ON THE CESÀRO SUMMABILITY OF INTEGRALS

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1. It is to be supposed in all that follows that $g(t)$ is integrable in every finite interval $(1, X)$†.

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† Throughout the paper, every integral over a finite range is a Lebesgue integral, and $\int^{\infty}$ denotes $\lim\limits_{x\to\infty}\int^{x}$, if this limit exists, finite or infinite.

We write, for $t \geqslant 1$,

$$I_0 g(t) = G_0(t) = g(t),$$

$$I_\alpha g(t) = G_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (t-u)^{\alpha-1} g(u)\,du \quad (\alpha > 0),$$

$$= (d/dt)^{-[\alpha]} G_{\alpha-[\alpha]}(t) \quad (\alpha < 0)^*,$$

$$m_\alpha g(t) = g_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} G_\alpha(t) \quad (\alpha \geqslant 0).$$

We shall apply the same system of notation to letters other than g, G.

It is well known that, for $\alpha > 0$, $G_\alpha(t)$ exists almost everywhere in $(1, \infty)$ (everywhere if $\alpha \geqslant 1$) and is integrable in every finite interval $(1, X)$; and that, for $\alpha > 0$, $\beta > 0$,

$$I_\beta G_\alpha(t) = G_{\alpha+\beta}(t),$$

whenever the right-hand side exists. Consequently, for $\alpha \geqslant 0$, $G_{\alpha+1}(t)$ is absolutely continuous†.

If, for $\alpha \geqslant 0$, $\Gamma(\alpha+1) t^{-\alpha} G_{\alpha+1}(t) \to l$ as $t \to \infty$, we write

$$\int_1^\infty g(t)\,dt = l(C, \alpha),$$

and say that the integral is summable (C, α) to l, and if in addition $t^{-\alpha} G_{\alpha+1}(t)$ is of bounded variation in $(1, \infty)$, we replace (C, α) by $|C, \alpha|$.

2. We shall prove the following theorems.

THEOREM 1. *For $\rho < 0$, $\lambda \geqslant \alpha \geqslant 0$, a necessary and sufficient condition that*

$$\int_1^\infty t^\rho g(t)\,dt = l\,(C, \lambda) \quad [or\ |C, \lambda|]$$

is that

$$\int_1^\infty t^{\rho-\alpha} G_\alpha(t)\,dt = \frac{\Gamma(-\rho)}{\Gamma(\alpha-\rho)}\, l(C, \lambda-\alpha) \quad [or \quad |C, \lambda-\alpha|].$$

THEOREM 2. *If $\rho < 0$, $\alpha < 0$, $\lambda \geqslant 0$, $G_{\alpha+1}(t)$ is absolutely continuous, and $\int_1^\infty t^\rho g(t)\,dt$ is summable (C, λ) [or $|C, \lambda|$], then $\int_1^\infty t^{\rho-\alpha} G_\alpha(t)\,dt$ is summable $(C, \lambda-\alpha)$ [or $|C, \lambda-\alpha|$].*

* At the point $t = 1$, d/dt denotes differentiation on the right.

† Where no interval of absolute continuity is specified, it is to be understood that the property pertains to every finite interval $(1, X)$.

THEOREM 3. *If $\rho > 0$ $(\rho \neq 1, 2, 3, \ldots)$, $\lambda \geqslant 0$, $\lambda - \alpha \geqslant 0$, $G_{\alpha+1}(t)$ is absolutely continuous, and $\int_1^\infty t^\rho g(t)\,dt$ is summable (C, λ) [or $|C, \lambda|$], then there are constants $s_1, s_2, \ldots, s_{[\rho]+1}$ such that**

$$\int_1^\infty t^{\rho-\alpha} \{G_\alpha(t) - \sum_{r=1}^{[\rho]+1} s_r t^{\alpha-r}\}\,dt \text{ is summable } (C, \lambda-\alpha) \quad [or\ |C, \lambda-\alpha|].$$

If ρ is a non-negative integer, the theorem holds only if $\rho - \alpha$ is a non-negative integer.

Analogous results for series, which include well known theorems of Hardy and Littlewood and of Andersen, have been established by Bosanquet†, who has also proved a result for Cesàro-Lebesgue integrals‡ similar to the first version of Theorem 1.

