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An O-Tauberian theorem and a high indices theorem for power series methods of summability*

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Abstract

We improve known Tauberian results concerning the power series method of summability J_p based on the sequence $\{p_n\}$ by removing the condition that p_n be asymptotically logarithmico-exponential. We also prove an entirely new Tauberian result for rapidly decreasing p_n.

1. *Introduction*

Suppose throughout that $\{p_n\}$ is a sequence of non-negative numbers, and that the power series

$$p(t) = \sum_{k=0}^{\infty} p_k t^k$$

has non-zero radius of convergence R. Let $\{a_n\}$ be a sequence of complex numbers, and let $s_n := \sum_{k=0}^{n} a_k$. The power series method J_p is defined as follows:

$$s_n \to s(J_p) \quad \text{if} \quad \sum_{k=0}^{\infty} p_k s_k t^k \text{ is convergent for } |t| < R,$$

and

$$\sigma_p(t) = \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k s_k t^k \to s$$

as $t \to R-$ through real values.

In this note we present a unified treatment of J_p-methods generated by 'smooth' sequences $\{p_n\}$. We deal with both so-called 'Abel-type' ($R < \infty$) and 'Borel-type' ($R = \infty$) methods. Although the Abel method itself ($p_n = 1$) is not included, the assumptions cover all smooth sequences $\{p_n\}$ with p_n growing faster than some positive power of n. The Abel method could be treated similarly, but methods like the logarithmic method

$$p_n = \frac{1}{n+1}$$

require another technique (see [**9**]).

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We prove two theorems (Theorems 1 and 2 below) concerning Tauberian conditions on $\{s_n\}$ under which $s_n \to s(J_p)$ implies $s_n \to s$. For the sake of clarity we describe, in this introduction, the following local version of Theorem 1:

If $p_n \sim e^{-g(n)}$ with the function g such that, for sufficiently large x, $g''(x)$ is continuous, positive and decreasing while $x^2 g''(x)$ is increasing; and if $a_n = s_n - s_{n-1} = O(\sqrt{g''(n)})$, and $s_n \to s(J_p)$, then $s_n \to s$.

This was known for logarithmico-exponential functions g (see [**2**, **4**, **7**, **8**, **10**]) with $g''(x) \to 0$ as $x \to \infty$. Theorem 2, our new 'high indices' theorem, shows that when $\liminf_{x\to\infty} g''(x) > 0$ a version of the above result holds without any monotonicity conditions on $g''(x)$. A corollary of this theorem is that if $g''(x)$ is ultimately monotonic and greater than some positive constant, then the Tauberian condition on a_n can be weakened to $a_n = O(e^{g''(n)/2})$.

2. *The O-Tauberian theorem*

In this section we prove the following Tauberian result.

THEOREM 1. *Suppose that the real function g satisfies the following condition:*

$$\text{(C)}\qquad \begin{cases} g \in C_2[x_0, \infty) \text{ for some } x_0 \in \mathbb{N}, g''(x) \text{ is positive and decreasing,} \\ \text{and } G(x) = x^2 g''(x) \text{ is increasing on } [x_0, \infty). \end{cases}$$

Let $p_n \sim e^{-g(n)}$ as $n \to \infty$, and let $l(x) = 1/\sqrt{g''(x)}$. Then

$$\lim_{\epsilon\to 0+} \omega(\epsilon) = 0 \quad \textit{with} \quad \omega(\epsilon) = \limsup_{n\to\infty} \max_{n \leqslant m \leqslant n+\epsilon l(n)} |s_{m+1} - s_n|, \tag{1}$$

and $s_n \to s(J_p)$ imply that $s_n \to s$.

Proof. If $\lim_{x\to\infty} g''(x) = \delta > 0$, then $l(n) \to \delta^{-\frac{1}{2}} < \infty$ and condition (1) reduces to $a_n = s_n - s_{n-1} = o(1)$; the required conclusion follows from Theorem 2 below (or from the o-result [**8**], corollary 1).

Suppose $\lim_{x\to\infty} G(x) = \delta < \infty$. Then, as $x \to \infty$,

$$g'(x) \to \eta < \infty, \text{ and so } x\{\eta - g'(x)\} = x \int_x^\infty \frac{G(t)}{t^2}\, dt \to \delta.$$

Hence $e^{-g(n)} = e^{-\eta n} n^{\delta} L(n)$ where L is a slowly varying function, since, for each $\lambda > 1$,

$$\frac{L(\lambda x)}{L(x)} = \exp\left(\int_x^{\lambda x} dt \int_t^\infty \frac{u^2 g''(u) - \delta}{u^2}\, du\right) \to 1 \quad \text{as} \quad x \to \infty.$$

Moreover, $l(n) \sim n\delta^{-\frac{1}{2}}$, and (1) reduces to the well-known Tauberian condition

$$\lim_{\lambda\to 1+} \limsup_{n\to\infty} \max_{n \leqslant m < \lambda n} |s_{m+1} - s_n| = 0,$$

which yields the required conclusion (see e.g. [**10**] and condition (1·5′) on p. 490 of [**7**] and the references there quoted).

