Automated Theorem Proving

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Sequence of lectures

1. Formal methods in mathematics
2. Automated theorem proving
3. Interactive theorem proving
4. Formal methods in analysis

Automated reasoning

Ideal: given an assertion, φ, either

- provide a proof that φ is true (or valid), or
- give a counterexample

Dually: given some constraints, φ, either

- provide a solution, or
- prove that there aren't any.

Partial solutions:

- search for proofs
- search for solutions

Automated reasoning

One can distinguish between:

- Domain-general methods and domain-specific methods
- Decision procedures and search procedures
- “Principled” methods and heuristics

Automated reasoning

Domain-general methods:

- Propositional theorem proving
- First-order theorem proving
- Equational reasoning
- Higher-order theorem proving
- Nelson-Oppen "combination" methods

Domain-specific methods:

- Linear arithmetic (integer, real, or mixed)
- Nonlinear real arithmetic (real closed fields, transcendental functions)
- Algebraic methods (such as Gröbner bases)

References

An excellent starting point:

- Harrison, John, *Handbook of Practical Logic and Automated Reasoning*, Cambridge University Press, 2009.

An authoritative reference:

- J. Allen Robinson and Andrei Voronkov, *Handbook of Automated Reasoning*, MIT Press and North Holland, 2001.

General message: automated methods do especially well on large, homogeneous problems, but often fail to capture even the most straightforward mathematical inferences.

Propositional logic

In this talk, I will focus on *classical* logic.

Start with variables $p, q, r, \ldots$. (Semantics: each can be either "true" or "false".)

Build compound formulas with $\wedge$, $\vee$, $\neg$, $\to$, for example

$$p \wedge q \wedge \neg r \to \neg(\neg p \vee s) \vee (\neg s \wedge q).$$

A formula φ is

- *satisfiable* if there is some assignment of truth values that makes is true,
- *valid* if every truth assignment makes it true.

Propositional logic

Note: φ is valid iff $\neg\varphi$ is unsatisfiable.

Challenge: given φ, prove φ, for find a falsifying assignment.

Dually: given φ, find a satisfying assignment, or establish that there is none.

Propositional logic

A variable p is called an *atomic* formula, p and $\neg p$ are *literals*

Normal forms:

- Negation normal form (NNF): built up from literals using only $\wedge$ and $\vee$. Example:

$$\neg p \vee \neg q \vee r \vee (p \wedge \neg s) \vee (\neg s \wedge q).$$

- Disjunctive normal form (DNF): $\bigvee \varphi_i$, where each φ_i is a conjunction of literals.
- Conjunctive normal form (CNF): $\bigwedge \varphi_i$, where each φ_i is a disjunction of literals.

Putting a formula in NNF is cheap, but putting a formula in DNF or CNF can yield exponential increase in length.

Propositional logic

Tseitin's trick (1968): given φ, one can find an equisatisfiable DNF φ' efficiently (length $O(n)$).

Dually, there is a short CNF φ'' that is valid iff φ is.

Idea: introduce new variables to define subformulas and avoid blowup.

Rephrased challenge:

- Decide whether a formula in DNF is satisfiable.
- Decide whether a set of clauses (disjunctions of literals) is satisfiable.
- Decide whether a formula in CNF is valid.

Propositional logic

I will describe three approaches

- tableau (cut-free) proofs
- resolution
- DPLL (Davis-Putnam-Logemann-Loveland)

Tableau proofs

Consider a 1-sided sequent calculus:

- Use formulas in negation normal form ($\wedge$, $\vee$, p, $\bar{p}$).
- Define $\neg\varphi$ by switching $\wedge$ and $\vee$, p and $\bar{p}$, e.g.

$$\neg(p \wedge (\bar{q} \vee r)) \quad \mapsto \quad \bar{p} \vee (q \wedge \bar{r}).$$

- A *sequent* is a finite set $\{\varphi_1, \ldots, \varphi_n\}$, read disjunctively.

Rules:

$$\Gamma, p, \bar{p} \qquad \frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi} \qquad \frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}$$

Tableau proofs

Notice that each rule is “bidirectional”: the conclusion is valid iff the hypothesis is.

Reading upwards, the $\wedge$ and $\vee$ rules remove one connective.

A sequent with only literals (variables and negated variables) is valid if and only if it is an axiom.

Applying the rules backwards then either yields a proof, or a counterexample.

Resolution theorem proving

Using Tseitin's trick, we can reduce the goal of proving φ to refuting a set of clauses.