3. In this section we establish some lemmas.

LEMMA 1§. *If*

$$t^{\rho+1} f(t) = \int_1^t u^\rho g(u)\,du, \tag{3.1}$$

where ρ is a real number and $t \geqslant 1$, then, for $\alpha > 0$,

$$t^{\rho+1-\alpha} F_\alpha(t) = \int_1^t u^{\rho-\alpha} G_\alpha(u)\,du. \tag{3.2}$$

Differentiating (3.1) we get

$$g(t) \equiv (\rho+1) f(t) + t f'(t).$$

Consequently

$$G_{\alpha+1}(t) = (\rho+1) F_{\alpha+1}(t) + I_{\alpha+1}\{t f'(t)\}$$

$$= (\rho+1) F_{\alpha+1}(t) + t I_{\alpha+1} f'(t) - (\alpha+1) I_{\alpha+2} f'(t).$$

It follows, since $f(t)$ is absolutely continuous and $f(1) = 0$, that

$$G_{\alpha+1}(t) = (\rho-\alpha) F_{\alpha+1}(t) + t F_\alpha(t). \tag{3.3}$$

Now differentiating (3.3) we get

$$G_\alpha(t) \equiv (\rho+1-\alpha) F_\alpha(t) + t F_\alpha'(t),$$

* Clearly the constants are unique.

† L. S. Bosanquet [1], *Journal London Math. Soc.*, 25 (1950), 72–80.

‡ L. S. Bosanquet [2], *Proc. London Math. Soc.* (2), 49 (1945), 40–62, Theorem 22.

§ *Cf.* Bosanquet [2], Theorem 21.

and hence

$$t^{p-\alpha} G_\alpha(t) \equiv \frac{d}{dt}\{t^{p+1-\alpha} F_\alpha(t)\}. \tag{3.4}$$

On integrating (3.4) we obtain (3.2), since, by (3.3), $F_\alpha(t)$ is absolutely continuous and $F_\alpha(1)=0$.

LEMMA 2. *If $\delta>0$ and n is a positive integer, then**

(i) $$\left|\frac{G_n(t)}{t^{\delta+n-1}}\right| \leqslant \frac{1}{(\delta)_n} \max_{1\leqslant u\leqslant t} \left|\frac{g(u)}{u^{\delta-1}}\right|,$$

(ii)† $$\int_1^\infty \left|\frac{G_n(u)}{u^{\delta+n}}\right| du \leqslant \frac{1}{(\delta)_n}\int_1^\infty \left|\frac{g(u)}{u^\delta}\right| du.$$

The results are obtained by inductive arguments based respectively on the inequalities

$$\left|\frac{G_n(t)}{t^{\delta+n-1}}\right| \leqslant t^{1-n-\delta}\int_1^t \left|\frac{G_{n-1}(u)}{u^{\delta+n-2}}\right| u^{\delta+n-2}\,du \leqslant \frac{1}{\delta+n-1}\max_{1\leqslant u\leqslant t}\left|\frac{G_{n-1}(u)}{u^{\delta+n-2}}\right|$$

and

$$\left|\int_1^\infty \frac{G_n(u)}{u^{\delta+n}}\right| du \leqslant \int_1^\infty u^{-\delta-n}\,du \int_1^u |G_{n-1}(v)|\,dv = \int_1^\infty |G_{n-1}(v)|\,dv \int_v^\infty u^{-\delta-n}\,du$$

$$= \frac{1}{\delta+n-1}\int_1^\infty \left|\frac{G_{n-1}(v)}{v^{\delta+n-1}}\right| dv.$$

LEMMA 3. *If $g(t)=o(1)$ as $t\to\infty$ [or is of bounded variation in $(1,\infty)$], then, for $\delta>0$, $t^{-\delta}\int_1^t u^{\delta-1} g(u)\,du = o(1)$ as $t\to\infty$ [or is of bounded variation in $(1,\infty)$].*

The first version is easily verified.

In the second version it is enough to suppose that $g(t)$ is positive, bounded and non-decreasing in $(1,\infty)$. Then, for $t>1$,

$$\frac{d}{dt}\left\{t^{-\delta}\int_1^t u^{\delta-1}g(u)\,du\right\} \equiv t^{-1}g(t) - \delta t^{-\delta-1}\int_1^t u^{\delta-1}g(u)\,du$$

$$\geqslant \delta t^{-\delta-1}\int_1^t u^{\delta-1}\{g(t)-g(u)\}\,du \geqslant 0,$$

and

$$0 < t^{-\delta}\int_1^t u^{\delta-1}g(u)\,du \leqslant \delta^{-1}g(t).$$

* $(\delta)_n = \delta(\delta+1)\ldots(\delta+n-1)$, and max denotes the essential upper bound.

† *Cf.* L. S. Bosanquet, *Proc. Edinburgh Math. Soc.* (2), 4 (1934), 12–17, Lemma 2.