Hence we may assume that

$$g''(x) \to 0 \quad \text{and} \quad G(x) = x^2 g''(x) \to \infty \quad \text{as} \quad x \to \infty. \tag{2}$$

In this case we can proceed along the same lines as in the proofs in [7], p. 500 and [**8**], p. 472, but use only condition (C) instead of the more restrictive conditions involving logarithmico-exponential functions of those proofs. The detailed asymptotic results required for the following proof are derived in Section 4 below. Let

$$q(t) = \sum_{k=x_0}^{\infty} e^{-g(k)} t^k.$$

Then the radius of convergence of this power series is $R = e^{\eta}$, where

$$-\infty < \eta = \lim_{x\to\infty} g'(x) \leqslant \infty.$$

Since $p_n \sim e^{-g(n)}$, the power series for $p(t)$ has the same radius of convergence, and (in much the same way as in [**1**]) we obtain from Lemma 2 in Section 4 below that

$$\Delta_n := \inf_{0<t<R} p(t)\, t^{-n} = p(t_n)\, t_n^{-n} \quad \text{with} \quad t_n \nearrow R$$

satisfies

$$\frac{\Delta_n}{p_n} \sim e^{g(n)} \inf_{u > x_0} \sqrt{\frac{2\pi}{g''(u)}}\, e^{-g(u) + ug'(u) - ng'(u)} \sim \sqrt{\frac{2\pi}{g''(n)}} = \sqrt{(2\pi)}\, l(n) \to \infty,$$

and hence

$$\sum_{n \leqslant k \leqslant n+\epsilon l(n)} \frac{p_k}{\Delta_k} \to \frac{\epsilon}{\sqrt{(2\pi)}} \quad \text{as} \quad n \to \infty.$$

We introduce a complex parameter α with $\beta = \operatorname{Re} \alpha > 0$, and consider the following expressions:

$$q_\alpha(t) = \sum_{k=x_0}^{\infty} e^{-\alpha g(k)} t^k = \sum_{k=x_0}^{\infty} e^{-g(k)} \mu_k(\alpha)\, t^k \quad \text{with} \quad \mu_k(\alpha) = e^{(1-\alpha) g(k)},$$

$$\tau_n(\alpha) = e^{\alpha g'(n)} \quad \text{and} \quad f_n(\alpha) = \frac{1}{q_\alpha(\tau_n(\alpha))} \sum_{k=x_0}^{\infty} s_k\, e^{-\alpha g(k)} \tau_n(\alpha)^k,$$

where $\{s_n\}$ is the given sequence of complex numbers satisfying condition (1) and $s_n \to s(J_p)$. Then, by [7], lemma 1 and [8], Lemma 4 (with $\lambda = \epsilon + 1, l(n, \lambda) = \epsilon l(n)$, $c = \sqrt{(2\pi)}$, we have the basic inequality

$$\limsup_{n\to\infty} |s_n - f_n(\alpha)| \leqslant \omega(\epsilon) + \frac{\omega(\epsilon)}{\epsilon} \sqrt{\frac{2\pi}{\alpha}} \quad \text{for real} \quad \alpha > 0, \epsilon > 0, \tag{3}$$

since

$$\phi(\alpha) = \lim_{n\to\infty} \frac{q_\alpha(\tau_n(\alpha))\, \tau_n(\alpha)^{-n}}{\Delta_n \mu_n(\alpha)} = \frac{1}{\sqrt{\alpha}}$$

by Lemma 2. Moreover, the above quoted results with $\mu_n = 1$, $\tau_n = t_n$, $\phi = 1$, and $\epsilon = 1$ reduce to

$$\limsup_{n\to\infty} |s_n - \sigma_p(t_n)| \leqslant \omega(1)(1 + \sqrt{(2\pi)}) < \infty. \tag{3'}$$

Since $\sigma_p(t_n) \to s$, the sequence $\{s_n\}$ is bounded, and then $p_n \sim e^{-g(n)}$ implies that $s_n \to s(J_q)$, i.e. $\sigma_q(t) \to s$ as $t \to R-$. By Lemma 2, we have that

$$\mu_n(\alpha) \sim \mu_n^*(\alpha) = \sqrt{\left(\frac{1-\alpha}{2\pi}\right)} \int_{x_0}^{\infty} \sqrt{g''(t)}\, e^{(1-\alpha)\{ng'(t)-tg'(t)+g(t)\}}\, dt = \int_0^{R^{1-\alpha}} u^n \, d\chi_\alpha(u)$$

for $0 < \alpha < 1$, where $\chi_\alpha(u)$ is non-decreasing (use the substitution $u = e^{(1-\alpha)g'(t)}$). Thus, for $0 < \alpha < 1$, $\{\mu_n^*(\alpha)\}$ is a 'moment sequence'.

