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MATRIX TRANSFORMATIONS OF POWER SERIES

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ABSTRACT. We consider the sequence of transforms (g_n) of a power series $\sum_{n=0}^{\infty} a_n z^n$ given by $g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k$. We establish necessary and sufficient conditions on the matrix (b_{nk}) for the sequence (g_n) to converge uniformly on compact subsets of the disk $D_P := \{z : |z| < P\}$ to a function holomorphic on D_P.

1. INTRODUCTION

Suppose throughout that $0 < P \le \infty$, $0 < R < \infty$, and that all sequences and matrices are complex with indices running through $0, 1, 2, \ldots$. We make the following definitions:

D_P is the disk $\{z : |z| < P\}$;
$\mathscr{E}$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\lim |a_n|^{\frac{1}{n+1}} = 0$;
$\mathscr{E}^{\beta}$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\limsup |a_n|^{\frac{1}{n+1}} < \infty$;
$\mathscr{E}_R$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\sum_{n=0}^{\infty} |a_n| R^n < \infty$;
$\mathbf{A}_R$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\limsup |a_n|^{\frac{1}{n+1}} = \frac{1}{R}$;
It will follow from the lemma (below) that $\mathscr{E}^{\beta}$ is the β-dual of $\mathscr{E}$.

The following are the first three of eight theorems we shall prove concerning matrix transformations of power series.

Theorem 1. *A matrix* $\mathbf{B} \equiv (b_{nk})$ *has the property that whenever the sequence* $\mathbf{a} \equiv (a_n) \in \mathscr{E}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k, \qquad n = 0, 1, \ldots,$$

converges uniformly on every compact subset of D_P, *each power series* $\sum_{k=0}^{\infty} b_{nk} a_k z^k$ *being convergent on* D_P, *if and only if*
(i) $\lim_{n\to\infty} b_{nk} =: b_k$ *for* $k = 0, 1, \ldots$;
(ii) $\sup_{n\ge 0, k\ge 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty$ *for each positive* $p < P$.
And then $\lim_{n\to\infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ *on* D_P.

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Theorem 2. *A matrix* $\mathbf{B} \equiv (b_{nk})$ *has the property that whenever the sequence* $\mathbf{a} \equiv (a_n) \in \mathbf{A}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k, \qquad n = 0, 1, \dots,$$

converges uniformly on every compact subset of D_P, *each power series* $\sum_{k=0}^{\infty} b_{nk} a_k z^k$ *being convergent on* D_P, *if and only if*

(i) $\lim_{n\to\infty} b_{nk} =: b_k$ *for* $k = 0, 1, \dots$;

(ii) $\sup_{n\ge 0, k\ge 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty$ *for each positive* $p < P$.

And then $\lim_{n\to\infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ *on* D_P.

Theorem 3. *A matrix* $\mathbf{B} \equiv (b_{nk})$ *has the property that whenever the sequence* $\mathbf{a} \equiv (a_n) \in \mathscr{E}$ *the sequence of functions* (g_n) *given by*

$$g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k, \qquad n = 0, 1, \dots,$$

converges uniformly on every compact subset of D_∞, *each power series* $\sum_{k=0}^{\infty} b_{nk} a_k z^k$ *being convergent on* D_∞, *if and only if*

(i) $\lim_{n\to\infty} b_{nk} =: b_k$ *for* $k = 0, 1, \dots$;

(ii) $\sup_{n\ge 0, k\ge 0} |b_{nk}|^{\frac{1}{k+1}} < \infty$.

And then $\lim_{n\to\infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ *on* D_∞.

These theorems show that if the series-to-sequence transform given by $\mathbf{B}$ is regular, then it is necessary in each case that $\lim_{n\to\infty} b_{nk} = b_k = 1$ for $k = 0, 1, \dots$, and this in turn implies that $P \le R$ in Theorems 1 and 2 (i.e., the sequence (g_n) cannot converge uniformly in any disk D_P with $P > R$). Regular sequence-to-sequence transforms of power series have been considered by Peyerimhoff [5] and Luh [4] among others. One of the novel features of our approach is that we deal with series-to-sequence transforms rather than sequence-to-sequence transforms.

Let (B_n) be a sequence of nonzero complex numbers. The associated Nörlund series-to-sequence matrix $\mathbf{N}_B$ is the triangular matrix (b_{nk}) with

$$b_{nk} := \begin{cases} \dfrac{B_{n-k}}{B_n} & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem is an immediate consequence of Theorem 2. The case $R = 1$ of Theorem KS is due to Karin Stadtmüller [6, Theorem 5]. Her method of proof is different from and more complicated than the one developed below.

Theorem KS. *The Nörlund matrix* $\mathbf{N}_B$ *has the property that whenever the sequence* $\mathbf{a} \equiv (a_n) \in \mathbf{A}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(z) := \frac{1}{B_n} \sum_{k=0}^{n} B_{n-k} a_k z^k, \qquad n = 0, 1, \dots,$$

converges uniformly on every compact subset of D_P, *if and only if*

$$\lim_{n\to\infty} \frac{B_{n-1}}{B_n} = b \quad \text{with} \quad |b| = \frac{R}{P}.$$

And then $\lim_{n\to\infty} g_n(z) = \sum_{k=0}^{\infty} a_k (bz)^k$ *on* D_P.

