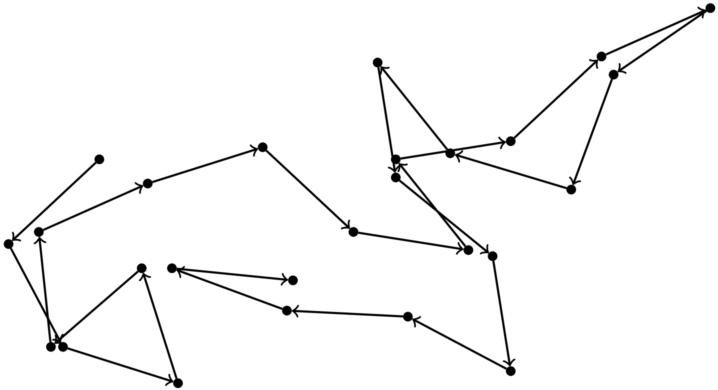

Figure: A 26-step random walk in the plane

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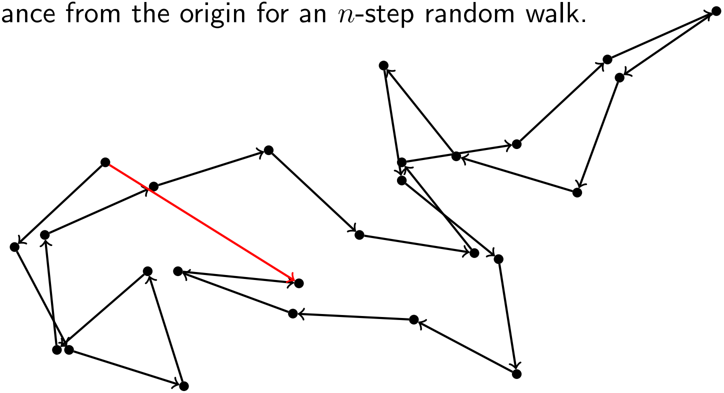

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We look at two functions:

- $p_n(x)$ the probability density function
- $W_n(s)$ the moment function

Probability Density Functions

For a continuous random variable X, the probability density function (pdf) describes the relative likelyhood that X takes on a given value.
The probability of X falling within a range of values is given by the integral of the pdf over that range.

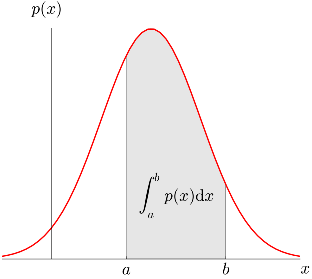

Figure: Probability of X taking on a value in the interval from a to b

Moment Functions

Definition

The s-th moment function of a real-valued continuous function $p(x)$ is

$$W(s) = \int_{-\infty}^{\infty} x^s p(x) \mathrm{d}x$$

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$$W(s) = \mathrm{E}[X^s]$$

where $\mathrm{E}[\cdot]$ is the expected value.

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The moments describe the shape of the distribution independent of translation.

The 2-Dimensional Case

In 2 dimensions we can represent a random walk in the following way

$$\sum_{k=1}^{n} e^{2\pi i x_k} \qquad \text{where } \mathbf{x} \in [0,1]^n$$

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Definition

The moments of the distance from the origin after an n-step random walk in 2-dimensions is given by

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi i x_k} \right|^s \mathrm{d}\mathbf{x}$$

Even Moments in 2 Dimensions

The even moments in 2 dimensions are all integral.

$$
\begin{aligned}
W_2(0;2k) &: 1, 2, 6, 20, 70, 252, 924, 3432, 12870, \ldots \\
W_3(0;2k) &: 1, 3, 15, 93, 639, 4653, 35169, 272835, 2157759, \ldots \\
W_4(0;2k) &: 1, 4, 28, 256, 2716, 31504, 387136, 4951552, 65218204, \ldots \\
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they are given by

$$W_n(2k) = \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, \ldots, k_n}^2 = \sum_{k_1+\cdots+k_n=k} \left(\frac{k!}{k_1! \cdot k_2! \cdots k_n!} \right)^2$$

which counts abelian squares (strings of length $2k$ over an n letter alphabet where the first k letters are a permutation of the last k letters.)