Suppose next that, for $0 < \alpha < 1$,

$$\tilde{q}(t) = q_\alpha^*(t) = \sum_{k=x_0}^{\infty} e^{-g(k)} \mu_k^*(\alpha)\, t^k,$$

and note that the radius of convergence of the power series is R^α. Let ϵ be an arbitrary positive number. Since $s_n \to s(J_q)$, there is a positive $t_0 < R$ such that $|\sigma_q(t) - s| < \epsilon$ for $t_0 < t < R$. Hence, for some $C > 0$, and $R^{\alpha-1} t_0 < t < R^\alpha$,

$$\begin{aligned}
|\sigma_{\tilde{q}}(t) - s| &= \left| \frac{1}{\tilde{q}(t)} \sum_{k=x_0}^{\infty} e^{-g(k)} \mu_k^*(\alpha)(s_k - s)\, t^k \right| \\
&\leqslant \frac{1}{\tilde{q}(t)} \int_0^{R^{1-\alpha}} |\sigma_q(tu) - s|\, q(tu)\, d\chi_\alpha(u) \\
&\leqslant \epsilon + \frac{1}{\tilde{q}(t)} \int_0^{t_0/t} |\sigma_q(tu) - s|\, q(tu)\, d\chi_\alpha(u) \\
&\leqslant \epsilon + C \frac{q(t_0)}{\tilde{q}(t)} \to \epsilon \quad \text{as} \quad t \to R^\alpha -,
\end{aligned}$$

the change of order of summation and integration implicit in the first inequality being justified because $s_n - s = O(1)$ implies that

$$\sum_{k=x_0}^{\infty} e^{-g(k)} |s_k - s|\, t^k \int_0^{R^{1-\alpha}} u^k \, d\chi_\alpha(u) < \infty.$$

Hence $s_n \to s(J_{q_\alpha^*})$ for $0 < \alpha < 1$ (cf. [**8**], lemma 6). Since $\mu_n^*(\alpha) \sim \mu_n(\alpha)$, $s_n = O(1)$, and $\tau_n(\alpha) \to R^\alpha -$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} f_n(\alpha) = s \quad \text{for} \quad 0 < \alpha \leqslant 1. \tag{4}$$

Finally, for $C = \sup_{n \geqslant 0} |s_n| < \infty$, and α complex with $\beta := \operatorname{Re} \alpha > 0$, we have

$$|f_n(\alpha)| \leqslant C \left| \frac{q_\beta(\tau_n(\beta))}{q_\alpha(\tau_n(\alpha))} \right| \leqslant C \sqrt{\left(\frac{|\alpha|}{|\beta|}\right)} \frac{|1 + R_{\beta,2}(n)|}{|1 + R_{\alpha,2}(n)|},$$

where, by Lemma 2, $R_{\alpha,2}(n)$ and $R_{\beta,2}(n)$ tend uniformly to 0 on compact subsets of the half-plane $\operatorname{Re} \alpha > 0$ as $n \to \infty$. Hence, the functions in the sequence $\{f_n(\alpha)\}$ are holomorphic and uniformly bounded on compact subsets of some region $U \supset (0, \infty)$. Therefore, by (3), (4), and Vitali's theorem (see [**11**], theorem 5·2·1),

$$\limsup_{n\to\infty} |s_n - s| \leqslant \omega(\epsilon) + \frac{\omega(\epsilon)}{\epsilon} \sqrt{\frac{2\pi}{\alpha}} \quad \text{for all} \quad \alpha > 0, \epsilon > 0.$$

Letting $\alpha \to \infty$ and then $\epsilon \to 0+$, we deduce from this and assumption (1) that $\lim_{n\to\infty} s_n = s$. ▌

A direct consequence of this theorem and of Theorem 2 (in the case that $g''(x)$ does not tend to 0 as $x \to \infty$) is the following local result.

COROLLARY. *Suppose that the function g satisfies* (C), *and that* $p_n \sim e^{-g(n)}$. *If* $a_n = s_n - s_{n-1} = O(\sqrt{g''(n)})$ *and* $s_n \to s(J_p)$, *then* $s_n \to s$.