Note. In view of Theorem 1, Theorem KS remains true if $\mathbf{A}_R$ is replaced by $\mathscr{E}_R$.

2. A preliminary result

Lemma. *A sequence* $\mathbf{b}$ *has the property that* $\sum_{n=0}^{\infty} b_n a_n$ *is convergent for each* $\mathbf{a} \in \mathscr{E}$ *if and only if* $\mathbf{b} \in \mathscr{E}^{\beta}$.

Proof. Sufficiency. If $\mathbf{b} \in \mathscr{E}^{\beta}$, then there exists a positive number M such that $|b_n| \le M^{n+1}$ for $n = 0, 1, \ldots$. Hence, if $\mathbf{a} \in \mathscr{E}$, then $\sum_{k=0}^{\infty} |b_k a_k| \le M \sum_{k=0}^{\infty} |a_k| M^k < \infty$.

Necessity. Assume $\mathbf{b} \notin \mathscr{E}^{\beta}$, i.e., $\limsup |b_n|^{\frac{1}{n+1}} = \infty$. Then there exists a strictly increasing sequence of positive integers (n_j) such that $0 < |b_{n_j}|^{\frac{1}{n_j+1}} \to \infty$. Choose

$$a_n := \begin{cases} \dfrac{1}{\sqrt{|b_n|}} & \text{if } n = n_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$|a_n|^{\frac{1}{n+1}} = \begin{cases} \left(\dfrac{1}{|b_n|^{\frac{1}{n+1}}} \right)^{1/2} & \text{if } n = n_j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\lim |a_n|^{\frac{1}{n+1}} = 0$, so $\mathbf{a} \in \mathscr{E}$. But

$$|b_{n_j} a_{n_j}| = \sqrt{|b_{n_j}|} = \left(|b_{n_j}|^{\frac{1}{n_j+1}} \right)^{\frac{n_j+1}{2}} \to \infty \quad \text{as } j \to \infty,$$

and therefore $\sum_{n=0}^{\infty} b_n a_n$ is not convergent. □

3. Proofs of Theorems 1, 2, and 3

Proof of Theorem 1. *Sufficiency.* We assume that

$$\begin{cases} \lim_{n\to\infty} b_{nk} =: b_k & \text{for } k = 0, 1, \ldots; \\ M(p) := \sup_{n \ge 0, k \ge 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty & \text{for } 0 < p < P. \end{cases}$$

Let $\mathbf{a} \in \mathscr{E}_R$. We have, for $n = 0, 1, \ldots$ and $|z| \le p < P$,

$$\left| \sum_{k=0}^{\infty} b_{nk} a_k z^k \right| \le \sum_{k=0}^{\infty} |b_{nk}||a_k| p^k \le M(p) \sum_{k=0}^{\infty} |a_k| R^k < \infty.$$

Hence the functions $g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k$ are holomorphic and uniformly bounded on D_p. Also $g_n^{(k)}(0) = k! b_{nk} a_k \to k! b_k a_k$ as $n \to \infty$ for $k = 0, 1, \ldots$. Further, from Cauchy's inequalities for the coefficients of power series we get that, for $|z| \le p_1 < p < P$, $n = 0, 1, \ldots$, and $k = 0, 1, \ldots$,

$$|b_{nk} a_k z^k| \le M(p, \mathbf{a})(p_1/p)^k, \quad \text{where } M(p, \mathbf{a}) := \sup_{n \ge 0} \max_{|z|=p} |g_n(z)| < \infty.$$

Therefore, by the Weierstrass M-test, $\lim_{n\to\infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ on D_P, and the sequence (g_n) is uniformly convergent on compact subsets of D_P.

Necessity. Let $a_k := 1/((k+1)^2 R^k)$ for $k = 0, 1, 2, \ldots$. Since $\mathbf{a} \in \mathscr{E}_R$, our assumption is that the series $g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k$ converges on D_P and that the sequence (g_n) is uniformly convergent on D_p for $0 < p < P$. By the Weierstrass double-series theorem, $\lim_{n\to\infty} b_{nk} a_k$ exists for $k = 0, 1, \ldots$. Since $a_k \neq 0$ for $k = 0, 1, \ldots$, it follows that the condition

$$\lim_{n\to\infty} b_{nk} =: b_k \quad \text{for } k = 0, 1, \ldots$$

must necessarily hold. Suppose now that p and $\tilde{p}$ are fixed and $0 < p < \tilde{p} < P$. Since the sequence (g_n) is uniformly convergent on $\bar{D}_{\tilde{p}}$, the closure of $D_{\tilde{p}}$, we have, for $|z| \leq \tilde{p}$ and $n = 0, 1, \ldots$, that $|g_n(z)| \leq M(\tilde{p}, \mathbf{a}) < \infty$. From Cauchy's inequalities for the coefficients of power series we get that

$$|b_{nk} a_k \tilde{p}^k| \leq M(\tilde{p}, \mathbf{a}) \quad \text{for } n = 0, 1, \ldots \text{ and } k = 0, 1, \ldots,$$

and hence that

$$\sup_{n\geq 0, k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k \leq M(\tilde{p}, \mathbf{a}) \sup_{k\geq 0} \left(\frac{p}{\tilde{p}}\right)^k (k+1)^2 < \infty.$$