Thus $t^{-\delta}\int_1^t u^{\delta-1}g(u)\,du$, being absolutely continuous, is a bounded non-decreasing function of t in $(1,\infty)$. Hence the result.*

4. The first version of the following lemma is contained in a result due to Bosanquet†. We shall, however, prove it by a new method, similar to that used in establishing the second version.

LEMMA 4. *If, for $\lambda>0$ and $p+\lambda>-1$, $t^{-p} m_\lambda g(t) = o(1)$ as $t\to\infty$ [or is of bounded variation in $(1,\infty)$], then, for $p+q>-1$, $t^{-p-q} m_\lambda\{t^q g(t)\} = o(1)$ as $t\to\infty$ [or is of bounded variation in $(1,\infty)$].*

For t such that

$$\int_1^t (t-u)^{\lambda-1}|g(u)|\,du < \infty \quad (t\geqslant 1), \tag{4.1}$$

we have

$$\frac{m_\lambda\{t^q g(t)\}}{t^{p+q}} = \frac{\lambda}{t^{p+q+\lambda}}\int_1^t (t-u)^{\lambda-1}u^q g(u)\,du$$

$$= \frac{\lambda}{t^{p+\lambda}}\int_1^t (t-u)^{\lambda-1}\left(1-\frac{t-u}{t}\right)^q g(u)\,du$$

$$= \frac{\lambda}{t^{p+\lambda}}\int_1^t (t-u)^{\lambda-1}g(u)\,du \sum_{n=0}^\infty \frac{(-q)_n (t-u)^n}{n!\,t^n}$$

$$= \sum_{n=0}^\infty \frac{\lambda(-q)_n}{n!\,t^{p+\lambda+n}}\int_1^t (t-u)^{\lambda+n-1}g(u)\,du$$

$$= \frac{\Gamma(\lambda+1)\,G_\lambda(t)}{t^{p+\lambda}} + \Gamma(\lambda+1)\sum_{n=1}^\infty \frac{(-q)_n(\lambda)_n}{n!}\frac{G_{\lambda+n}(t)}{t^{p+\lambda+n}}; \tag{4.2}$$

the inversion being justified by (4.1), since $(-q)_n$ is of one sign for all n sufficiently large. We write

$$\beta(t) = \sum_{n=1}^\infty \frac{(-q)_n(\lambda)_n}{n!}\frac{G_{\lambda+n}(t)}{t^{p+\lambda+n}}. \tag{4.3}$$

First version. Since the validity of (4.1), for all sufficiently large t, is implicit in the hypothesis, it is sufficient, in virtue of (4.2), to prove that $\beta(t)=o(1)$ as $t\to\infty$.

* This proof was suggested to me by Dr. J. Cossar.

† L. S. Bosanquet, *Journal London Math. Soc.*, 23 (1948), 35–38, Lemma 1. The replacement of O by o presents no difficulty.

Now

$$\frac{G_{\lambda+n}(t)}{t^{p+\lambda+n}} = t^{-p-\lambda-n}\int_1^t \frac{G_{\lambda+n-1}(u)}{u^{p+\lambda+n-1}}\, u^{p+\lambda+n-1}\,du \quad (n = 1, 2, 3, \ldots) \qquad (4.4)$$

and, since $t^{-p-\lambda} G_\lambda(t) = o(1)$ as $t\to\infty$ and $p+\lambda+1>0$, it follows, by Lemma 3 and induction, that

$$\frac{G_{\lambda+n}(t)}{t^{p+\lambda+n}} = o(1) \text{ as } t\to\infty \quad (n = 1, 2, 3, \ldots). \qquad (4.5)$$

There is thus a constant M such that, for all $t \geqslant 1$,

$$\left|\frac{G_{\lambda+1}(t)}{t^{p+\lambda+1}}\right| \leqslant \frac{M}{p+\lambda+1},$$

and hence, by Lemma 2(i), with n, δ replaced by $n-1$, $p+\lambda+2$,

$$\left|\frac{G_{\lambda+n}(t)}{t^{p+\lambda+n}}\right| \leqslant \frac{M}{(p+\lambda+1)_n} \quad (t\geqslant 1,\ n = 1, 2, 3, \ldots). \qquad (4.6)$$

Also, since $p+\lambda+1 > \lambda - q$,

$$\sum_{n=1}^{\infty}\left|\frac{(-q)_n\,(\lambda)_n}{n!\,(p+\lambda+1)_n}\right| < \infty^*. \qquad (4.7)$$

It follows from (4.6) and (4.7) that the series defining $\beta(t)$ in (4.3) converges uniformly with respect to t in $(1, \infty)$, and thus, by (4.5), $\beta(t) = o(1)$ as $t\to\infty$.