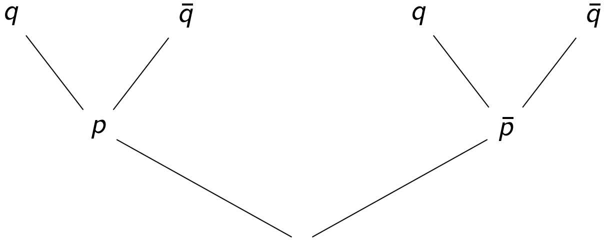
Let Γ, Δ stand for clauses, e.g. $\{p, q, \bar{r}, \bar{s}, t\}$.

Use the resolution rule:

$$\frac{\Gamma \vee \varphi \qquad \Delta \vee \neg\varphi}{\Gamma \vee \Delta}$$

Keep deriving new clauses in this way, until we obtain the empty clause $\{\}$, or cannot make any more progress.

DPLL



- Try to decide satisfiability of a set of clauses.
- Keep track of which clauses are not yet satisfied.
- Use unit propagation.
- Backtrack when a clause is unsatisfiable.

SAT solvers

Most modern SAT solvers use variants of DPLL.

Innovations:

- non-chronological backtracking
- conflict-driven clause learning
- efficient data structures and implementation tricks

Modern SAT solvers can handle tens of thousands of variables and millions of clauses.

First-order theorem proving

First-order logic adds relations $r(x, y, z)$, functions $f(x, y)$, $g(x)$, and quantifiers $\exists x\ \varphi(x), \forall x\ \varphi(x)$.

Formulas can still be put in negation normal form.

For tableau search, add the following rule for the universal quantifier:

Some rules for working backwards:

$$\frac{\Gamma, \varphi(a)}{\Gamma, \forall x\ \varphi(x)}$$

First-order theorem proving

What about the existential quantifier?

$$\frac{\Gamma, \exists x\ \varphi(x), \varphi(?t(a, b, c, \ldots))}{\Gamma, \exists x\ \varphi(x)}$$

Notes:

- $?t$ can be instantiated to any term involving the other parameters.
- It's best to delay the choice.
- More than one term may be needed.
- All the background knowledge is lumped into Γ.

Unification

Suppose you know that for every x and y, $A(x, f(x, y)) \to B(x, y)$.

Suppose you also know that for every w and z, $A(g(w), z)$.

Then you can conclude $B(g(w), y)$ by solving $x = g(w)$ and $z = f(g(w), y)$.

Theorem (Robinson)

There is an algorithm that determines whether a set of pairs $\{(s_1, t_1), \ldots, (s_n, t_n)\}$ *of first-order terms has a unifier, and, if it does, finds a most general unifier.*

Skolemization

If φ is the formula $\forall x\, \exists y\, \forall z\, \exists w\, \theta(x, y, z, w, u)$, the *Skolem normal form* φ^S is the formula

$$\forall x, z\, \theta(x, g(x, u), z, f(x, g(x, u), z, u), u)$$

Dually, the Herbrand normal form ψ^H of ψ replaces the universal quantifiers.

- $\vdash \varphi^S \to \varphi$
- If $\varphi^S \vdash \alpha$ then $\varphi \vdash \alpha$.
- $\vdash \psi \to \psi^H$
- If $\Delta \vdash \psi^H$ then $\Delta \vdash \psi$.

Putting it all together: $T \vdash \varphi$ if and only if $T^H \vdash \varphi^S$.

Resolution

Herbrand's theorem (1930): $T^H \vdash \varphi^S$ if and only if there is a *propositional proof* of a disjunction of instances of φ^S from instances of T^H.

Resolution tries to prove $\perp$ from $T^H \cup \{\neg\varphi^S\}$:

- Leave the universal quantifiers implicit.
- Put all formulas in conjunctive normal form, and split up conjuncts.
- So, the goal is to prove $\perp$ from *clauses*, i.e. disjunctions of atomic formulas and literals.
- Use the resolution rule:
$$\frac{\Gamma \vee \varphi \qquad \Delta \vee \neg\varphi}{\Gamma \vee \Delta}$$
More generally, use unification to instantiate clauses to the form above.

Resolution

Main loop:

1. Use resolution to generate new clauses.
2. Check for redundancies (subsumption) and delete clauses.

Issues:

- How much effort to put into each phase?
- How to choose new clause (biggest, widest, heaviest, ...)?
- How to handle equality? (paramodulation, superposition)
- How to handle other equivalence relations, transitive relations?
- How to distinguish different kinds of information (like sort information)?
- How to incorporate domain specific information, like arithmetic, or AC operations?

