JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **188**, 209–218 (1994)

On Some Trigonometric and Exponential Lattice Sums*

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Submitted by Bruce C. Berndt

Received June 17, 1993

Certain two-dimensional trigonometric lattice series, the ordinary convergence of which involves unresolved questions of a deep and delicate number-theoretic nature, are shown to be summable by a special Abelian method and their Abelian sums are obtained. This is done by first evaluating an absolutely convergent exponential lattice series and then analytically extending its sum.

1. INTRODUCTION

We consider the trigonometric sums

$$S_{\pm}(\theta) := \sum_{\substack{m,n=-\infty \\ m^2+n^2\neq 0}}^{\infty} \frac{(\pm 1)^{m+n}}{\sqrt{m^2+n^2}} \sin(\sqrt{m^2+n^2}\,\theta) \qquad (\theta \text{ real})$$

$$= \sum_{m,n}{}' \frac{(\pm 1)^{m+n}}{r} \sin(r\theta) \qquad \text{where } r := \sqrt{m^2+n^2} \geq 0,$$

* This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

0022-247X/94 $6.00

and $C_\pm(\theta)$, the corresponding sums with $\cos(r\theta)$ in place of $\sin(r\theta)$. We use the notation $\Sigma_{m,n}$ for sums over all lattice points (m, n) and Σ' to indicate that the point $(0, 0)$ is omitted. It is not clear, ab initio, in what sense these sums exist. When absolute convergence is not assured it will be convenient to sum over expanding circles, so that

$$S_\pm(\theta) = \sum_{n=1}^{\infty} \frac{(\pm 1)^n c_n}{\sqrt{n}} \sin(\sqrt{n}\theta),$$

where c_n is the number of ways of expressing n as the sum of two integer squares, and similarly for $C_\pm(\theta)$. These series, however, may or may not converge in the ordinary sense. Since

$$\sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

it is certainly possible that they may be conditionally convergent or even divergent. In fact, since

$$\sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n}} (\cos^2(\sqrt{n}\theta) + \sin^2(\sqrt{n}\theta)) = \infty,$$

it follows that either

$$\sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n}} |\cos(\sqrt{n}\theta)| = \infty \qquad \text{or} \qquad \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n}} |\sin(\sqrt{n}\theta)| = \infty.$$

The problem of determining whether or not the series for $S_\pm(\theta)$ and $C_\pm(\theta)$ actually converge involves deep number-theoretic questions which we cannot answer. What we shall show, inter alia, is that

$$S_-(\theta)(\tilde{A}) = -\theta \qquad \text{for } |\theta| < \sqrt{2}\pi, \tag{1}$$

$$S_+(\theta)(\tilde{A}) = \frac{2\pi}{\theta} - \theta \qquad \text{for } 0 < |\theta| < 2\pi, \tag{2}$$

where $\tilde{A}$ denotes the Abelian method of summability defined as follows:

$$\sum_{n=1}^{\infty} a_n = \ell(\tilde{A}) \qquad \text{if } \lim_{x\to 0^+} \sum_{n=1}^{\infty} a_n e^{-x\sqrt{n}} = \ell.$$

The method is regular (see [3, Section 4.7]), i.e., if $\sum_{n=1}^{\infty} a_n = \ell$ then $\sum_{n=1}^{\infty} a_n = \ell(\tilde{A})$. We shall also show that

$$C_-(\theta)(\tilde{A}) = \sum_{n=0}^{\infty} \binom{2n}{n} \gamma\left(n + \frac{1}{2}\right)\left(\frac{\theta}{4\pi}\right)^{2n} \qquad \text{for } |\theta| < \sqrt{2}\pi, \tag{3}$$

$$C_+(\theta)(\tilde{A}) = \sum_{n=0}^{\infty} \binom{2n}{n} \delta\left(n + \frac{1}{2}\right)\left(\frac{\theta}{4\pi}\right)^{2n} \qquad \text{for } 0 < |\theta| < 2\pi, \tag{4}$$