The Probability Density Function

In 1905, Lord Rayleigh gave an asymptotic form for large n

$$p_n(x) \sim \frac{2x}{n} \exp\left(\frac{-x^2}{n}\right) \quad \text{as } n \to \infty$$

a Rayleigh distribution with mean $\sqrt{\frac{n\pi}{4}}$.

The Probability Density Function

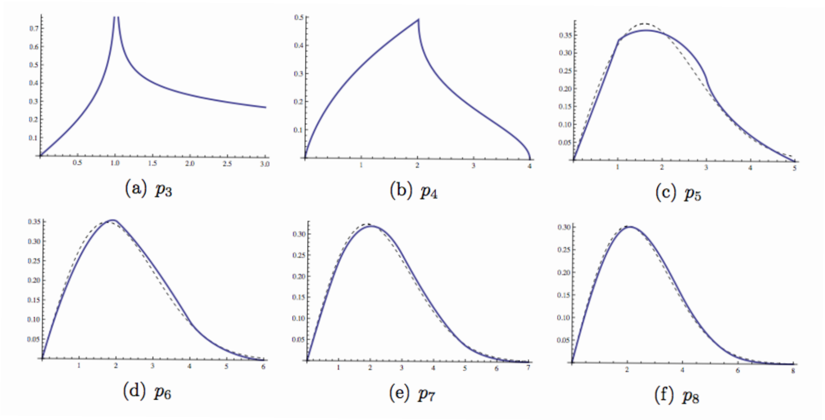

Figure: $p_n(x)$ for $n = 3, 4, \ldots, 8$

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For walks of 7 steps or more this is a very good approximation.

For this reason we will restrict ourselves to n-step walks where $2 \leq n \leq 6$ (hence the name "short" walks).

Gamma Function

Definition

The Gamma function is an extension of the factorial function such that for a positive integer n

$$\Gamma(n) = (n-1)!$$

For complex numbers z with positive real part it can be defined by (Euler's definition)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$$

Bessel Functions of the First Kind

Definition

The Besel function of the first kind $\mathrm{J}_\nu(x)$ is a solution to the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2-\nu^2)y = 0$$

we can define them by their taylor series around $x = 0$

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

Bessel Functions of the First Kind

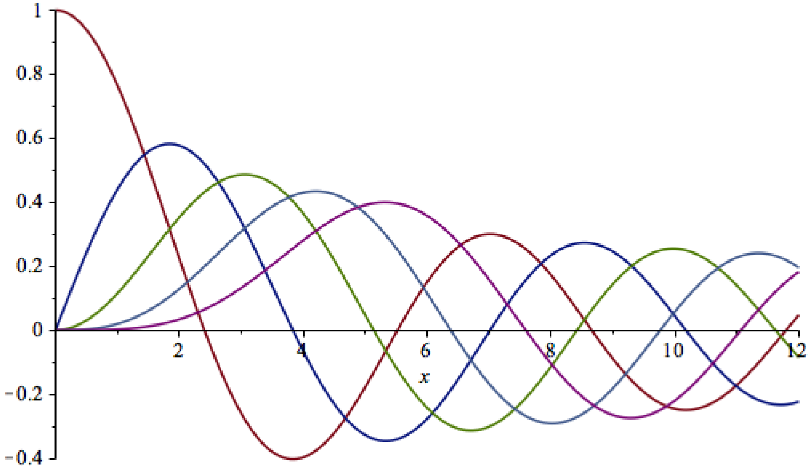

Figure: $J_\nu(x)$ for $\nu = 0, 1, 2, 3, 4$

Towards Higher Dimensions

For walks in $d \geq 2$ dimensions, we define

$$\nu = \frac{d}{2} - 1$$

notice that when $d = 2$ we have $\nu = 0$.

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notice that when $d = 2$ we have $\nu = 0$.

We also define

$$\mathrm{j}_\nu(x) = \nu! \left(\frac{2}{x}\right)^\nu \mathrm{J}_\nu(x)$$

where $J_\nu(x)$ is the Bessel function of the first kind.