Comparison

General approaches to theorem proving:

- global / top-down (e.g. tableaux): goal directed, works backwards to construct a proof (or countermodel)
- local / bottom-up (e.g. resolution): start with a set of facts, reason forwards to derive additional facts

Equality reasoning

It is reasonable to simplify terms:

- $x + 0 = x$
- $x > 0 \to |x| = x$
- $y \neq 0 \to (x/y) * y = x$
- $x + (z + (y + 0) + x) = x + x + y + z$

There are stand-alone equational systems, but equational reasoning is also built-in to first-order systems.

Higher-order unification

Sometimes mathematics requires higher-order unification:

$$P(0) \wedge \forall x\, (P(x) \to P(x+1)) \to \forall x\, P(x)$$

or

$$\sum_{x \in A} (f(x) + g(x)) = \sum_{x \in A} f(x) + \sum_{x \in A} g(x)$$

Notes:

- Second-order unification is undecidable (Goldfarb).
- Huet's algorithm is complete.
- Miller patterns are a decidable fragment.

Decision procedures

Full first-order theory:

- Quantifier elimination (integer / linear arithmetic, *RCF*, *ACF*)
- "Global" methods (Cooper, CAD)
- Reductions to Rabin's *S2S*
- Feferman-Vaught (product structures)

Sometimes it is enough to focus on the universal fragment:

- Some theories are only decidable at this level (e.g. uninterpreted functions)
- Can be more efficient (integer / linear arithmetic).
- Can use certificates.
- A lot of mathematical reasoning is close to quantifier-free.
- These can be *combined*.

Combining decision procedures

Theorem (Nelson-Oppen, 1979)

Suppose T_1 and T_2 are "stably infinite" and there is a decision procedure for their universal consequences. Suppose that the languages are disjoint, except for the equality symbol. Then the set of universal consequences of $T_1 \cup T_2$ is decidable.

In particular, if T_1 and T_2 have only infinite models, they are stably infinite.

This allows you to design decision procedures for individual theories and then put them together.

Combining decision procedures

First idea: one can "separate variables" in universal formulas.

That is, $\forall \vec{x}\, \varphi(\vec{x})$ is equivalent to $\forall \vec{y}\, (\varphi_1(\vec{y}) \vee \varphi_2(\vec{y}))$, where φ_1 is in the language of T_1, and φ_2 is in the language of T_2.

To do this, just introduce new variables to name subterms.

Second idea: the Craig interpolation theorem.

Theorem (Craig, 1957)

Suppose ψ_1 is a sentence in L_1 and ψ_2 is a sentence in L_2, such that $\vdash \psi_1 \to \psi_2$. Then there is a sentence θ in $L_1 \cap L_2$ such that

- $\vdash \psi_1 \to \theta$
- $\vdash \theta \to \psi_2$

Combining decision procedures

Let φ be any universal sentence, equivalent to $\forall \vec{x}\ (\varphi_1(\vec{x}) \vee \varphi_2(\vec{x}))$.

Then $T_1 \cup T_2 \vdash \varphi$ if and only if there is θ in the common language, such that

- $T_1 \cup \{\neg\varphi_1(\vec{x})\} \vdash \theta(\vec{x})$
- $T_2 \cup \{\neg\varphi_2(\vec{x})\} \vdash \neg\theta(\vec{x})$

We can assume θ is in disjunctive normal form. All that each disjunct can do is declare certain variables equal to one another, and others unequal!

Use the decision procedures for T_1 and T_2 to test each possibility.

Combining decision procedures

Nelson-Oppen methods are based on this idea.

- A fast propositional SAT solver "core" tries to build a satisfying assignment.
- Individual decision procedures examine proposals, and report conflicts.
- The SAT solver incorporates this information into the search.
- Some systems go beyond the universal fragment, for example, instantiating universal axioms in sensible ways.

For example, SMT solvers user methods for integer/linear arithmetic that support backtracking search.

Combining decision procedures

SMT solvers are both constraint solvers and theorem provers.

They incorporate domain-specific methods in a domain-general framework.

They are modular and extensible.

There are used to verify hardware and software, but also to synthesize objects.

Summary

Formal methods provide languages for

- expressing mathematical background,
- making mathematical assertions, and
- describing mathematical objects.

They provide general ways of

- searching for proofs, and
- searching for objects.

Combination methods provide ways of incorporating domain-specific methods.

Thesis: the matematical potential has not yet been realized.