Therefore the condition

$$\sup_{n\geq 0, k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \quad \text{for all positive } p < P$$

is also necessary. □

Proof of Theorem 2. *Sufficiency.* We assume that

$$\begin{cases} \lim_{n\to\infty} b_{nk} =: b_k & \text{for } k = 0, 1, \ldots; \\ M(p) := \sup_{n\geq 0, k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty & \text{for } 0 < p < P. \end{cases}$$

Let $\mathbf{a} \in \mathbf{A}_R$. For $0 < p < P$ choose r so that $0 < r < R$ and $\frac{p}{r} < \frac{P}{R}$. Now choose p_1 such that $0 < p_1 < P$ and $\frac{p}{r} = \frac{p_1}{R}$. We have, for $|z| \leq p$, that

$$\begin{aligned} \left|\sum_{k=0}^{\infty} b_{nk} a_k z^k\right| &\leq \sum_{k=0}^{\infty} |b_{nk}||a_k| p^k = \sum_{k=0}^{\infty} |b_{nk}| \left(\frac{p}{r}\right)^k |a_k| r^k \\ &= \sum_{k=0}^{\infty} |b_{nk}| \left(\frac{p_1}{R}\right)^k |a_k| r^k \leq M(p_1) \sum_{k=0}^{\infty} |a_k| r^k < \infty. \end{aligned}$$

Hence the functions $g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k$ are uniformly bounded on D_p for $0 < p < P$. Also $g_n^{(k)}(0) = k! b_{nk} a_k \to k! b_k a_k$ as $n \to \infty$ for $k = 0, 1, \ldots$. Further, from Cauchy's inequalities for the coefficients of power series we get that, for $|z| \leq p_1 < p < P$, $n = 0, 1, \ldots$ and $k = 0, 1, \ldots$,

$$|b_{nk} a_k z^k| \leq M(p, \mathbf{a})(p_1/p)^k, \quad \text{where } M(p, \mathbf{a}) := \sup_{n\geq 0} \max_{|z|=p} |g_n(z)| < \infty.$$

Therefore, by the Weierstrass M-test, $\lim_{n\to\infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ on D_P, and the sequence (g_n) is uniformly convergent on compact subsets of D_P.

Necessity. Let $a_k := 1/R^k$ for $k = 0, 1, 2, \ldots$. Since $\mathbf{a} \in \mathbf{A}_R$, our assumption is that the series $g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k$ converges on D_P and that the sequence (g_n) is uniformly convergent on D_p for $0 < p < P$. By the Weierstrass double-series theorem, $\lim_{n\to\infty} b_{nk}a_k$ exists for $k = 0, 1, \ldots$. Since $a_k \neq 0$ for $k = 0, 1, \ldots$, it follows that the condition

$$\lim_{n\to\infty} b_{nk} =: b_k \quad \text{for } k = 0, 1, \ldots$$

must necessarily hold. Suppose now that p is fixed and $0 < p < P$. Since the sequence (g_n) is uniformly convergent on $\bar{D}_p$, we have, for $|z| \leq p$ and $n = 0, 1, \ldots$, that $|g_n(z)| \leq M(p, \mathbf{a}) < \infty$. From Cauchy's inequalities for the coefficients of power series we get that

$$|b_{nk}|\left(\frac{p}{R}\right)^k = |b_{nk}a_k p^k| \leq M(p, \mathbf{a}) \quad \text{for } n = 0, 1, \ldots \text{ and } k = 0, 1, \ldots.$$

Therefore, the condition

$$\sup_{n\geq 0,\, k\geq 0} |b_{nk}|\left(\frac{p}{R}\right)^k < \infty \quad \text{for all positive } p < P$$

is also necessary. □

Proof of Theorem 3. *Sufficiency.* We assume that

$$\begin{cases} \lim_{n\to\infty} b_{nk} = b_k & \text{for } k = 0, 1, \ldots, \\ M := \sup_{n\geq 0,\, k\geq 0} |b_{nk}|^{\frac{1}{k+1}} < \infty. \end{cases}$$

Let $\mathbf{a} \in \mathscr{E}$. We have, for $|z| \leq R < \infty$, that

$$\left|\sum_{k=0}^{\infty} b_{nk}a_k z^k\right| \leq \sum_{k=0}^{\infty} |b_{nk}||a_k|M^k \leq M\sum_{k=0}^{\infty} |a_k|(MR)^k < \infty.$$

Hence the functions $g_n(z) := \sum_{k=0}^{\infty} b_{nk}a_k z^k$ are entire and are uniformly bounded on each closed disk $\bar{D}_R$. Also $g_n^{(k)}(0) = k!b_{nk}a_k \to k!b_k a_k$ as $n \to \infty$ for $k = 0, 1, \ldots$. Further, from Cauchy's inequalities for the coefficients of power series we get that, for $|z| \leq p < R$, $n = 0, 1, \ldots$ and $k = 0, 1, \ldots$,

$$|b_{nk}a_k z^k| \leq M(R, \mathbf{a})\left(\frac{p}{R}\right)^k, \quad \text{where } M(R, \mathbf{a}) := \sup_{n\geq 0} \max_{|z|=R} |g_n(z)| < \infty.$$

Therefore, by the Weierstrass M-test, $\lim_{n\to\infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ on D_∞, and the sequence (g_n) is uniformly convergent on compact subsets of D_∞.