Remark. For real sequences $\{s_n\}$, the statement of Theorem 1 remains true when assumption (1) is replaced by the corresponding one-sided Tauberian condition. This follows essentially from [**5**], theorem 1 or [**6**], Satz 3·2 where the boundedness of $\{s_n\}$ is derived from the one-sided condition via a technique due to Vijayaraghavan (see [**3**]).

3. *The high indices theorem*

In this section we prove the following 'high indices' result to supplement Theorem 1.

THEOREM 2. *Suppose that the real function g satisfies the following condition:*

(H) $\quad g \in C_2[x_0, \infty)$ *for some* $x_0 \in \mathbb{N}$, *and* $g''(x) \geqslant \delta > 0$ *on* $[x_0, \infty)$.

Let $p_n \sim e^{-g(n)}$ *as* $n \to \infty$, *and let*

$$\left.\begin{aligned}
&A_n = \min(e^{\alpha_n}, e^{\beta_n}) \text{ where} \\
&\alpha_n = g(n) - g(n-1) - g'(n-1), \quad \beta_n = g(n-1) - g(n) + g'(n).
\end{aligned}\right\} \tag{5}$$

Then $a_n = s_n - s_{n-1} = O(A_n)$ *and* $s_n \to s(J_p)$ *imply that* $s_n \to s$.

Remarks. (i) Observe that

$$\alpha_n = \int_{n-1}^{n} (n-u)\, g''(u)\, du, \quad \beta_n = \int_{n-1}^{n} (u - n + 1)\, g''(u)\, du \quad \text{for} \quad n > n_0, \tag{6}$$

and hence that

$$\frac{\delta}{2} \leqslant \tfrac{1}{2} \min_{[n-1,n]} g''(u) \leqslant \min(\alpha_n, \beta_n) \leqslant \max(\alpha_n, \beta_n) \leqslant \tfrac{1}{2} \max_{[n-1,n]} g''(u).$$

Thus, if g'' is monotonic, then $a_n = O(A_n)$ reduces to $a_n = O(e^{g''(n)/2})$. Of course, $a_n = O(1)$ always implies $a_n = O(A_n)$ without any additional monotonicity condition.

(ii) Next, condition (H) implies that the radius of convergence for the J_p-method under consideration is ∞, and also that $1 \leqslant \Delta_n / p_n = O(1)$ where $\Delta_n = \inf_{t>0} p(t)\, t^{-n}$, by [**8**], remark 5(iii) or by estimates similar to those developed below. Moreover, the limitation result [**1**], theorem L2 yields that, subject to (H) holding, $s_n \to s$ whenever $s_n \geqslant 0$ and $s_n \to s(J_p)$. Thus, when (H) holds, the J_p-method is equivalent to convergence for real non-negative sequences, and this underlies the use of the terminology 'high indices' to describe Theorem 2. This equivalence to convergence, however, does not hold for oscillating sequences; e.g.

$$s_n = \frac{(-1)^n}{n!\, p_n}$$

is unbounded, but $$\sigma_p(t) = \frac{e^{-t}}{p(t)} \to 0 \quad \text{as} \quad t \to \infty$$

so that $s_n \nrightarrow 0(J_p)$.

We use the following lemma in the proof of Theorem 2.

LEMMA 1. *Suppose that* (H) *holds, that* $g''(x) \geqslant \eta \geqslant \delta$ *on* $[x_0, \infty)$, *and that*

$$t_n = e^{g'(n)}, \quad I_n = \sum_{k=x_0}^{\infty} e^{-g(k)} |s_k - s_n| \, t_n^{k-n} \, e^{g(n)}.$$

Then $$I_n \leqslant C \quad \textit{if} \quad a_n = O(A_n), \quad \textit{and} \quad I_n \leqslant Ce^{-\eta/2} \quad \textit{if} \quad s_n = O(1), \tag{7}$$

for all $n > x_0$ *and some positive constant* C *depending on* δ *but not on* η.

Proof. Let
$$h(t;n) = g(n) - g(t) + (t-n)\, g'(n) = -\int_n^t (t-u)\, g''(u)\, du.$$

This is denoted by $h_2(t;n)$ in Section 4 below. Then

$$h(t;n) \leqslant -\frac{\eta}{2}(t-n)^2 \leqslant -\frac{\delta}{2}(t-n)^2, \quad \text{and} \quad I_n = \sum_{k=x_0}^{\infty} |s_k - s_n| \, e^{h(k;n)} = \Sigma_1 + \Sigma_2,$$

where $$\Sigma_1 = \sum_{k=x_0}^{n-1} |s_k - s_n| \, e^{h(k;n)}, \quad \Sigma_2 = \sum_{k=n+1}^{\infty} |s_k - s_n| \, e^{h(k;n)}.$$

By C and $C(\delta)$ we denote constants which may depend on δ but not on η, and which may be different on different occasions.

