The Theorem of Copeland and Erdős on Normal Numbers

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Review Of Normality

Definition

A number α is **normal** with respect to the base β, provided each of the digits $0, 1, 2, \ldots, \beta - 1$ occurs with a limiting relative frequency of $1/\beta$, and each of the β^k sequences of k digits occurs with the relative frequency $1/\beta^k$.

Definition

The **natural density** of the subset $A \subset \mathbb{N}$, denoted $d(A)$, is defined as

$$d(A) = \lim_{x \to \infty} \frac{N(x)}{x}$$

where $N(x) := \#\{a : a \in A, a \leqslant x\}$.

Example

The natural density of the even numbers is $1/2$.

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- Normality: The Known and Unkown
- Copeland-Erdős Theorem
- Questions of Strong Normality

Normality: The Known and Unkown

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With this definition in mind, we begin with the unknown:

Conjecture (Borel, 1950)

All real irrational algebraic numbers are normal.

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Normal Numbers

Some of the most **well-known normal numbers** discovered so far include:

1933	$0.123456789\ldots$	Champernowne's Number
1946	$0.23571113\ldots$	Copeland-Erdős Number
1956	$0.f(1)f(2)f(3)\ldots$	Davenport-Erdős Numbers
1973	$\sum_{k=1}^{\infty} \frac{1}{b^{c^k} c^k}$	Stoneham Numbers

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Copeland-Erdős Theorem

Theorem (Copeland-Erdős, 1946)

If $a_1, a_2, a_3, \ldots$ is an increasing sequence of integers such that for every $\theta < 1$ the number of a_i's up to N exceeds N^{θ} provided N is sufficiently large, then the infinite decimal

$$0.a_1 a_2 a_3 \ldots$$

is normal with respect to the base β in which these integers are expressed.

Lemma (Copeland-Erdős, 1946)

The number of integers up to N (N sufficiently large) which are not (ε, k) normal with respect to a given base β is less than N^δ where $\delta = \delta(\varepsilon, k, \beta) < 1$.

To understand this lemma, we must first be familiar with (ε, k) normality:

Definition

A number α (in the base β) is said to be (ε, k) normal if any combination of k digits appears consecutively among the digits of α with a relative frequency between $\beta^{-k} - \varepsilon$ and $\beta^{-k} + \varepsilon$.

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Let x be such that $\beta^{x-1} \leqslant N < \beta^x$, where β^x refers to a number (base$-\beta$) consisting of x digits.

We introduce the notation $\beta_j = (\beta - 1)^{x-j} \binom{x}{j}$, where β_j counts the number of numbers (up to N) which have a single digit, for instance 0, occurring a total of j times amongst their x digits.

Since a given base β has β digits, and since β_j counts the occurrences of only a single digit, there are at most

$$\beta \left(\sum_{j < x(1-\varepsilon)/\beta} \beta_j + \sum_{j > x(1+\varepsilon)/\beta} \beta_j \right)$$

numbers up to N which have less than $x(1-\varepsilon)/\beta$ or more than $x(1+\varepsilon)/\beta$ 0's,1's,...,$(\beta-1)$'s.

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In order to prove the lemma for $(\varepsilon, 1)$ normality, we have to show that

$$\beta \left(\sum_{j < x(1-\varepsilon)/\beta} \beta_j + \sum_{j > x(1+\varepsilon)/\beta} \beta_j \right) < N^\delta.$$

We first require some intermediate inequalities.

We have from the properties of the binomial expansion:

$$\sum_{j < x(1-\varepsilon)/\beta} \beta_j < (x+1)\beta_{r_1}, \qquad \sum_{j > x(1+\varepsilon)/\beta} \beta_j < (x+1)\beta_{r_2}$$

where

$$r_1 = \lfloor (1-\varepsilon)x/\beta \rfloor, \quad r_2 = \lfloor (1+\varepsilon)x/\beta \rfloor$$

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By repeated application of the relation

$$\beta_{j+1}/\beta_j = (x - j)/(j + 1)(\beta - 1)$$

we obtain

$$\beta_{r_1}\rho_1^{\varepsilon x/2} < \beta^x, \quad \beta_{r_2}\rho_2^{\varepsilon x/2} < \beta^x$$

where

$$\rho_1 = (x - r_1)/(r_1 + 1)(\beta - 1), \quad \rho_2 = (x - r_2)/(r_2 + 1)(\beta - 1)$$

and where $\rho_1, \rho_2 > 1$ for x sufficiently large.