Second version. Since in this case $G_\lambda(t)$ exists, and thus (4.1) holds, for all $t\geqslant 1$, it is sufficient, in view of (4.2), to prove that $\beta(t)$ is of bounded variation in $(1, \infty)$.

Since $G_{\lambda+1}(t)$ is absolutely continuous and $G_{\lambda+1}(1) = 0$, there is a function $h(t)$, integrable in every finite interval $(1, X)$, such that

$$\frac{G_{\lambda+1}(t)}{t^{p+\lambda+1}} = \int_1^t \frac{h(u)}{u^{p+\lambda+2}}\,du. \qquad (4.8)$$

Thus, by Lemma 1,

$$\frac{G_{\lambda+n}(t)}{t^{p+\lambda+n}} = \int_1^t \frac{H_{n-1}(u)}{u^{p+\lambda+n-1}}\,du \quad (n = 1, 2, 3, \ldots). \qquad (4.9)$$

Now from (4.4), with $n = 1$, and the hypothesis, it follows, by Lemma 3, that $t^{-p-\lambda-1} G_{\lambda+1}(t)$ is of bounded variation in $(1, \infty)$. Hence, by (4.8),

* K. Knopp, *Infinite series*, p. 299.

there is a finite number M for which

$$\int_1^\infty \left|\frac{h(u)}{u^{p+\lambda+2}}\right| du = \frac{M}{p+\lambda+1},$$

and thus, by Lemma 2(ii), with n, δ replaced by $n-1$, $p+\lambda+2$,

$$\int_1^\infty \left|\frac{H_{n-1}(u)}{u^{p+\lambda+n+1}}\right| du \leqslant \frac{M}{(p+\lambda+1)_n} \quad (n = 1, 2, 3, \ldots). \qquad (4.10)$$

Then, by (4.7) and (4.10),

$$\sum_{n=1}^{\infty}\left|\frac{(-q)_n\,(\lambda)_n}{n!}\right|\int_1^\infty \left|\frac{H_{n-1}(u)}{u^{p+\lambda+n+1}}\right| du < \infty. \qquad (4.11)$$

In view now of (4.3), (4.9) and (4.11), we have

$$\beta(t) = \sum_{n=1}^{\infty}\frac{(-q)_n\,(\lambda)_n}{n!}\int_1^t \frac{H_{n-1}(u)}{u^{p+\lambda+n+1}}\,du = \int_1^t du \sum_{n=1}^{\infty}\frac{(-q)_n\,(\lambda)_n}{n!}\,\frac{H_{n-1}(u)}{u^{p+\lambda+n+1}};$$

and since, by (4.11), the final integral is of bounded variation in $(1, \infty)$, this completes the proof.

5. *Proof of Theorem* 1* (*first version*). *Necessity.* We write

$$t^{p+1}f(t) = \int_1^t u^p g(u)\,du. \qquad (5.1)$$

The hypothesis is then equivalent to

$$m_\lambda\{t^{p+1}f(t) - l\} = o(1) \text{ as } t\to\infty.$$

Since $\rho < 0$, it follows by Lemma 4, that

$$t^{p+1} m_\lambda\{f(t) - lt^{-p-1}\} = o(1) \text{ as } t\to\infty.$$

Hence

$$t^{p+1-\alpha} m_{\lambda-\alpha}\{F_\alpha(t) - lI_\alpha t^{-p-1}\} = o(1) \text{ as } t\to\infty.$$

Since $\rho < \lambda$, a further application of Lemma 4 now gives

$$m_{\lambda-\alpha}\{t^{p+1-\alpha} F_\alpha(t) - lt^{p+1-\alpha} I_\alpha t^{-p-1}\} = o(1) \text{ as } t\to\infty. \qquad (5.2)$$

It is familiar that, since $\rho<0$, $t^{p+1-\alpha} I_\alpha t^{-p-1} \to \Gamma(-\rho)/\Gamma(\alpha-\rho)$ as $t\to\infty$, and thus, in view of (5.1) and Lemma 1, the result follows from (5.2).

Sufficiency. We may reverse the argument to obtain the required result.

A similar proof, in which the terms multiplied by l do not appear, can be used to establish the second version of the theorem.

* *Cf.* Bosanquet [2], Theorem 22.