Suppose first that $a_n = O(A_n)$. Then, by (6),

$$\Sigma_1 \leqslant C \sum_{k=x_0}^{n-1} (n-k) \exp\left(-\int_k^n (u-k)\, g''(u)\, du + \max_{k+1 \leqslant j \leqslant n} \int_{j-1}^{j} (u-j+1)\, g''(u)\, du\right)$$
$$\leqslant C \sum_{k=x_0}^{n-1} (n-k) \exp\left(-\int_{k+1}^n g''(u)\, du\right) \leqslant C \sum_{k=0}^{\infty} (k+1)\, e^{-\delta k} = C(\delta),$$

and

$$\Sigma_2 \leqslant C \sum_{k=n+1}^{\infty} (k-n) \exp\left(-\int_n^k (k-u)\, g''(u)\, du + \max_{n+1 \leqslant j \leqslant k} \int_{j-1}^{j} (j-u)\, g''(u)\, du\right)$$
$$\leqslant C \sum_{k=x_0}^{n-1} (k-n) \exp\left(-\int_n^{k-1} g''(u)\, du\right) \leqslant C \sum_{k=0}^{\infty} (k+1)\, e^{-\delta k} = C(\delta).$$

Hence $I_n \leqslant C$.

Suppose next that $s_n = O(1)$. Then

$$\Sigma_1 \leqslant C \sum_{k=x_0}^{n-1} \exp(-\eta(k-n)^2/2) \leqslant Ce^{-\eta/2},$$

and $$\Sigma_2 \leqslant C \sum_{k=n+1}^{\infty} \exp(-\eta(k-n)^2/2) \leqslant Ce^{-\eta/2}. \quad ∎$$

Remark. Observe that $\alpha_{n+1} = -h(n+1;n)$, $\beta_n = -h(n-1;n)$. Therefore, for any given n and any $\gamma > 0$, $I_n \geqslant \gamma$ if $|a_n| \geqslant \gamma e^{\beta_n}$ or $|a_{n+1}| \geqslant \gamma e^{\alpha_{n+1}}$. This shows that the condition $a_n = O(A_n)$ of Lemma 1 cannot be weakened, and throws light on the form of condition (5) in Theorem 2.

Proof of Theorem 2. Suppose that $s_n \to s(J_p)$ and $a_n = O(A_n)$. Let x_1 be a sufficiently large but fixed integer in $[x_0, \infty)$. Again C will denote possibly different positive constants. Since $t_n = e^{g'(n)} \to \infty$ and $p_n \sim e^{-g(n)}$, we have that

$$\limsup_{n\to\infty} |s_n - s| = \limsup_{n\to\infty} |s_n - \sigma_p(t_n)| \leqslant T_1 + T_2,$$

where $$T_1 = \limsup_{n\to\infty} \frac{1}{p(t_n)} \sum_{k=0}^{x_1} p_k |s_k - s_n| \, t_n^k,$$

and $$T_2 = \limsup_{n\to\infty} \frac{1}{p(t_n)} \sum_{k=x_1}^{\infty} p_k |s_k - s_n| \, t_n^k.$$

Then $T_1 \leqslant C \limsup_{n\to\infty} (ne^{c_n})$ with

$$c_n = g(n) + (x_1 - n)\, g'(n) - g(x_1) + \max_{x_1+1 \leqslant j \leqslant n} \int_{j-1}^{j} (u-j+1)\, g''(u)\, du$$
$$\leqslant -\int_{x_1+1}^{n} g''(u)\, du \leqslant -\delta(n - x_1 - 1),$$

so that $T_1 = 0$. Further

$$T_2 \leqslant 2 \limsup_{n\to\infty} \sum_{k=x_0}^{\infty} e^{-g(k)} |s_k - s_n| \, t_n^{k-n} \, e^{g(n)} \leqslant C,$$

by Lemma 1. Hence the sequence $\{s_n\}$ is bounded. Since $s_n \to s(J_p)$ we now have that

$$s_n = O(1); \quad \text{and} \quad s_n \to s(J_q), \quad \text{where} \quad q(t) = \sum_{k=x_0}^{\infty} e^{-g(k)} t^k. \tag{8}$$

As in the proof of Theorem 1, we introduce a complex parameter α with $\beta = \operatorname{Re} \alpha > -\frac{1}{4}\delta$, and we consider the functions

$$\mu_k(\alpha) = e^{-\alpha k^2}, \quad \tau_n(\alpha) = e^{g'(n) + 2\alpha n}, \quad q_\alpha(t) = \sum_{k=x_0}^{\infty} e^{-g(k)} \mu_k(\alpha)\, t^k,$$

and $$f_n(\alpha) = \frac{1}{q_\alpha(\tau_n(\alpha))} \sum_{k=x_0}^{\infty} s_k \, e^{-g(k) - \alpha k^2} \tau_n(\alpha)^k.$$