In Higher Dimensions

Definition

The probability density function of the distance to the origin in $d \geq 2$ dimensions after $n \geq 2$ steps is

$$p_n(\nu; x) = \frac{1}{2^\nu \nu!} \int_0^\infty (tx)^{\nu+1} \mathrm{J}_v(tx) \mathrm{j}_\nu^n(t) \mathrm{d}t \qquad \text{for } x > 0$$

In Higher Dimensions

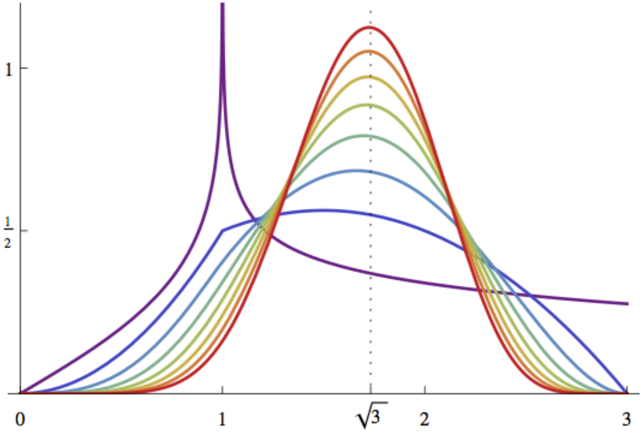

Figure: $p_3(\nu, x)$ for $\nu = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}$

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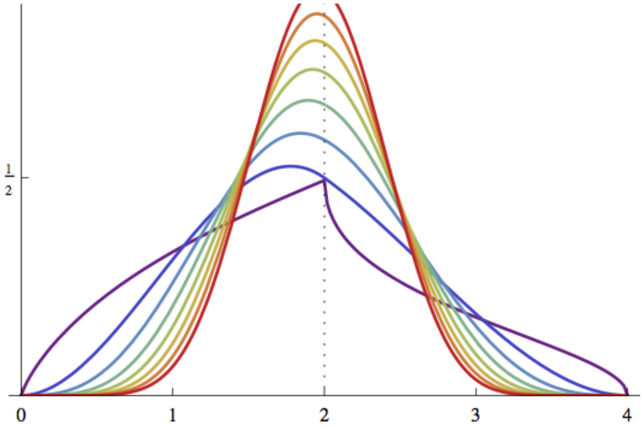

Figure: $p_4(\nu, x)$ for $\nu = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}$

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Asymptotically, for $x > 0$, as $\nu \to \infty$

$$p_n(\nu; x) \sim \frac{2^{-\nu}}{\Gamma(\nu+1)} \left(\frac{2\nu+1}{n}\right)^{\nu+1} x^{2\nu+1} \exp\left(-\frac{2\nu+1}{2n} x^2\right)$$

a Chi distribution with mean $\sqrt{\frac{2n}{2\nu+1}} \frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu+1)} \to \sqrt{n}$ as $\nu \to \infty$.

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The proof follows from

$$\mathrm{j}_\nu(t) \sim \exp\left(\frac{-t^2}{4\nu+2}\right) \text{ as } \nu \to \infty$$

The Moment function

By definition the moment function is

$$W_n(\nu; s) = \int_0^\infty x^s p_n(\nu; x)\mathrm{d}x$$

Theorem

Let $n \geq 2$ and $d \geq 2$. For any nonnegative integer k,

$$W_n(\nu; s) = \frac{2^{s-k+1}\Gamma(\frac{s}{2} + \nu + 1)}{\Gamma(\nu + 1)\Gamma(k - \frac{s}{2})} \int_0^\infty x^{2k-s-1} \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathrm{j}_\nu^n(x)\mathrm{d}x$$

Combinatorial Intepretation of the Moments

Theorem

The even moments of an n-step random walk in d dimensions are

$$W_n(\nu;2k) = \frac{(k+\nu)!\nu!^{n-1}}{(k+n\nu)!} \sum_{k_1+\cdots+k_n=k} \binom{k}{k1,\ldots,k_n} \binom{k+n\nu}{k_1+\nu,\ldots,k_n+\nu}$$

Proof.