6. We shall require the following lemma in the proof of Theorem 2.

LEMMA 5. *If ρ is a real number, $\lambda \geqslant 0$, $g(t)$ is absolutely continuous, and $\int_1^\infty t^\rho g(t)\,dt$ is summable (C, λ) [or $|C, \lambda|$], then $\int_1^\infty t^{\rho+1} g'(t)\,dt$ is summable $(C, \lambda+1)$ [or $|C, \lambda+1|$].*

We have

$$\int_1^t u^{\rho+1} g'(u)\,du = t^{\rho+1} g(t) - g(1) - (\rho+1) \int_1^t u^\rho g(u)\,du.$$

The result follows, since a well known and easily proved consequence of the main hypothesis is that $m_{\lambda+1}\{t^{\rho+1} g(t)\} = o(1)$ as $t \to \infty$ [and is of bounded variation in $(1, \infty)$].

*Proof of Theorem 2 (first version)**. We write $\delta = \alpha - [\alpha]$, and so $0 \leqslant \delta < 1$. Since $G_{\alpha+1}(t) = (d/dt)^{-[\alpha]-1} G_\delta(t)$ is absolutely continuous, so is $G_\delta(t)$, and thus

$$\int_1^\infty u^{\rho+1-\delta} G_{\delta-1}(u)\,du = t^{\rho+1-\delta} G_\delta(t) - G_\delta(1) - (\rho+1-\delta) \int_1^t u^{\rho-\delta} G_\delta(u)\,du. \quad (6.1)$$

It follows from our main hypothesis that $\int_1^\infty u^\rho g(u)\,du$ is summable $(C, \lambda+1)$, and thus, by Theorem 1,

$$\int_1^\infty u^{\rho-\delta} G_\delta(u)\,du \text{ is summable } (C, \lambda+1-\delta). \quad (6.2)$$

Another consequence of this hypothesis is the result stated in the proof of Lemma 5; namely†

$$m_{\lambda+1}\{t^{\rho+1} g(t)\} = o(1) \text{ as } t \to \infty.$$

Proceeding now as in the proof of Theorem 1, we obtain

$$m_{\lambda+1-\delta}\{t^{\rho+1-\delta} G_\delta(t)\} = o(1) \text{ as } t \to \infty. \quad (6.3)$$

In view of (6.1), (6.2), and (6.3),

$$\int_1^\infty u^{\rho+1-\delta} G_{\delta-1}(u)\,du \text{ is summable } (C, \lambda+1-\delta),$$

and the result is now obtained by $-[\alpha]-1$ applications of Lemma 5.

* The proof of the second version is similar.

† I am indebted to Dr. Bosanquet for pointing out that this result could be used in the proof.

7. We require another lemma.

LEMMA 6*. *If, for $\rho > 0$ and $\lambda \geqslant 1$, $\int_1^\infty t^\rho g(t)\,dt$ is summable (C, λ) [or $|C, \lambda|$], then there is a constant s such that†*

$$\int_1^\infty t^{\rho-1}\{G_1(t) - s\}\,dt \text{ is summable } (C, \lambda-1) \text{ [or } |C, \lambda-1|].$$

We suppose first that s is an arbitrary constant, and fix its value in the course of the proof.

We write

$$v(t) = \int_1^t u^\rho g(u)\,du, \quad w(t) = \int_1^t u^{\rho-1}\{G_1(u) - s\}\,du,$$

and

$$\phi(t) = v_\lambda'(t) - s\lambda(t-1)^{\lambda-1} t^{-1-\lambda}. \quad (7.1)$$

We shall first establish the following identities, for $\lambda \geqslant 1$, $t \geqslant 1$.

$$v_\lambda(t) = \lambda w_{\lambda-1}(t) - (\rho+\lambda) w_\lambda(t) + s(1-1/t)^\lambda. \quad (7.2)$$

$$v_\lambda(t) = t w_\lambda'(t) - \rho w_\lambda(t) + s(1-1/t)^\lambda. \quad (7.3)$$

$$t^{1-\rho} w_\lambda'(t) = \int_1^t u^{-\rho} \phi(u)\,du. \quad (7.4)$$

We have

$$\begin{aligned} t^\lambda v_\lambda(t) &= \int_1^t (t-u)^\lambda u^\rho g(u)\,du \\ &= \Big[u^\rho (t-u)^\lambda \{G_1(u) - s\}\Big]_1^t \\ &\qquad - \int_1^t \{G_1(u) - s\}\{\rho u^{\rho-1}(t-u)^\lambda - \lambda u^\rho (t-u)^{\lambda-1}\}\,du \\ &= s(t-1)^\lambda - \rho t^\lambda w_\lambda(t) + \lambda \int_1^t (t-u)^{\lambda-1} u^\rho \{G_1(u) - s\}\,du \\ &= s(t-1)^\lambda - \rho t^\lambda w_\lambda(t) + \lambda t^\lambda \{w_{\lambda-1}(t) - w_\lambda(t)\}; \end{aligned}$$

from which (7.2) follows.