where $\gamma(s) := 4\zeta(s)\beta(s)(2^s - 1)$ and $\delta(s) := 4\zeta(s)\beta(s)$ with

$$\zeta(s) := 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots \qquad (\text{re } s > 1)$$

$$= \frac{1}{1 - 2^{1-s}}(1 - 2^{-s} + 3^{-s} - 4^{-s} + \cdots) \qquad (\text{re } s > 0),$$

$$\beta(s) := 1 - 3^{-s} + 5^{-s} - 7^{-s} + \cdots \qquad (\text{re } s > 0).$$

We shall describe in Section 2 below how $S_\pm(\theta)$ and $C_\pm(\theta)$ behave in the $\tilde{A}$-sense for θ outside the "principal domains" specified in (1), (2), (3), and (4). Note that these equations only give $\tilde{A}$-values of the sine and cosine series, but that whenever the series do converge they must converge to the said values. We cannot infer ordinary convergence from $\tilde{A}$-convergence. However, had we had in our definition of the Abelian summability method λ_n instead of $\sqrt{n}$ with $\lambda_n > 0$ and $\liminf \lambda_{n+1}/\lambda_n > 1$, then, by the "high indices" theorem [3, Theorem 114], Abelian summability would have implied convergence.

The results in this note provide answers, in slightly rescaled form, to Problem 92–11* in SIAM Review. Moreover, on computing the right-hand series in (3) to high precision, we found that $C_-(2.5)$ cannot be zero as conjectured in the problem. Computation shows that the unique zero of that right-hand series in $(0, \sqrt{2}\pi)$ is in fact $\theta = 2.504259\ldots$ Computation of the original defining lattice sum (in whatever sense) is much too slow a process to practically determine the value of the zero to even three digit accuracy. After our note was submitted for publication the September 1993 issue of SIAM Review presented a complete formal solution for Problem 92–11* by Boersma and De Doelder which, however, made no explicit mention of the delicate convergence questions. Other responses to the problem are briefly discussed in this issue.

It is interesting to observe the following differences in behavior at the origin between those of the above lattice series that involve the alternating term $(-1)^{m+n}$ and those that do not.

$$\lim_{\theta\to 0}\{S_-(\theta)(\tilde{A})\} = S_-(0) = 0 \qquad \text{and} \qquad \lim_{\theta\to 0}\{C_-(\theta)(\tilde{A})\} = C_-(0) = \gamma\left(\frac{1}{2}\right)$$

(see [4] and [1, Section 9.2]); but

$$\lim_{\theta\to 0+}\{S_+(\theta)(\tilde{A})\} = \infty > S_+(0) = 0 \qquad \text{and}$$

$$\lim_{\theta\to 0}\{C_+(\theta)(\tilde{A})\} = \delta\left(\frac{1}{2}\right) < C_+(0) = \infty.$$

2. The Proofs: An Exponential sum

Let

$$E_\varepsilon(a) := \sum_{\substack{m,n=-\infty \\ m^2+n^2\neq 0}}^{\infty} \frac{\cos(2\pi\varepsilon_1 m)\cos(2\pi\varepsilon_2 n)}{\sqrt{m^2+n^2}}\, e^{-a\sqrt{m^2+n^2}}$$

$$= \sum_{m,n}{}' \frac{\cos(2\pi\varepsilon_1 m)\cos(2\pi\varepsilon_2 n)}{r}\, e^{-ar},$$

where $a > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon := (\varepsilon_1, \varepsilon_2)$, and $r := \sqrt{m^2+n^2} \geq 0$. The sums are absolutely convergent, so the order of summation is immaterial. Further, since (see [1, p. 39, Exercise 4])

$$\frac{e^{-ar}}{r} = \frac{a}{2\sqrt{\pi}} \int_0^\infty t^{-3/2} e^{-t-a^2r^2/4t}\, dt,$$

we have (as in [2, p. 1417]) that $E_\varepsilon(a) = \lim_{\delta\to 0+} E_\varepsilon(a, \delta)$ where

