# An Investigation Into Gram Matrices Of Rectangular $\pm 1$ Matrices

Joshua Hartigan
Supervisor: Judy-anne Osborn

# Gram Matrices

- Here's a ± matrix $R = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \end{pmatrix}$
- And here's its' Gram matrix $G = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$
- In general, the Gram matrix is

$$G = RR^T$$

# Why bother?

- Gram matrices relate to determinants and high determinants are interesting to combinatorialists and statisticians

# Context

- A lot of work has been done on square $\pm 1$ matrices, their Gram matrices and their determinants
- We decided to investigate rectangular $\pm 1$ matrices and were going to look at determinants but got interested in Gram matrices along the way for their own sake

# The first thing we tried

- We started with random $\pm 1$ matrices, computed their Gram matrices and looked at what we got
- We found Gram matrices like this:

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -3 \\ -1 & -3 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -3 \\ 1 & -3 & 3 \end{pmatrix}$$

# Things we noticed for Gram matrices of k x n ±1 matrices

- 'n's on the diagonal
- Symmetry
- All entries either even or odd, and from the set $\{-n, -n+2, \ldots, n\}$

- And we can prove them all, so it's a Theorem

# e.g. Proof of symmetry

- Take any k x n matrix, called R:

$$R = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix}$$

Our definition of Gram matrices is that $G = RR^T$

So, to get the ijth entry of the Gram matrix, we take the dot product of row i with row j, i.e:

$G_{ij} = r_i \cdot r_j$

Similarly, for entry $G_{ji} = r_j \cdot r_i = r_i \cdot r_j = G_{ij}$

Hence, Gram matrices are always symmetric.

# Next, we were more systematic

- We considered 2 x n $\pm 1$ matrices for n=1..10
- And 3 x n case
- And 4 x n case
- And 5 x n case

- And then the computer went crazy

# Computing the frequency

- With the previous theorem, we focused on the entries on the right hand side of the main diagonal
- As all of these entries came from the set {-n, -n+2,…, n}, we could code these entries in their respective base and add them up, giving each matrix its own ID and allowing us to find the frequency each matrix occurred

# Encoding Grams example

- Take the Gram matrix G= $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -3 \\ -1 & -3 & 3 \end{pmatrix}$

This comes from a 3x3 ±1 matrix, so the possible entries off the main diagonal come from the set {-3, -1, 1, 3}->{0,1,2,3} in base 4. Doing the appropriate sum allows us to create an ID for each distinct Gram:

$$((\frac{1}{2}) + (\frac{3}{2})) \times 4^2 + ((\frac{-1}{2}) + (\frac{3}{2})) \times 4 + ((\frac{-3}{2}) + \frac{3}{2})$$

$$= (2) \times 4^2 + (1) \times 4 + 0$$

$$= 36$$

# Here's some data

$$\begin{pmatrix} \pm \\ \pm \end{pmatrix} : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} ; \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \pm & \pm \\ \pm & \pm \end{pmatrix} : \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} ; \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} ; \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \pm & \pm & \pm \\ \pm & \pm & \pm \end{pmatrix} : \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} ; \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} ; \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} ; \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

Curiously, all possible Grams occurred subject to our Theorem

# More data

- Furthermore, they occurred with the following frequencies:

$$\begin{pmatrix}\pm\\\pm\end{pmatrix} : \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} ; \begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

2 2

$$\begin{pmatrix}\pm & \pm\\\pm & \pm\end{pmatrix} : \begin{pmatrix}2 & 2\\2 & 2\end{pmatrix} ; \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} ; \begin{pmatrix}2 & -2\\-2 & 2\end{pmatrix}$$

4 8 4

$$\begin{pmatrix}\pm & \pm & \pm\\\pm & \pm & \pm\end{pmatrix} : \begin{pmatrix}3 & 3\\3 & 3\end{pmatrix} ; \begin{pmatrix}3 & 1\\1 & 3\end{pmatrix} ; \begin{pmatrix}3 & -1\\-1 & 3\end{pmatrix} ; \begin{pmatrix}3 & -3\\-3 & 3\end{pmatrix}$$

8 24 24 8

# Here's those frequencies again

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 2 | | 2 | | | |
| | | 4 | | 8 | | 4 | | |
| | 8 | | 24 | | 24 | | 8 | |
| 16 | | 64 | | 96 | | 64 | | 16 |

…Anyone notice anything?