Necessity. We assume that for each $\mathbf{a} \in \mathscr{E}$ the series $g_n(z) := \sum_{k=0}^{\infty} b_{nk}a_k z^k$ is convergent on D_∞ and that the sequence (g_n) is uniformly convergent on compact subsets of D_∞. By the Weierstrass double-series theorem, $\lim_{n\to\infty} b_{nk}a_k$ exists for $k = 0, 1, \ldots$. Since there is an $\mathbf{a} \in \mathscr{E}$ such that $a_k \neq 0$ for $k = 0, 1, \ldots$, it follows that the condition

$$\lim_{n\to\infty} b_{nk} =: b_k \quad \text{for } k = 0, 1, \ldots$$

must necessarily hold.

Suppose that $\mathbf{a} \in \mathscr{E}$. Since the sequence (g_n) is uniformly convergent on $\bar{D}_R$, we have, for $|z| \leq R$ and $n = 0, 1, \ldots$, that $|g_n(z)| \leq M(R, \mathbf{a}) < \infty$. From Cauchy's inequalities for the coefficients of power series we get that

$$(1) \qquad |b_{nk}a_k R^k| \leq M(R, \mathbf{a}) \quad \text{for } n = 0, 1, \ldots \text{ and } k = 0, 1, \ldots.$$

Also, since $\sum_{k=0}^{\infty} b_{nk} a_k$ is convergent whenever $\mathbf{a} \in \mathscr{E}$, we have, by the lemma, that

$$M_n := \sup_{k \geq 0} |b_{nk}|^{\frac{1}{k+1}} < \infty \quad \text{for } n = 0, 1, \ldots.$$

Assume now that

$$\sup_{n \geq 0} \sup_{k \geq 0} |b_{nk}|^{\frac{1}{k+1}} = \sup_{n \geq 0} M_n = \infty.$$

This implies that there exists a strictly increasing sequence of positive integers (n_j) such that $M_{n_j} \to \infty$. This in turn implies that there exists a sequence of nonnegative integers (k_j) such that

$$(*) \qquad |b_{n_j, k_j}|^{\frac{1}{k_j+1}} > \tfrac{1}{2} M_{n_j} \to \infty \quad \text{as } j \to \infty.$$

We show now that the sequence (k_j) is not bounded. Assume that it is bounded. Then there is a positive integer k^* such that $0 \leq k_j \leq k^*$. Since $\lim_{n \to \infty} b_{nk} = b_k$ for $k = 0, 1, \ldots, k^*$, it follows that the set of numbers $(b_{nk})_{n \geq 0, 0 \leq k \leq k^*}$ is bounded and hence that the set of numbers $(|b_{nk}|^{\frac{1}{k+1}})_{n \geq 0, 0 \leq k \leq k^*}$ is bounded. But this contradicts $(*)$. Therefore, the sequence (k_j) is not bounded. We can suppose (by considering a subsequence if necessary) that the sequence is strictly increasing. Choose

$$a_k := \begin{cases} 1/(|b_{n_j, k}|)^{\frac{k+1}{2}} & \text{if } k = k_j, \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$|a_{k_j}|^{\frac{1}{k_j+1}} = \frac{1}{\sqrt{|b_{n_j, k_j}|}} < \left(\frac{1}{\frac{1}{2} M_{n_j}} \right)^{\frac{k_j+1}{2}} \to 0 \quad \text{as } j \to \infty.$$

Therefore $\mathbf{a} \in \mathscr{E}$, but

$$|b_{n_j, k_j}| a_{k_j} = \sqrt{|b_{n_j, k_j}} \to \infty \quad \text{as } j \to \infty,$$

which contradicts (1). Thus the condition

$$\sup_{n \geq 0, k \geq 0} |b_{nk}|^{\frac{1}{k+1}} < \infty$$

is also necessary. □

4. Additional theorems

In this section we prove some theorems showing that the disk of convergence D_P specified in Theorem 2 cannot be enlarged when the matrix $\mathbf{B}$ satisfies conditions (i) and (ii) of that theorem together with certain other conditions.