Then, by (8), $f_n(0) = \sigma_q(\tau_n(0)) \to s$ as $n \to \infty$, and since, for $n \geqslant 0$, $\alpha < 0$,

$$\mu_n(\alpha) = e^{-\alpha n^2} = \frac{1}{2\sqrt{(-\alpha\pi)}} \int_0^\infty t^n \exp\left(\frac{\log^2 t}{4\alpha}\right) \frac{dt}{t}$$

(so that $\{\mu_n(\alpha)\}$ is a 'moment sequence'), we can conclude as in the proof of Theorem 1 that

$$\lim_{n\to\infty} f_n(\alpha) = s \quad \text{for} \quad -\frac{\delta}{4} < \alpha < 0. \tag{9}$$

Moreover, since

$$(g(x)+\alpha x^2)'' \geqslant \eta(\alpha) := \delta + 2\alpha \geqslant \frac{\delta}{2} \quad \text{for} \quad \alpha > -\frac{\delta}{4},$$

it follows from (8) and Lemma 1 that

$$\limsup_{n\to\infty} |f_n(\alpha) - s_n| \leqslant C e^{-\alpha} \quad \text{for} \quad \alpha > -\frac{\delta}{4}, \tag{10}$$

C being a constant depending on δ but not on α. Next we shall show that Vitali's theorem can be applied to the sequence $\{f_n(\alpha)\}$. To this end let

$$h_\alpha(t;n) = h(t;n) - \alpha(t-n)^2 = -\tfrac{1}{2}(g''(\xi)+2\alpha)(t-n)^2$$

with

$$h(t;n) = g(n) - g(t) + (t-n)\, g'(n)$$

(as in the proof of Lemma 1). Then

$$f_n(\alpha) = \frac{g_n(\alpha)}{h_n(\alpha)},$$

where

$$g_n(\alpha) = \sum_{k=x_0}^{\infty} s_k\, e^{h_\alpha(k;n)}, \quad \text{and} \quad h_n(\alpha) = \sum_{k=x_0}^{\infty} e^{h_\alpha(k;n)}.$$

Then, since $s_n = O(1)$, we obtain that

$$|g_n(\alpha)| \leqslant C \text{ for all } n \geqslant x_0, \text{ and all complex } \alpha \text{ with } \operatorname{Re}\alpha > -\delta/4,$$

and

$$h_n(\alpha) \geqslant e^{h_\alpha(n;n)} = 1 \text{ for all } n \geqslant x_0, \quad \text{and all real } \alpha.$$

Moreover

$$|h_n'(\alpha)| \leqslant \sum_{k=-\infty}^{\infty} (k-n)^2\, e^{-\delta(k-n)^2/4} = \tfrac{1}{2} C_\delta$$

for all $n \geqslant x_0$, and all complex α with $\operatorname{Re}\alpha > -\delta/4$. Hence if

$$G = \{\alpha \in \mathbb{C} \mid \operatorname{Re}\alpha > -\tfrac{1}{4}\delta, |\operatorname{Im}\alpha| < C_\delta^{-1}\},$$

we get that $|h_n(\alpha)| \geqslant \frac{1}{2}$ for $\alpha \in G$, $n \geqslant x_0$. Therefore $\{f_n(\alpha)\}$ is a sequence of functions holomorphic and uniformly bounded in G. Hence, by Vitali's theorem, it follows from (9) and (10) that

$$\limsup_{n\to\infty} |s_n - s| \leqslant C e^{-\alpha} \quad \text{for all real} \quad \alpha > -\frac{\delta}{4}.$$

Letting $\alpha \to \infty$, we obtain the required result that $s_n \to s$. ∎

4. *Asymptotics*

In the following lemma we state the asymptotic results required in Section 2 in the form of strict inequalities which may be useful elsewhere.

LEMMA 2. *Suppose that the real function g satisfies the following condition:*

$$\text{(C)} \qquad \begin{cases} g \in C_2[x_0,\infty) \textit{ for some } x_0 \in \mathbb{N},\ g''(x) \textit{ is positive and} \\ \textit{decreasing, and } G(x) := x^2 g''(x) \textit{ is increasing on } [x_0,\infty). \end{cases}$$

Let α be a complex number with $\beta = \operatorname{Re}\alpha > 0$, and let

$$f_\alpha(z) = \sum_{k=x_0}^{\infty} e^{-\alpha g(k)} z^k \quad \textit{with} \quad z = z(x,\alpha) = e^{\alpha g'(x)}.$$