Replace k by $k+1$ and set $s=2k$, we obtain

$$\begin{aligned} W_n(\nu;2k) &= \frac{2^k(k+\nu)!}{\nu!} \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}}x\right)^k \mathrm{j}_\nu^n(x)\mathrm{d}x \\ &= \left[\frac{(k+\nu)!}{\nu!}\left(-\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathrm{j}_\nu^n(x)\right]_{x=0} \end{aligned}$$

□

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$$W_n(\nu;2k)=\left[\frac{(k+\nu)!}{\nu!}\left(-\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathrm{j}_\nu^n(x)\right]_{x=0}$$

$$j_\nu(x)=\nu!\sum_{m\geq 0}\frac{(-x^2/4)^m}{m!(m+\nu)!}$$

□

Moment Recursion

For positive integers n_1, n_2, half-integer ν and nonnegative integer k

$$W_{n_1+n_2}(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)(j+\nu)!} W_{n_1}(\nu; 2j) W_{n_2}(\nu; 2(k-j))$$

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In particular when $n_2 = 1$ we have $W_{n_2}(\nu, s) = 1$ we obtain the recursive relation

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This gives a nice way of computing even moments of walks.

Even Moments in 2 and 4 dimensions

The even moments in 2 and 4 dimensions are all integral.

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$$W_6(1;2k) \ : \ 1, 6, 51, 546, 6906, 99156, 1573011, 27045906, 496875786, \ldots$$

Catalan Numbers

Definition

For integers $n > 0$ the Catalan numbers C_n are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad \text{for } n \geq 0$$

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	2	5	14	42	132	429	1430	4862	16796

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The Catalan numbers come up in many combinatorial problems:

- triangulation of convex polygons with $n+2$ sides
- lattice paths from $(0,0)$ to (n,n) below the diagonal
- rooted binary trees with n leaves
- etc.

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Narayana Numbers

Definition

For integers $0 \leq k \leq n$ the Narayana numbers $N(k, j)$ are

$$N(k,j) = \frac{1}{j+1}\binom{k}{j}\binom{k+1}{j}$$

$_k\backslash^j$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	3	1			
3	1	6	6	1		
4	1	10	20	10	1	
5	1	15	50	50	15	1

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0	1						1
1	1	1					2
2	1	3	1				5
3	1	6	6	1			14
4	1	10	20	10	1		42
5	1	15	50	50	15	1	132

the Catalan numbers!

Closed Form for Even Moments

Recall the recursion

$$W_n(\nu;2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)(j+\nu)!} W_{n-1}(\nu;2j)$$

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Definition

For integers ν and k (even moments in even dimensions) we define the matrix $A(\nu)$ by

$$A_{k,j}(\nu) = \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!}$$

notice that in the case where $\nu = 1$, $A(1)$ is the Narayana triangle.

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The moments $W_n(\nu; 2k)$ are given by the sum of the entries in the k-th row of $A(\nu)^{n+1}$.

Narayana Numbers

$$A(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 1 & 3 & 1 & 0 & \\ 1 & 6 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix} : \begin{matrix} 1 \\ 2 \\ 5 \\ 14 \\ \vdots \end{matrix} \qquad A(1)^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \\ 12 & 9 & 1 & 0 & \\ 57 & 72 & 18 & 1 & \\ \vdots & & & & \ddots \end{bmatrix} : \begin{matrix} 1 \\ 4 \\ 22 \\ 148 \\ \vdots \end{matrix}$$

$W_2(1;2k)$: $1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$

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Even Moments of a Three Step Walk

Theorem

The nonnegative even moments for a 3-step walk in d dimensions is

$$W_3(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \binom{k+\nu}{j} \binom{2j+2\nu}{j} \binom{j+\nu}{j}^{-2}$$

Its Ordinary Generating Function is

$$\sum_{k=0}^{\infty} W_3(\nu; 2k) x^k = \frac{(-1)^\nu}{\binom{2\nu}{\nu}} \frac{(1-\frac{1}{x})^{2\nu}}{1+3x} {}_2F_1\left({\frac{1}{3}, \frac{2}{3} \atop 1+\nu} \middle| \frac{27x(1-x)^2}{(1+3x)^3} \right) - q(1/x)$$

where $q(x)$ is a polynomial such that $q(1/x)$ is the principal part of the hypergeometric term on the right.