Theorem 4. *Suppose that P and R are positive numbers, and that $\mathbf{B} \equiv (b_{nk})$ is a normal infinite matrix (i.e., $b_{nk} = 0$ for $k > n$ and $b_{nn} \neq 0$) satisfying*

$$M(p) := \sup_{n \geq 0,\ k \geq 0} |b_{nk}| \left(\frac{p}{R} \right)^k < \infty \quad \textit{for } 0 < p < P.$$

Then, for each $\mathbf{a} \in \mathbf{A}_R$ *and each* $R_1 \geq P$,

$$\limsup_{n\to\infty} \max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^{\frac{1}{n}} \leq \frac{R_1}{P}.$$

Proof. Choose $R_1 \geq P$, and suppose $\mathbf{a} \in \mathbf{A}_R$. Let $0 < \lambda < 1$, and take $p := \lambda P$. Then $0 < p < P$. Since $\limsup |a_k|^{\frac{1}{k+1}} = \frac{1}{R}$, there is a positive constant $c(\lambda)$ such that

$$|a_k| \leq \frac{c(\lambda)}{(\lambda R)^k} \quad \text{for } k \geq 0.$$

Now for $|z| = R_1$ we have

$$\left| \sum_{k=0}^{n} b_{nk} a_k z^k \right| \leq \sum_{k=0}^{n} |b_{nk}| \left(\frac{p}{R}\right)^k |a_k| R^k \left(\frac{R_1}{p}\right)^k$$
$$\leq M(p) c(\lambda) \sum_{k=0}^{n} \left(\frac{R}{\lambda R}\right)^k \left(\frac{R_1}{\lambda P}\right)^k = M(p) c(\lambda) \sum_{k=0}^{n} \left(\frac{R_1}{\lambda^2 P}\right)^k.$$

Since $R_1/(\lambda^2 P) > R_1/P \geq 1$, it follows that

$$\limsup_{n\to\infty} \max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^{\frac{1}{n}} \leq \lim_{n\to\infty} \left(\sum_{k=0}^{n} \left(\frac{R_1}{\lambda^2 P}\right)^k \right)^{\frac{1}{n}} = \frac{R_1}{\lambda^2 P}.$$

Letting $\lambda \nearrow 1$ we get

$$\limsup_{n\to\infty} \max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^n \right|^{\frac{1}{n}} \leq \frac{R_1}{P}. \quad \square$$

Remark. Assume that a normal matrix $\mathbf{B}$ satisfies

$$M(p) := \sup_{n\geq 0,\ k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \quad \text{for } 0 < p < P.$$

Then

$$|b_{nn}|^{\frac{1}{n}} \frac{p}{R} \leq M(p)^{\frac{1}{n}} \to 1 \quad \text{as } n \to \infty,$$

and hence

$$\limsup_{n\to\infty} |b_{nn}|^{\frac{1}{n}} \leq \frac{R}{p} \quad \text{for each positive } p < P.$$

Letting $p \nearrow P$ we get

$$\limsup_{n\to\infty} |b_{nn}|^{\frac{1}{n}} \leq \frac{R}{P}.$$

This suggests that it is not inappropriate to impose the condition

$$\lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P},$$

as we do in the following theorem.

Theorem 5. *Let* $\mathbf{B}$ *be a normal matrix. Suppose that*

$$\lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P},$$

where P *and* R *are positive numbers. Then for each* $\mathbf{a} \in \mathbf{A}_R$ *and each* $R_1 \geq P$ *we have*

$$\limsup_{n\to\infty} \max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^{\frac{1}{n}} \geq \frac{R_1}{P}.$$

Proof. Assume that the conclusion of the theorem is not true. Then there is an $\mathbf{a}^* \in \mathbf{A}_R$ and an $R_1 \geq P$ such that

$$\limsup_{n\to\infty} \max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{nk} a_k^* z^k \right|^{\frac{1}{n}} < \frac{R_1}{P}.$$

Therefore, there exists a positive $\tilde{R} < R_1$ such that, for all n sufficiently large,

$$\max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{nk} a_k^* z^k \right|^{\frac{1}{n}} \leq \frac{\tilde{R}}{P}, \quad \text{and hence} \quad \max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{nk} a_k^* z^k \right| \leq \left(\frac{\tilde{R}}{P}\right)^n.$$

Applying the Cauchy inequalities to the function $g_n(z) := \sum_{k=0}^{n} b_{nk} a_k^* z^k$ we get in particular that, for all large n,

$$|b_{nn}||a_n^*|R_1^n \leq \left(\frac{\tilde{R}}{P}\right)^n, \text{ and therefore } |b_{nn}|^{\frac{1}{n}}|a_n^*|^{\frac{1}{n}} R_1 \leq \frac{\tilde{R}}{P}.$$

From the last inequality we get that

$$\frac{\tilde{R}}{P} \geq \limsup_{n\to\infty} \left(|b_{nn}|^{\frac{1}{n}}|a_n^*|^{\frac{1}{n}} R_1\right) = R_1 \lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} \cdot \limsup_{n\to\infty} |a_n^*|^{\frac{1}{n}} = \frac{R_1}{P}.$$

But this is a contradiction since $0 < \tilde{R} < R_1$. Hence the conclusion of the theorem must hold. □

The next two theorems generalize results about regular and nonregular Nörlund matrices due respectively to Luh [3] and K. Stadtmüller [6, Theorems 6 and 7]. The first of these theorems, which follows immediately from Theorems 4 and 5, shows, inter alia, that the sequence (g_n) specified in Theorem 2 cannot converge uniformly in any disk D_{P_1} with $P_1 > P$ when $\mathbf{B}$ is a normal matrix satisfying condition (ii) of Theorem 2 together with the diagonal condition of Theorem 5.