Then, for all $x \geqslant 2x_0$,

$$\sqrt{\frac{2\pi}{\alpha}} \int_{x_0}^{\infty} \sqrt{g''(t)}\, e^{\alpha[xg'(t) - tg'(t) + g(t)]}\, dt = e^{\alpha g(x)}(1 + R_{\alpha,1}(x))$$

and

$$f_\alpha(z) = \sqrt{\frac{2\pi}{\alpha}} \frac{1}{\sqrt{g''(x)}} e^{-\alpha[g(x) - xg'(x)]}(1 + R_{\alpha,2}(x))$$

with

$$|R_{\alpha,1}(x)| \leqslant C \frac{\sqrt{|\alpha|}}{\beta} \frac{|\alpha|}{\beta} \frac{1}{\sqrt{G(x)}}, \quad |R_{\alpha,2}(x)| \leqslant 2 \frac{\sqrt{|\alpha|}}{\beta} |\alpha| \sqrt{g''(x)} + C \frac{\sqrt{|\alpha|}}{\beta} \frac{|\alpha|}{\beta} \frac{1}{\sqrt{G(x)}},$$

where C is an absolute constant, e.g. $C = 40$; and $\sqrt{\alpha}$ denotes the principal branch of the square root.

Proof. We use the inequalities

$$|e^{\alpha t} - e^{\alpha u}| \leqslant \frac{|\alpha|}{\beta} |e^{\beta t} - e^{\beta u}|, \quad |e^t - e^u| \leqslant |t-u|\, e^{\max(t,u)} \tag{11}$$

for real t, u and $\beta = \operatorname{Re}\alpha > 0$. Next, we define, for $t \geqslant x_0$, $x \geqslant x_0$,

$$h_1(t;x) = g(t) - g(x) + (x-t)\, g'(t), \quad h_2(t;x) = g(x) - g(t) - (x-t)\, g'(x),$$

and we write $h(t) = h(t;x)$ for either function whenever the distinction between the use of h_1 or h_2 is immaterial. We have

$$R_{\alpha,1}(x) = \sqrt{\frac{\alpha}{2\pi}} \int_{x_0}^{\infty} \sqrt{g''(t)}\, e^{\alpha h_1(t;x)}\, dt - 1,$$

$$R_{\alpha,2}(x) = \sqrt{\frac{\alpha}{2\pi}} \sqrt{g''(x)} \sum_{k=x_0}^{\infty} e^{\alpha h_2(k;x)} - 1,$$

and

$$\left.\begin{aligned} h(x) = h'(x) = 0, \quad h_1'(t) = (x-t)\, g''(t), \quad h_2'(t) = g'(x) - g'(t), \\ h''(t) = -g''(t), \quad h(t) = -\tfrac{1}{2} g''(\xi)(t-x)^2 \end{aligned}\right\} \tag{12}$$

for some ξ between t and x. Observe that

$$h_1(t) = \int_x^t (x-u)\, g''(u)\, du = -\tfrac{1}{2} g''(\xi)(x-t)^2.$$

Also, it follows from the monotonicity of $g''(x)$ and $G(x) = x^2 g''(x)$ that

$$\left| \frac{g''(t)}{g''(x)} - 1 \right| \leqslant 4 \frac{|t-x|}{x} \quad \text{for all} \quad t \geqslant x_0, \quad x \geqslant x_0 \quad \text{with} \quad |t-x| \leqslant \frac{x}{4}. \tag{13}$$

Since the asymptotic form of the inequalities asserted in Lemma 2 can be established by Laplace's well-known method, we only briefly summarize the steps required to obtain the strict form.

Step 1. Using the fact that $h_2(t)$ is increasing on $[x_0, x]$, decreasing on $[x, \infty)$, and always $\leqslant 0$, we obtain, in view also of (11), that for all $x \geqslant x_0$,

$$\left|\sqrt{\frac{\alpha}{2\pi}}\sqrt{g''(x)}\left(\sum_{k=x_0}^{\infty} e^{\alpha h_2(k;x)} - \int_{x_0}^{\infty} e^{\alpha h_2(t;x)}\,dt\right)\right| \leqslant 2\frac{\sqrt{|\alpha|}}{\beta}|\alpha|\sqrt{g''(x)}.$$

In the following put $\delta = \delta(x) = x/8$, and suppose $x \geqslant 2x_0$.