* See G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 27 (1928), 327–348, Theorem 2, for the case λ an integer of the first version. See also A. F. Andersen, *Proc. London Math. Soc.* (2), 27 (1928), 39–71, Hardy and Littlewood, *loc. cit.*, C. E. Winn, *Journal London Math. Soc.*, 7 (1932), 227–230, and L. S. Bosanquet and H. C. Chow, *Journal London Math. Soc.*, 16 (1941), 42–48, for series analogues.

† Since $\int_1^\infty t^{\rho-1}\,dt = \infty$, s is unique.

Now

$$tw_\lambda'(t) = \Gamma(\lambda+1)\,t(d/dt)\{t^{-\lambda}\,W_\lambda(t)\} = \Gamma(\lambda+1)\{t^{1-\lambda}\,W_{\lambda-1}(t) - \lambda t^{-\lambda}\,W_\lambda(t)\}$$

$$= \lambda\{w_{\lambda-1}(t) - w_\lambda(t)\}. \qquad (7.5)$$

Substituting (7.5) in (7.2), we get (7.3).

Differentiating (7.3), we get

$$v_\lambda'(t) \equiv (1-\rho)\,w_\lambda'(t) + tw_\lambda''(t) + s\lambda(t-1)^{\lambda-1}t^{-1-\lambda},$$

and hence, in view of (7.1),

$$t^{-\rho}\phi(t) \equiv t^{-\rho}\,\{(1-\rho)\,w_\lambda'(t) + tw_\lambda''(t)\} = (d/dt)\{t^{1-\rho}\,w_\lambda'(t)\}.$$

Identity (7.4) now follows, since, by (7.3), $w_\lambda'(t)$ is absolutely continuous and $w_\lambda'(1) = 0$.

Proof of Lemma 6. Since either hypothesis ensures the convergence of $\int_1^\infty u^{-\rho}\,v_\lambda'(u)\,du$, we may now fix s so that $\int_1^\infty u^{-\rho}\,\phi(u)\,du = 0$*.

It follows then from (7.4) that, for $t \geqslant 1$,

$$t^{1-\rho}\,w_\lambda'(t) = -\int_t^\infty u^{-\rho}\,\phi(u)\,du. \qquad (7.6)$$

First version. By hypothesis $v_\lambda(t)$ tends to a finite limit as $t \to \infty$, and consequently, $\int_1^\infty \phi(u)\,du$ is convergent. It follows then from (7.6) that $tw_\lambda'(t) = o(1)$ as $t \to \infty$, and hence, by (7.3), $w_\lambda(t)$ tends to a finite limit. The result now follows from (7.2).

Second version. By hypothesis $\int_1^\infty |v_\lambda'(t)|\,dt < \infty$, and hence

$$\int_1^\infty |\phi(u)|\,du < \infty.$$

* If $\int_1^\infty u^{-\rho}\phi(u)\,du = 0$, then, by (7.1),

$$\int_1^\infty u^{-\rho}\,v_\lambda'(u)\,du = s\lambda\int_1^\infty (1-1/u)^{\lambda-1}\,u^{-\rho-2}\,du = s\lambda\int_0^1 (1-t)^{\lambda-1}\,t^\rho\,dt.$$

Thus $$s = \frac{\Gamma(\lambda+\rho+1)}{\Gamma(\lambda+1)\,\Gamma(\rho+1)}\int_1^\infty u^{-\rho}\,v_\lambda'(u)\,du = \frac{\Gamma(\lambda+\rho+1)}{\Gamma(\lambda+1)\,\Gamma(\rho)}\int_1^\infty u^{-\rho-1}\,v_\lambda(u)\,du,$$

Now, by (7.6),

$$\int_1^\infty |w_\lambda'(t)|\,dt \leqslant \int_1^\infty t^{\rho-1}\,dt\int_t^\infty u^{-\rho}\,|\phi(u)|\,du = \int_1^\infty u^{-\rho}\,|\phi(u)|\,du\int_1^u t^{\rho-1}\,dt$$

$$\leqslant \frac{1}{\rho}\int_1^\infty |\phi(u)|\,du < \infty.$$

Thus $w_\lambda(t)$ is of bounded variation in $(1, \infty)$, and the result follows from (7.2).

8. *Proof of Theorem* 3* (*first version*)†.

CASE 1. *Suppose that* $\rho > 1$ $(\rho \neq 2, 3, \ldots)$, $\alpha > -1$, *and assume the theorem with* ρ *replaced by* $\rho - 1$.