Theorem 6. *Suppose that* P *and* R *are positive numbers and that* $\mathbf{B}$ *is a normal matrix satisfying*

$$M(p) := \sup_{n\geq 0,\ k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \text{ for } 0 < p < P \quad \text{and} \quad \lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P}.$$

Then, for each $\mathbf{a} \in \mathbf{A}_R$ *and each* $R_1 \geq P$,

$$\limsup_{n\to\infty} \max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^{\frac{1}{n}} = \frac{R_1}{P}.$$

The next theorem shows that the circle $|z| = R_1$ in the conclusion of Theorem 6 can be replaced by any arc of that circle when condition (i) of Theorem 2 is also satisfied.

Theorem 7. *Suppose that P and R are positive numbers and that* $\mathbf{B}$ *is a normal matrix such that*

$$\lim_{n\to\infty} b_{nk} =: b_k \quad \textit{for } k = 0, 1, \ldots, \textit{ where } b_k \neq 0 \textit{ for } k > k^*;$$

$$M(p) := \sup_{n\geq 0,\ k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \quad \textit{for } 0 < p < P, \quad \textit{and} \quad \lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P}.$$

Then, for each $\mathbf{a} \in \mathbf{A}_R$ *and each* $R_1 \geq P$,

$$\limsup_{n\to\infty} \max_{z\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k z^k\right|^{\frac{1}{n}} = \frac{R_1}{P},$$

where Γ *is any closed non-trivial arc of* $|z| = R_1$.

Proof. By Theorem 6 we know that

$$\limsup_{n\to\infty} \max_{z\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k z^k\right|^{\frac{1}{n}} \leq \frac{R_1}{P}.$$

Hence it is enough to prove that, for every $\mathbf{a} \in \mathbf{A}_R$,

$$\limsup_{n\to\infty} \max_{z\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k z^k\right|^{\frac{1}{n}} \geq \frac{R_1}{P}, \tag{2}$$

which we now proceed to do.

Case 1. $R_1 = P$: Suppose (2) is not true. Then for some $\mathbf{a}^* \in \mathbf{A}_R$ we have

$$\limsup_{n\to\infty} \max_{z\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k^* z^k\right|^{\frac{1}{n}} < \frac{R_1}{P} = 1.$$

It follows that there exists a positive number $q < 1$ such that, for all n sufficiently large,

$$\sup_{z\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k^* z^k\right| < q^n.$$

Given $\epsilon > 0$ we get from Theorem 6 that, for all n sufficiently large,

$$\max_{|z|=P} \left|\sum_{k=0}^{n} b_{nk} a_k^* z^k\right| \leq 2^{\epsilon n}.$$

For $0 < r < P$ we have, by Nevanlinna's N-constants theorem (see [1, Theorem 18.3.3]), that there exists a positive number $\theta < 1$ (depending on r but not on ϵ) such that, for all large n,

$$\max_{|z|=r} \left|\sum_{k=0}^{n} b_{nk} a_k^* z^k\right| \leq \left(q^\theta 2^{(1-\theta)\epsilon}\right)^n.$$

Since we can choose $\epsilon > 0$ so small that $q^\theta 2^{(1-\theta)\epsilon} < 1$, it follows that

$$\max_{|z|=r} \left|\sum_{k=0}^{n} b_{nk} a_k^* z^k\right| \to 0 \quad \text{as } n \to \infty.$$

By the Weierstrass double-series theorem we get that

$$0 = \lim_{n\to\infty} b_{nk}a_k^* = b_k a_k^* \quad \text{for } k = 0, 1, \dots .$$

Since $\mathbf{a}^* \in \mathbf{A}_R$, we have that $a_k^* \neq 0$ for some $k > k^*$. Hence $b_k = 0$ for such a k. But this contradicts the assumption that $b_k \neq 0$ for $k > k^*$. Therefore (2) must hold when $R_1 = P$.

Case 2. $R_1 > P$: Assume that (2) is not true. Then there exists a sequence $\mathbf{a}^* \in \mathbf{A}_R$ and a number $\tilde{R}$ such that $P < \tilde{R} < R_1$ and

$$\limsup_{n\to\infty} \max_{z\in\Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k^* z^k \right|^{\frac{1}{n}} \leq \frac{\tilde{R}}{P}.$$

Hence given $\epsilon > 0$ we have, for all sufficiently large n,

$$\max_{z\in\Gamma} \left| z^{-n} \sum_{k=0}^{n} b_{nk} a_k^* z^k \right| \leq \left(\frac{\tilde{R}}{P} \cdot \frac{1}{R_1} \right)^n 2^{\epsilon n} = \left(\frac{\tilde{R}}{R_1} \right)^n \left(\frac{2^\epsilon}{P} \right)^n .$$