Step 2. Using (11), (12), and (13), and substituting $u = \sqrt{(\beta g''(x))}\,(t-x)$, we get (for both h_1 and h_2) that

$$\left|\sqrt{\frac{\alpha}{2\pi}}\sqrt{g''(x)}\int_{x-\delta}^{x+\delta} (e^{\alpha h(t;x)} - e^{-\alpha g''(x)(t-x)^2/2})\,dt\right|$$
$$\leqslant \frac{\sqrt{|\alpha|}}{\beta}\frac{|\alpha|}{\beta}\frac{2}{\sqrt{(2\pi)}}\frac{1}{\sqrt{G(x)}}\int_{-\infty}^{\infty}|u|^3 e^{-u^2/4}\,du \leqslant 16\frac{\sqrt{|\alpha|}}{\beta}\frac{|\alpha|}{\beta}\frac{1}{\sqrt{G(x)}}.$$

Step 3. Using mostly (13), and substituting $u = \sqrt{(\beta g''(x))}\,(t-x)$ as above, we obtain

$$\left|\sqrt{\frac{\alpha}{2\pi}}\int_{x-\delta}^{x+\delta} (\sqrt{g''(x)} - \sqrt{g''(t)})\,e^{\alpha h_1(t;x)}\,dt\right| \leqslant 8\frac{\sqrt{|\alpha|}}{\beta}\frac{|\alpha|}{\beta}\frac{1}{\sqrt{G(x)}}.$$

Step 4. Since $\sqrt{\dfrac{2\pi}{\alpha}} = \displaystyle\int_{-\infty}^{\infty} e^{-\alpha u^2/2}\,du$, the above substitution leads to

$$\left|\sqrt{\frac{\alpha}{2\pi}}\sqrt{g''(x)}\int_{x-\delta}^{x+\delta} e^{-\alpha g''(x)(t-x)^2/2}\,dt - 1\right| \leqslant 8\frac{\sqrt{|\alpha|}}{\beta}\frac{|\alpha|}{\beta}\frac{1}{\sqrt{G(x)}}.$$

Step 5. Since

$$0 \leqslant \frac{\sqrt{g''(t)}}{h_1'(t;x)} \leqslant \frac{8}{\sqrt{G(x)}} \quad\text{and}\quad 0 \leqslant \frac{\sqrt{g''(x)}}{h_2'(t;x)} \leqslant \frac{8}{\sqrt{G(x)}} \quad \text{for} \quad x_0 \leqslant t \leqslant x-\delta,$$

it follows that both

$$\left|\sqrt{\frac{\alpha}{2\pi}}\int_{x_0}^{x-\delta}\sqrt{g''(t)}\,e^{\alpha h_1(t;x)}\,dt\right| \quad\text{and}\quad \left|\sqrt{\frac{\alpha}{2\pi}}\sqrt{g''(x)}\int_{x_0}^{x-\delta} e^{\alpha h_2(t;x)}\,dt\right|$$

are less than or equal to

$$\sqrt{\frac{|\alpha|}{2\pi}}\frac{8}{\sqrt{G(x)}}\int_{x_0}^{x-\delta} h'(t)\,e^{\beta h(t)}\,dt \leqslant 4\frac{\sqrt{|\alpha|}}{\beta}\frac{|\alpha|}{\beta}\frac{1}{\sqrt{G(x)}}.$$

Step 6. Since

$$0 \leqslant \frac{\sqrt{g''(t)}}{-h_1'(t;x)} \leqslant \frac{9}{\sqrt{G(x)}} \quad \text{for} \quad t \geqslant x+\delta,$$

we get (as in Step 5) that

$$\left|\sqrt{\frac{\alpha}{2\pi}}\int_{x+\delta}^{\infty}\sqrt{g''(t)}\,e^{\alpha h_1(t;x)}\,dt\right| \leqslant 4\frac{\sqrt{|\alpha|}}{\beta}\frac{|\alpha|}{\beta}\frac{1}{\sqrt{G(x)}}.$$

Step 7. Since

$$h_2(t;x) \leqslant -\int_{x+\delta}^{t} (g'(u) - g'(x))\,du \leqslant -\tfrac{1}{12}(t-x-\delta)\,xg''(x) \quad \text{for} \quad t \geqslant x+\delta$$

by (12) and (13), it follows that

$$\left|\sqrt{\frac{\alpha}{2\pi}}\sqrt{g''(x)}\int_{x+\delta}^{\infty} e^{\alpha h_2(t;x)}\,dt\right| \leqslant \sqrt{\frac{|\alpha|}{2\pi}}\sqrt{g''(x)}\int_{x+\delta}^{\infty} e^{-xg''(x)(t-x-\delta)\beta/12}\,dt$$
$$\leqslant 6\frac{\sqrt{|\alpha|}}{\beta}\frac{|\alpha|}{\beta}\frac{1}{\sqrt{G(x)}}.$$

Combining the inequalities of Steps 1 to 7 leads directly to the asserted inequalities (with constant $C = 40$). No effort was made to improve on the value of the constant, but the dependency on α and $\beta = \operatorname{Re}\alpha$ was carefully considered, because this played an essential part in the application in Section 2. ∎

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