Solving Partial Differential Equations with Finite Element Methods

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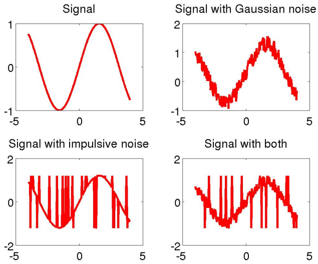
"The art of doing mathematics consists in finding that special case which contains all the germs of generality" by D. Hilbert (1862-1943).

Table of Contents

Noisy Signal

An Image Processing Problem

- When a signal or an image is transmitted over some faulty communication lines, they become corrupted.
- The exact nature of corruption is not known a priori. However, the corruption can be modelled as Gaussian or impulsive noise or the mixture of both.



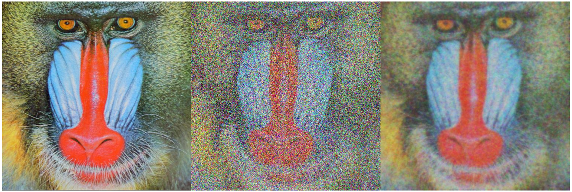
- Only Gaussian noise $\Rightarrow$ Use Fourier or Wavelets smoothing
- Only impulsive $\Rightarrow$ Detect the impulses, remove them and reconstruct using interpolation
- Mixture of both $\Rightarrow$ Detect the impulses, remove them and interpolate but how to filter the Gaussian noise? Fourier and Wavelets perform badly on smoothing scattered data

Let $\mathcal{T}$ be the operator modelling the faulty transmission line. Modelling the signal as a vector $\mathbf{f}$, the new signal is $\mathbf{z} = \mathcal{T}\mathbf{f}$ or $z_i = \mathcal{T} f_i$ component-wise.

Filtering the Mixture of Gaussian and Impulsive Noise

Use the finite element smoothing for the mixture noise. The basic idea is to take into account the derivative of the signal. Here $\{z_i\}_{i=1}^N$ is the corrupted signal but $u(x,y)$ is the recoverd signal. We want to solve the minimisation problem with V as the **finite element space**:

$$\min_{u\in V} \underbrace{\sum_{i=0}^{N}(u(x_i,y_i)-z_i)^2}_{\text{control the error}} + \lambda \underbrace{\int_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy}_{\text{control the smoothness}}$$



Fluid Flow

For a prescribed body force $\boldsymbol{f} \in [L^2(\Omega)]^d$, the Stokes equations with homogeneous Dirichlet boundary condition on the boundary of Ω reads

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in} \quad \Omega \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega \end{aligned} \tag{1}$$

with $\boldsymbol{u} = \boldsymbol{0}$ on the boundary of Ω. where $\boldsymbol{u}$ is the velocity, p is the pressure, and ν denotes the viscosity of the fluid.

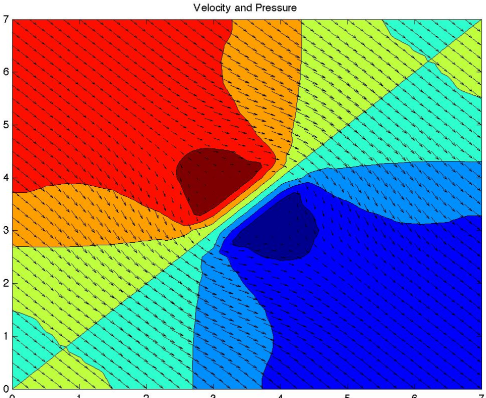
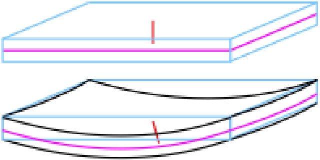


Plate Deformation

Mixed formulation of Reissner–Mindlin plate with clamped boundary condition is to find $(\boldsymbol{\phi}, u, \boldsymbol{\zeta}) \in [H_0^1(\Omega)]^2 \times H_0^1(\Omega) \times [L^2(\Omega)]^2$ such that

$$\begin{aligned}
\int_\Omega \mathcal{C}\boldsymbol{\epsilon}(\boldsymbol{\phi}) : \boldsymbol{\epsilon}(\boldsymbol{\psi})\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{\psi} - \nabla v) \cdot \boldsymbol{\zeta}\, d\boldsymbol{x} &= \ell(v), \quad (\boldsymbol{\psi}, v) \in [H_0^1(\Omega)]^2 \times H_0^1(\Omega), \\
\int_\Omega (\boldsymbol{\phi} - \nabla v) \cdot \boldsymbol{\beta}\, d\boldsymbol{x} - \frac{t^2}{\lambda(1-t^2)}(\boldsymbol{\zeta}, \boldsymbol{\beta}) &= 0, \quad \boldsymbol{\beta} \in [L^2(\Omega)]^2
\end{aligned}$$

u: transverse displacement
ϕ: rotation of the transverse normal vector
ζ: shear stress (Lagrange multiplier)



Thank You!!!

Thank You