Further, from Theorem 6 we get that, for all large n,

$$\max_{|z|=P} \left| z^{-n} \sum_{k=0}^{n} b_{nk} a_k^* z^k \right| \leq \left(\frac{2^\epsilon}{P} \right)^n$$

and

$$\max_{|z|=R_1} \left| z^{-n} \sum_{k=0}^{n} b_{nk} a_k^* z^k \right| \leq \left(\frac{2^\epsilon}{P} \right)^n .$$

Let $g_n(z) := \sum_{k=0}^n b_{nk} a_k^* z^k$, and let $P < r < R_1$. Then, by Nevanlinna's N-constants theorem, there exist positive constants $\theta_1, \theta_2, \theta_3$ (depending on r but not on ϵ) such that $\theta_1 + \theta_2 + \theta_3 = 1$ and

$$\max_{|z|=r} \left| \frac{g_n(z)}{z^n} \right| \leq \left(\frac{\tilde{R}}{R_1} \frac{2^\epsilon}{P} \right)^{n\theta_1} \left(\frac{2^\epsilon}{P} \right)^{n\theta_2} \left(\frac{2^\epsilon}{P} \right)^{n\theta_3} = \left(\frac{\tilde{R}}{R_1} \right)^{n\theta_1} \left(\frac{2^\epsilon}{P} \right)^n$$

for all sufficiently large n. Hence, choosing $\epsilon > 0$ so small that $(\tilde{R}/R_1)^{\theta_1} 2^\epsilon < 1$, we get

$$\limsup_{n\to\infty} \max_{|z|=r} |g_n(z)|^{\frac{1}{n}} \leq \left(\frac{\tilde{R}}{R_1} \right)^{\theta_1} 2^\epsilon \frac{r}{P} < \frac{r}{P}.$$

Since $r > P$, the last inequality contradicts the conclusion of Theorem 5. Hence (2) must hold when $R_1 > P$. □

The next theorem deals with the possibility of pointwise convergence of the sequence $(g_n(z))$ specified in Theorem 2 outside the convergence disk D_P. It generalizes results due to Lejá [2] and Stadtmüller [6, Theorem 8] about regular and nonregular Nörlund matrices respectively. Both authors mistakenly assumed that their proofs were valid when, in the notation of the following theorem, $R = 1$ and sequence (a_n) is bounded. The example $a_n := 1/(n+1)$ shows that their method of proof cannot be used in this case. The difficulty is avoided in our Theorem 8 by the imposition of the limsup condition.

Theorem 8. *Suppose that P and R are positive numbers and that* $\mathbf{B}$ *is a normal matrix such that*

$$\lim_{n\to\infty} b_{nk} =: b_k \quad \text{for } k = 0, 1, \ldots, \text{ where } b_k \neq 0 \text{ for } k > k^*;$$

$$M(p) := \sup_{n\geq 0,\ k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \quad \text{for } 0 < p < P; \qquad \lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P},$$

and

$$|b_{nk}| \leq c(\tilde{R})|b_{nn}| \left(\frac{P}{\tilde{R}}\right)^{n-k} \quad \text{for } 0 < \tilde{R} < R \text{ and } 0 \leq k \leq n.$$

Suppose that $\mathbf{a} \in \mathbf{A}_R$ *and that* $\limsup_{n\to\infty} |a_n|R^n > 0$. *Let*

$$g_n(z) := \sum_{k=0}^{n} b_{nk} a_k z^k.$$

Then $\limsup_{n\to\infty} |g_n(z)|^{\frac{1}{n}} \leq 1$ *for at most a finite number of points z satisfying $|z| > P_1 > P$, and hence, in particular, the sequence (g_n) can converge at most at a finite number of points z satisfying $|z| > P_1 > P$.*

Proof. Let $c_n := a_n R^n$ where $\mathbf{a} \in \mathbf{A}_R$, and let $\limsup_{n\to\infty} |c_n| > c > 0$. Define

$$M := \begin{cases} 1 & \text{if } \sup_{n\geq 0} |c_n| = \infty, \\ c^{-1} \sup_{n\geq 0} |c_n| & \text{otherwise.} \end{cases}$$

By considering the unbounded monotonic sequence (d_n) where $d_n := \max_{0\leq k\leq n} |c_k|$ when $\max_{n\geq 0} |c_n| = \infty$, we see that there is a strictly increasing sequence of positive integers (n_k) integers such that

$$|c_n| \leq M|c_{n_k}| \quad \text{for } 0 \leq n < n_k, \quad \text{and} \quad |c_{n_k}| > c.$$

Since $\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 1$, we have

$$1 \geq \limsup_{k\to\infty} |c_{n_k}|^{\frac{1}{n_k}} \geq \liminf_{k\to\infty} |c_{n_k}|^{\frac{1}{n_k}} \geq \lim_{k\to\infty} c^{\frac{1}{n_k}} = 1,$$

so $\lim_{k\to\infty} |c_{n_k}|^{\frac{1}{n_k}} = 1$. Whenever $c_n \neq 0$, let

$$\tilde{g}_n(z) := \sum_{j=0}^{n} \frac{b_{nj}}{b_{nn}} \frac{c_j}{c_n} \left(\frac{z}{R}\right)^{j-n} = \frac{g_n(z)}{b_{nn} c_n (z/R)^n}; \tag{3}$$