It is well known and simply proved that

$$I_{\alpha+1}\{tg(t)\} = tG_{\alpha+1}(t) - (\alpha+1)\,G_{\alpha+2}(t) \qquad (8.1)$$

whenever $G_{\alpha+1}(t)$ exists. Since $G_{\alpha+1}(t)$ is by hypothesis absolutely continuous, (8.1) holds for all $t \geqslant 1$, and

$$I_{\alpha+1}\{tg(t)\} \text{ is absolutely continuous.} \qquad (8.2)$$

Hence, differentiating (8.1), we get

$$I_\alpha\{tg(t)\} \equiv tG_\alpha(t) - \alpha G_{\alpha+1}(t). \qquad (8.3)$$

Now $\int_1^\infty t^{\rho-1}.\,tg(t)\,dt$ is summable (C, λ), and thus, in view of (8.2) and our assumption, there are constants $a_1, a_2, \ldots, a_{[\rho]}$ such that

$$\int_1^\infty t^{\rho-1-\alpha}\left(I_\alpha\{tg(t)\} - \sum_{r=1}^{[\rho]} a_r t^{\alpha-r}\right)dt \text{ is summable } (C, \lambda-\alpha). \qquad (8.4)$$

By Lemma 6, since $\int_1^\infty t^\rho g(t)\,dt$ is summable $(C, \lambda+1)$, there is a constant a such that $\int_1^\infty t^{\rho-1}\{G_1(t) - a\}\,dt$ is summable (C, λ). Since $\alpha > -1$, $I_{\alpha+1}\{G_1(t) - a\}$ is absolutely continuous. Thus, in view of our assumption, there are constants $b, b_1, b_2, \ldots, b_{[\rho]}$ such that

$$\int_1^\infty t^{\rho-1-\alpha}\,\{G_{\alpha+1}(t) - b(t-1)^\alpha - \sum_{r=1}^{[\rho]} b_r t^{\alpha-r}\}\,dt \text{ is summable } (C, \lambda-\alpha). \qquad (8.5)$$

* *Cf.* Bosanquet [1], Theorem 2.

† The proof of the second version is similar.

Also, since $\alpha > -1$,

$$\int_1^\infty t^{\rho-1-\alpha} \left| (t-1)^\alpha - \sum_{r=0}^{[\rho]} \frac{(-\alpha)_r}{r!} t^{\alpha-r} \right| dt < \infty. \tag{8.6}$$

In view now of (8.3), (8.4), (8.5) and (8.6), there are constants $s_1, s_2, \ldots, s_{[\rho]+1}$ such that

$$\int_1^\infty t^{\rho-\alpha} \{G_\alpha(t) - \sum_{r=1}^{[\rho]+1} s_r t^{\alpha-r}\} dt \text{ is summable } (C, \lambda-\alpha), \tag{8.7}$$

and thus the proof of Case 1 can be completed by induction, once the following case is established.

CASE 2. *Suppose that $0 < \rho < 1$ and $\alpha > -1$.* We argue as in Case 1, justifying (8.4) and (8.5), from which the sum terms are now omitted, by Theorem 1, when $\alpha \geqslant 0$, and by Theorem 2, when $-1 < \alpha < 0$. Then (8.7) is the required result.

CASE 3. *Suppose that $\rho > 0$ $(\rho \neq 1, 2, \ldots)$, and $\alpha \leqslant -1$.* Let m denote the positive integer for which $-1 < \alpha + m \leqslant 0$. Then, in view of the results already established, there are constants $a_1, a_2, \ldots, a_{[\rho]+1}$ such that

$$\int_1^\infty t^{\rho-\alpha-m} \{G_{\alpha+m}(t) - \sum_{r=1}^{[\rho]+1} a_r t^{\alpha+m-r}\} dt \text{ is summable } (C, \lambda-\alpha-m),$$

and the result follows by m applications of Lemma 5.

CASE 4. *Suppose that ρ and $\rho-\alpha$ are non-negative integers.* The result is obtained by $-\alpha$ applications of Lemma 5, when $\alpha \leqslant 0$, and α applications of Lemma 6, when $\alpha > 0$.