It follows that

$$\beta_{r_1} < \left(\rho_1^{-\varepsilon/2}\beta\right)^x, \quad \beta_{r_2} < \left(\rho_2^{-\varepsilon/2}\beta\right)^x$$

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Hence

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$$< \beta(x+1) \left[\left(\rho_1^{-\varepsilon/2} \beta \right)^x + \left(\rho_2^{-\varepsilon/2} \beta \right)^x \right]$$

$$< \beta^{\delta(x-1)} \leqslant N^\delta$$

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Consider the digits $b_0, b_1, \ldots$ of a number $m \leqslant N$ grouped as follows:

$$b_0, b_1, \ldots, b_{k-1}; b_k, \ldots, b_{2k-1}; b_{2k}, \ldots, b_{3k-1}; \ldots$$

Each of these groups represents a single digit of m when m is expressed in the base β^k. Hence, there are at most N^δ integers $m \leqslant N$ for which the frequency among these groups of a given combination of k digits falls outside the interval from $\beta^{-k} - \varepsilon$ to $\beta^{-k} + \varepsilon$.

The same holds for

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Consider the numbers $a_1, a_2, \ldots$ of the increasing sequence up to the largest $a_i \leq N$, where $N = \beta^n$. By a counting argument, these numbers altogether have at least $n(1 - \varepsilon) \cdot (N^\theta - N^{1-\varepsilon})$ digits.

Let f_N be the relative frequency of the digit 0. It follows from the lemma that the number of a_i's for which the frequency of the digit 0 exceeds $\beta^{-1} + \varepsilon$ is at most N^δ, and hence

$$f_N < \beta^{-1} + \varepsilon + \frac{nN^\delta}{n(1 - \varepsilon)(N^\theta - N^{1-\varepsilon})}$$
$$= \beta^{-1} + \varepsilon + \frac{N^{\delta-\theta}}{(1 - \varepsilon)(1 - N^{1-\varepsilon-\theta})}$$

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Since we are permitted to take θ greater than δ and greater than $1 - \varepsilon$, it follows that $\lim_{N \to \infty} f_N$ is at most $\beta^{-1} + \varepsilon$ and hence at most β^{-1}.

A similar result holds for the digits $1, 2, \ldots, \beta - 1$ and hence each of these digits must have a limiting relative frequency of exactly β^{-1}.

In a similar manner, it can be shown that the limiting relative frequency of any combination of k digits is β^{-k}. $\qquad\square$

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Questions of Strong Normality

Strong Normality

Definition

A number α is **simply strongly normal** to the base β, if for each $k \in \{0, 1, \ldots, \beta - 1\}$, we have

$$\limsup_{n \to \infty} \frac{m_k(n) - n/\beta}{\sqrt{2n \log \log n}} = \frac{\sqrt{\beta - 1}}{\beta} \qquad \text{and}$$

$$\liminf_{n \to \infty} \frac{m_k(n) - n/\beta}{\sqrt{2n \log \log n}} = -\frac{\sqrt{\beta - 1}}{\beta}$$

where $m_k(n) := \#\{i : a_i = k, i \leqslant n\}$.

A number is **strongly normal** to the base β if it is simply strongly normal in each base $\beta^j, j = 1, 2, 3, \ldots$, and is **absolutely strongly normal** if it is strongly normal to every base.

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$$\lim_{x \to \infty} \frac{N(x)}{x} = \frac{1}{\beta^k}$$

where $N(x)$ is the number of occurrences of $a_1 a_2 \ldots a_k$ in the first x digits of α.

Some interesting results arising from these definitions are:

- A number which is strongly normal to the base β is normal to the base β.
- Champernowne's base-β number is not strongly normal to the base β.

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Conjecture

The number $\alpha\ (= 0.a_1 a_2 a_3 \dots)$ formed from the concatenation of the increasing sequence $a_1, a_2, a_3, \dots$ is not strongly normal, provided that the sequence of integers is dense enough, that is, $N(x) > x^\theta$ for every $\theta < 1$ and sufficiently large x.

- **Heuristic:** This conjecture is put forward as a consequence of Champernowne's number failing to be strongly normal. We believe that all the other concatenation numbers should also fail this strong normality test, the reason being that these sequences are just too dense – there are too few integers being excluded. In this way, we see these concatenation numbers as being basically the same as Champernowne's number – too structured – and thus we conjecture that these numbers should fail to be strongly normal.

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