and let

$$h_k(w) := \tilde{g}_{n_k}\left(\frac{1}{w}\right). \tag{4}$$

Assume that z^* is a point such that $|z^*| > P_1$ and $\limsup_{n\to\infty} |g_n(z^*)|^{\frac{1}{n}} \leq 1$. Since

$$\begin{aligned} \lim_{k\to\infty} \left| b_{n_k, n_k} c_{n_k} \left(\frac{z^*}{R}\right)^n \right|^{\frac{1}{n}} &= \lim_{k\to\infty} |b_{n_k, n_k}|^{\frac{1}{n_k}} \cdot \lim_{k\to\infty} |c_{n_k}|^{\frac{1}{n_k}} \cdot \frac{|z^*|}{R} \\ &\geq \frac{R}{P} \frac{P_1}{R} = \frac{P_1}{P} > 1, \end{aligned}$$

it follows from (3) and (4) that $\limsup_{k\to\infty} |\tilde{g}_{n_k}(z^*)|^{\frac{1}{n_k}} < 1$ and hence that

$$\lim_{k\to\infty} h_k(w^*) = 0 \quad \text{where } w^* := 1/z^*. \tag{5}$$

Suppose $|w| \le 1/P^*$ where $P_1 > P^* > P$. Then we have, for $0 < \tilde{R} < R$,

$$|h_k(w)| \le \sum_{j=0}^{n_k} c(\tilde{R}) \left(\frac{P}{\tilde{R}}\right)^{n_k - j} M \left(\frac{R}{P^*}\right)^{n_k - j} = c(\tilde{R}) M \sum_{j=0}^{n_k} \left(\frac{P}{P^*}\frac{R}{\tilde{R}}\right)^{n_k - j}.$$

Choose $\tilde{R} < R$ so close to R that $0 < \frac{P}{P^*}\frac{R}{\tilde{R}} < 1$. Then

$$|h_k(w)| \le \frac{c(\tilde{R})M}{1 - \frac{P}{P^*}\frac{R}{\tilde{R}}} < \infty \quad \text{for } |w| \le \frac{1}{P^*} < \frac{1}{P} \text{ and } k \ge 0.$$

This means that the sequence $(h_k(w))$ is uniformly bounded for $|w| \le 1/P^*$. Suppose now that there are infinitely many points z_r with $|z_r| > P_1 > P^*$ such that $\limsup_{n\to\infty} | g_n(z_r)|^{\frac{1}{n}} \le 1$. Then by (5)

$$\lim_{k\to\infty} h_k(w_r) = 0 \quad \text{for } w_r := 1/z_r.$$

By Vitali's theorem (see [7, Theorem 5.2.1]) the sequence $(h_k(w))$ converges uniformly to 0 on compact subsets of $D_{\frac{1}{P^*}}$. In particular,

$$0 = \lim_{k\to\infty} h_{n_k}(0) = 1,$$

which is a contradiction. Hence there are at most finitely many points z such that $|z| > P_1$ and $\limsup_{n\to\infty} |g_n(z)|^{\frac{1}{n}} \le 1$. □

5. Construction

In this section we construct a Nörlund matrix $\mathbf{N}_B$ satisfying the hypotheses of Theorem 8 with $P = 1$ such that the corresponding sequence of transforms (g_n) of the power series $\sum_{k=0}^{\infty}(z/R)^k$ converges at N points outside the convergence disk D_1.

Let $p(z)$ be a polynomial of degree N defined by

$$p(z) := \sum_{k=0}^{\infty} p_k z^k := (z + \alpha_1)(z + \alpha_2)\cdots(z + \alpha_N),$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N < 1$. Define the Nörlund matrix $\mathbf{N}_B \equiv (b_{nk})$ by setting

$$b_{nk} := \frac{B_{n-k}}{B_n} \quad \text{for } 0 \le k \le n, \quad \text{where } B_n := \frac{1}{R^n}\sum_{k=0}^{n} p_k.$$

Then, for $a_k := 1/R^k$, $w = 1/z$, and $n \geq N$,

$$\begin{aligned} g_n(z) :&= \sum_{k=0}^{n} b_{nk} a_k z^k = \frac{1}{B_n} \sum_{k=0}^{n} B_{n-k} \left(\frac{z}{R}\right)^k \\ &= \frac{z^n}{B_n R^n} \sum_{k=0}^{n} B_k (Rw)^k = \frac{z^n}{B_n R^n} \sum_{k=0}^{n} w^k \sum_{j=0}^{k} p_j \\ &= \frac{z^n}{B_n R^n} \sum_{j=0}^{n} p_j \sum_{k=j}^{n} w^k = \frac{z^n}{B_n R^n} \sum_{j=0}^{n} p_j \frac{w^j - w^{n+1}}{1-w} \\ &= \frac{z^n}{B_n R^n} \frac{p(w)}{1-w} - \frac{w}{1-w}. \end{aligned}$$

Hence, for every $n \geq N$, we have $g_n(z) = z/(1-z)$ whenever $p(w) = 0$, and this occurs when $z = -1/a_k$, $k = 1, 2, \ldots, N$.

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