The exceptional case. Suppose that ρ is a non-negative integer and that $\rho-\alpha \neq 0, 1, 2, \ldots$. Assume that, whenever $G_{\alpha+1}(t)$ is absolutely continuous and $\int_1^\infty t^\rho g(t)\, dt$ is summable (C, λ), there are constants $s_1, s_2, \ldots, s_{\rho+1}$ such that

$$\int_1^\infty t^{\rho-\alpha} \left\{ G_\alpha(t) - \sum_{r=1}^{\rho+1} s_r t^{\alpha-r} \right\} dt \textit{ is summable } (C)^*. \tag{8.8}$$

Now let $m = \max(-[\alpha], 0)$, $\eta(t) = (d/dt)^{\rho+1}\{1/\log(t+1)\}$, and take

$$g(t) = t^{-\rho-m-1} I_m I_{-m} \{t^{\rho+m+1} \eta(t)\} \quad (t \geqslant 1). \tag{8.9}$$

It follows from (8.9) that

$$t^\rho g(t) = t^\rho \eta(t) + O(t^{-2}) = O\big(1/\{t \log^2(t+1)\}\big); \tag{8.10}$$

* (C) denotes (C, μ) for some $\mu > 0$.

and that there are constants $\mu_1, \mu_2, \ldots, \mu_{\rho+1}$ such that

$$G_{\rho+1}(t) = 1/\log(t+1) + \sum_{r=1}^{\rho+1} \mu_r (t-1)^{\rho+1-r} + O(t^{-1}). \tag{8.11}$$

It is clear from (8.9) that, when $m \geqslant 1$,

$$G_0(1) = G_{-1}(1) = \ldots = G_{1-m}(1) = 0,$$

and thus, for $m \geqslant 0$,

$$G_{\alpha+1}(t) = I_{1-m} G_{\alpha+m}(t) = I_{1-m} I_{\alpha+2m} G_{-m}(t) = I_{\alpha+m+1} G_{-m}(t).$$

Hence, since $\alpha+m+1 \geqslant 1$, $G_{\alpha+1}(t)$ is absolutely continuous. Also, by (8.10), $\int_1^\infty t^\rho g(t)\, dt$ is absolutely convergent, and so is summable (C, λ).

Suppose now that $\alpha \geqslant \rho+1$. It follows from (8.8) that there are constants $a_1, a_2, \ldots, a_{\rho+1}$ such that*

$$\int_1^\infty t^{\rho-\alpha} \left\{ G_\alpha(t) - \sum_{r=1}^{\rho+1} a_r (t-1)^{\alpha-r} \right\} dt \text{ is summable } (C).$$

Hence, by Theorem 2, with ρ, α replaced by $\rho-\alpha$, $\rho+1-\alpha$, there are constants $\alpha_1, \alpha_2, \ldots, \alpha_{\rho+1}$ such that

$$\int_1^\infty t^{-1} \left\{ G_{\rho+1}(t) - \sum_{r=1}^{\rho+1} \alpha_r (t-1)^{\rho+1-r} \right\} dt \text{ is summable } (C).$$

However, in contradiction to this, it is evident from (8.11) that the final integral is strictly divergent.

Suppose finally that $\alpha < \rho+1$, and let n be the non-negative integer for which $\rho+1 > \alpha+n > \rho$. It follows from (8.8), after n applications of Lemma 6, that there are constants $b_1, b_2, \ldots, b_{\rho+1}, c_1, c_2, \ldots, c_n$ such that

$$\int_1^\infty t^{\rho-\alpha-n} \left\{ G_{\alpha+n}(t) - \sum_{r=1}^{\rho+1} b_r (t-1)^{\alpha+n-r} - \sum_{r=1}^{n} c_r (t-1)^{n-r} \right\} dt^\dagger \text{ is summable } (C). \tag{8.12}$$

Hence, by Theorem 1, with ρ, α replaced by $\rho-\alpha-n$, $\rho+1-\alpha-n$, there are constants $\beta_1, \beta_2, \ldots, \beta_{\rho+1}, \gamma_1, \gamma_2, \ldots, \gamma_n$ such that

$$\int_1^\infty t^{-1} \left\{ G_{\rho+1}(t) - \sum_{r=1}^{\rho+1} \beta_r (t-1)^{\rho+1-r} - \sum_{r=1}^{n} \gamma_r (t-1)^{\rho+1-\alpha-r} \right\} dt \text{ is summable } (C), \tag{8.13}$$

* Here and in (8.12), we make use of the result:

$$\int_1^\infty t^{\rho-\beta} \left| t^{\beta-r} - \sum_{\nu=r}^{\rho+1} \binom{\beta-r}{\nu-r} (t-1)^{\beta-\nu} \right| dt < \infty \quad (\beta > \rho,\ r = 1, 2, \ldots, \rho+1).$$

† Here and in (8.13), the second sum disappears when $n = 0$.

and this also is incompatible with (8 . 11); and thus the assumption cannot be valid.

This completes the proof of the theorem.

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