# Here's what we noticed

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | (1 | | | | 1) | |
| 4 | (1 | | 2 | | 1) | |
| 8 | (1 | 3 | | 3 | 1) | |
| 16 | (1 | 4 | 6 | 4 | 1) | |

Pascal's Triangle in disguise!

# We can explain the powers of 2!

Multiplying a column by −1 doesn't change the Gram for a 2 x n R−matrix!

- Proof:

Let's begin with any 2 x n matrix R= $\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \end{pmatrix}$

Now, take any column and multiply by −1:

R' = $\begin{pmatrix} -a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ -a_{2,1} & a_{2,2} & \cdots & a_{2,n} \end{pmatrix}$

Finding the Gram: G= $\begin{pmatrix} -a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ -a_{2,1} & a_{2,2} & \cdots & a_{2,n} \end{pmatrix} \begin{pmatrix} -a_{1,1} & -a_{2,1} \\ a_{1,2} & a_{2,2} \\ \vdots & \vdots \\ a_{1,n} & a_{2,n} \end{pmatrix}$

# We can explain the powers of 2!

- G= $\begin{pmatrix} (-a_{1,1})^2 + a_{1,2}^2 + \cdots + a_{1,n}^2 & (-a_{1,1})(-a_{2,1}) + a_{1,2}a_{2,2} + \cdots + a_{1,n}a_{2,n} \\ (-a_{2,1})(-a_{1,1}) + a_{2,2}a_{1,2} + \cdots + a_{2,n}a_{1,n} & (-a_{2,1})^2 + a_{2,2}^2 + \cdots + a_{2,n}^2 \end{pmatrix}$

$$= \begin{pmatrix} a_{1,1}^2 + a_{1,2}^2 + \cdots + a_{1,n}^2 & a_{1,1}a_{2,1} + a_{1,2}a_{2,2} + \cdots + a_{1,n}a_{2,n} \\ a_{2,1}a_{1,1} + a_{2,2}a_{1,2} + \cdots + a_{2,n}a_{1,n} & a_{2,1}^2 + a_{2,2}^2 + \cdots + a_{2,n}^2 \end{pmatrix}$$

Which is the same Gram that comes from a $\pm 1$ matrix where the first column isn't multiplied by $-1$. There are $2^n$ choices of sign change of columns.

# We can explain the powers of 2!

- As multiplying columns by −1 doesn't change the resulting Gram matrix, we can reduce the number of R-matrices used to find all Grams by making every entry in the first row +1.

- So we made our program more efficient by applying this.

# We can explain Pascal's Triangle too!

Remember, first row all +1s now!

Then look at the number of ways to put −1 in the second row:

2x1 case:

| + | − |
|---|---|
| 0 "–" | 1 "–" |

Binomial coefficients: $\binom{1}{0}$ $\binom{1}{1}$

# We can explain Pascal’s Triangle too!

- 2x2 case:

| 0 “–” | 1 “–” | 2 “–”s |
|---|---|---|
| [ ][ ] | [–][ ] | [–][–] |
| | [ ][–] | |
| $\binom{2}{0}$ | $\binom{2}{1}$ | $\binom{2}{2}$ |

2x3 case:

| 0 “–” | 1 “–” | 2 “–”s | 3 “–”s |
|---|---|---|---|
| [ ][ ][ ] | [–][ ][ ] | [–][–][ ] | [–][–][–] |
| | [ ][–][ ] | [–][ ][–] | |
| | [ ][ ][–] | [ ][–][–] | |
| $\binom{3}{0}$ | $\binom{3}{1}$ | $\binom{3}{2}$ | $\binom{3}{3}$ |

# Conclusion: Pascal's Triangle

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 1 | | 1 | | | |
| | | 1 | | 2 | | 1 | | |
| | 1 | | 3 | | 3 | | 1 | |
| 1 | | 4 | | 6 | | 4 | | 1 |

# 3xn is more mysterious

Interestingly, not all possible Grams occur.

3x1 case: Out of 8 possible Grams, only 4 occur each with a frequency of 2

3x2 case: Out of 27 possible Grams, only 10 occur with a frequency of: 4 8 4 8 8 8 8 4 8 4

3x3 case: Out of 64 possible Grams, only 20 occur with a frequency of:

8 24 24 24 8 24 48 24 24 48 24 24 24 48 48 24 24 8 24 24 8

# Frequencies

- Once again, we can take out powers of 2 and now end up with something which contains Pascals triangle:

```
                              2  (1  1  1  1)
                    4  (1  2  1  2  2  2  2  1  2  1)
8  (1  3  3  1  3  6  3  3  6  3  3  3  6  6  3  3  1  3  3  1)
```

This can be explained in a similar way to that of the 2xn cases, it just has an extra row of possible $\pm 1$s!

# Explaining those frequencies

| |
|---|
| + + + ..................................+ |
| |
| |

The first row is all +1s

# Explaining those frequencies

<table>
<tr><td colspan="2">+ + + ...............................+</td></tr>
<tr><td>+..................+</td><td>−...................−</td></tr>
<tr><td colspan="2"></td></tr>
</table>

Now we'll arrange the second row so all the +1s are on the left.

# Explaining those frequencies

<table>
<tr><td colspan="4">+ + + ..............................+</td></tr>
<tr><td colspan="2">+..................+</td><td colspan="2">−..................−</td></tr>
<tr><td>+......+</td><td>−......−</td><td>+......+</td><td>−......−</td></tr>
</table>

In the third row, within each "block", arrange all +1s on the left.

# Explaining those frequencies

<table>
<tr><td colspan="4">+ + + ...............................+</td></tr>
<tr><td colspan="2">+...................+</td><td colspan="2">−.....................−</td></tr>
<tr><td>+......+</td><td>−......−</td><td>+......+</td><td>−......−</td></tr>
</table>

2nd row: k minuses, means $\binom{n}{k}$ possibilities

3rd row: i minuses in the left block (length n−k), and j minuses in the right block means $\binom{n-k}{i}\binom{k}{j}$ possibilities.

# 3xn frequency multi-set

$$\{\binom{n}{k}\binom{n-k}{i}\binom{k}{j};0 \le i,j,k \le n\}$$

e.g. when n=2:

1 2 1 2 2 2 2 1 2 1

$$\binom{2}{0}\binom{2}{0};\ \binom{2}{0}\binom{2}{1};\ \binom{2}{0}\binom{2}{2}$$

$$\binom{2}{1}\binom{1}{0}\binom{1}{0};\ \binom{2}{1}\binom{1}{0}\binom{1}{1};\ \binom{2}{1}\binom{1}{1}\binom{1}{0};\ \binom{2}{1}\binom{1}{1}\binom{1}{1}$$

$$\binom{2}{2}\binom{2}{0};\ \binom{2}{2}\binom{2}{1};\ \binom{2}{2}\binom{2}{2}$$

# Frequencies multi-set summary

- 2xn: $\{\binom{n}{k};0 \leq k \leq n\}$
- 3xn: $\{\binom{n}{k}\binom{n-k}{i}\binom{k}{j};0 \leq i,j,k \leq n\}$

# Frequencies multi-set summary

- 2xn: $\{\binom{n}{k}; 0 \le k \le n\}$
- 3xn: $\{\binom{n}{k}\binom{n-k}{i}\binom{k}{j}; 0 \le i, j, k \le n\}$
- 4xn: $\{\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{k_1}{k_3}\binom{n-k_1-k_2}{k_4}\binom{k_2}{k_5}\binom{k_1-k_3}{k_6}\binom{k_3}{k_7}; \forall k_i\}$

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# Frequencies multi-set summary

- 2xn: $\{\binom{n}{k}; 0 \le k \le n\}$
- 3xn: $\{\binom{n}{k}\binom{n-k}{i}\binom{k}{j}; 0 \le i, j, k \le n\}$
- 4xn: $\{\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{k_1}{k_3}\binom{n-k_1-k_2}{k_4}\binom{k_2}{k_5}\binom{k_1-k_3}{k_6}\binom{k_3}{k_7}; \forall k_i\}$

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This is nice, but gets intricate…

…so we decided to look at a simpler question

# Frequencies aside, how many Grams are there?

- For 3xn, remember we had (empirically)
  - n=1: 4 out of 8
  - n=2: 10 out of 27
  - n=3: 20 out of 64
  - n=4: 35 out of 729

# Frequencies aside, how many Grams are there?

- For 3xn, remember we had (empirically)
  - n=1: 4 out of 8
  - n=2: 10 out of 27
  - n=3: 20 out of 64
  - n=4: 35 out of 125

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 1 | | | | | | | |
| | | | | | | 1 | | 1 | | | | | | |
| | | | | | 1 | | 2 | | 1 | | | | | |
| | | | | 1 | | 3 | | 3 | | 1 | | | | |
| | | | 1 | | 4 | | 6 | | 4 | | 1 | | | |
| | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | | |
| | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 | |
| 1 | | 7 | | 21 | | 35 | | 35 | | 21 | | 7 | | 1 |

# Gram counting formulas

2x1 case: $2=\binom{2}{1}$

2x2 case: $3=\binom{3}{1}$

2x3 case: $4=\binom{4}{1}$

$$\binom{n+1}{1}$$

3x1 case: $4=\binom{4}{3}$

3x2 case: $10=\binom{5}{3}$

3x3 case: $20=\binom{6}{3}$

$$\binom{n+3}{3}$$

Still empirical

# Following diagonals on Pascal's Triangle

1 ← 2xn

1 1

1 2 1 ← 3xn

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

# Following diagonals on Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

2xn

3xn

4xn ?

# Counting formula conjecture

We have:

- 2xn: #Grams $= \binom{n+1}{1}$
- 3xn: #Grams $= \binom{n+3}{3}$

# We thought we had a conjecture for 4xn too:

Because…

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

# We thought we had a conjecture for 4xn too:

But…

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

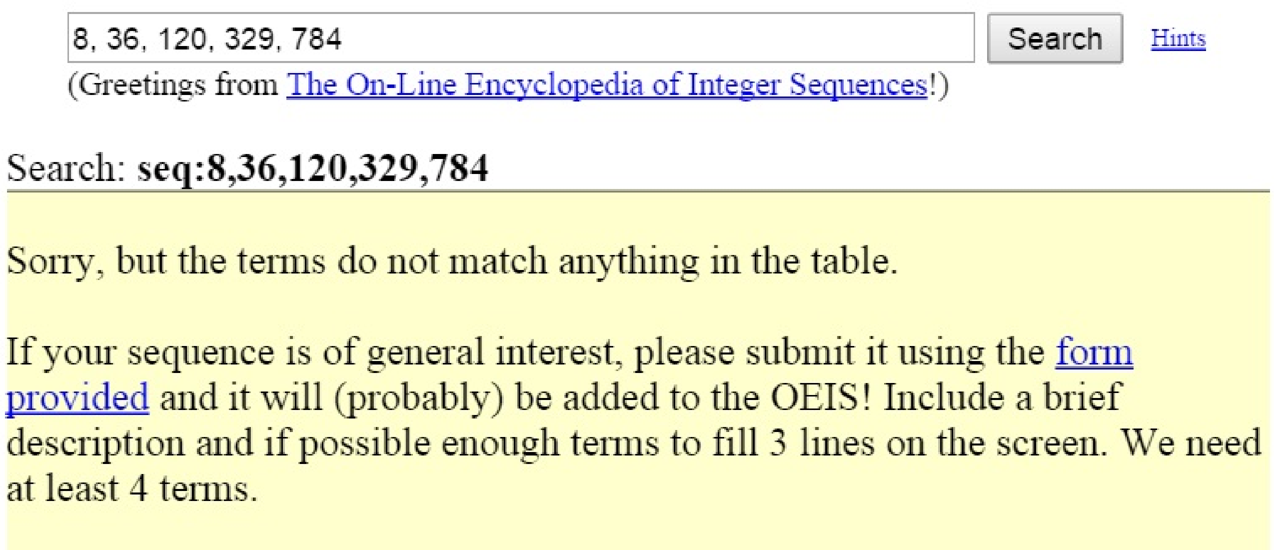
1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 ~~330~~ 165 55 11 1

1 12 66 220 495 792 924 ~~792~~ 495 220 66 12 1

# The actual sequence is

- 8, 36, 120, 329, 784
- Unfortunately, this does not occur in Sloan's online Encyclopedia of Integer Sequences:



# Program limitations

- We are beginning to hit the limits of how far we can investigate using our C-program. For example, the 6x3 case is causing the program to crash

# So we still have mysteries to investigate further!

Thanks for your attention