$$E_\varepsilon(a, \delta) := \frac{a}{2\sqrt{\pi}} \sum_{m,n}{}' \frac{\cos(2\pi\varepsilon_1 m)\cos(2\pi\varepsilon_2 n)}{r} \int_\delta^\infty t^{-3/2} e^{-t-a^2r^2/4t}\, dt$$

$$= \frac{a}{2\sqrt{\pi}} \int_\delta^\infty t^{-3/2} e^{-t} \left(\sum_{m,n} \cos(2\pi\varepsilon_1 m)\cos(2\pi\varepsilon_2 n)\, e^{-a^2r^2/4t} - 1\right) dt,$$

the interchange of order of Σ' and lim being justified by the uniform convergence of the series. By Poisson's formula (see [1, Section 2.2]), the final sum is equal to

$$\frac{4\pi t}{a^2} \sum_{m,n} e^{-4\pi^2 r_\varepsilon^2 t/a^2} \qquad \text{where } r_\varepsilon^2 := (m + \varepsilon_1)^2 + (n + \varepsilon_2)^2,\ r_\varepsilon > 0.$$

Integration now yields (as in [2])

$$E_\varepsilon(a, \delta) := \sum_{m,n} \frac{\operatorname{erfc}(\sqrt{\delta(1 + 4\pi^2 r_\varepsilon^2/a^2)})}{\sqrt{r_\varepsilon^2 + a^2/4\pi^2}} - \frac{a}{\sqrt{\pi\delta}} e^{-\delta} + a \operatorname{erfc}(\sqrt{\delta}),$$

where $\operatorname{erfc}(x) = (2/\sqrt{\pi}) \int_x^\infty e^{-t^2}\, dt$. We can express the sum involving $\operatorname{erfc}(\cdot)$ as

$$\sum_{m,n} \frac{\operatorname{erfc}(2\pi r \sqrt{\delta/a^2})}{r_\varepsilon} - \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 := \sum_{m,n} \operatorname{erfc}(2\pi r_\varepsilon \sqrt{\delta/a^2}) \left(\frac{1}{r_\varepsilon} - \frac{1}{\sqrt{r_e^2 + a^2/4\pi^2}}\right)$$
$$\to \sum_{m,n} \left(\frac{1}{r_\varepsilon} - \frac{1}{\sqrt{r_\varepsilon^2 + a^2/4\pi^2}}\right) \qquad \text{as } \delta \to 0+,$$

and

$$\Sigma_2 := \sum_{m,n} \frac{\operatorname{erfc}(\sqrt{\delta(1 + 4\pi^2 r_\varepsilon^2/a^2)}) - \operatorname{erfc}(2\pi r_\varepsilon \sqrt{\delta/a^2})}{\sqrt{r_\varepsilon^2 + a^2/4\pi^2}}$$
$$= O(\delta^{1/4}) \sum_{m,n} \frac{\delta^{1/4}}{r_\varepsilon^2} e^{-4\delta\pi^2 r_\varepsilon^2/a^2} = O(\delta^{1/4}) \sum_{m,n} \frac{1}{r_\varepsilon^{5/2}}$$
$$= o(1) \qquad \text{as } \delta \to 0+.$$

Hence

$$E_\varepsilon(a, \delta) = \sum_{m,n} \frac{\operatorname{erfc}(2\pi r_\varepsilon \sqrt{\delta/a^2})}{r_\varepsilon} - \frac{a}{\sqrt{\pi\delta}} e^{-\delta} + a \operatorname{erfc}(\sqrt{\delta})$$
$$- \sum_{m,n} \left(\frac{1}{r_\varepsilon} - \frac{1}{\sqrt{r_\varepsilon^2 + a^2/4\pi^2}}\right) + o(1) \qquad \text{as } \delta \to 0+.$$

Further, for $b > 0$, $b \neq a$,

$$E_\varepsilon(b, \delta b^2/a^2) = \sum_{m,n} \frac{\operatorname{erfc}(2\pi r_\varepsilon \sqrt{\delta/a^2})}{r_\varepsilon} - \frac{a}{\sqrt{\pi\delta}} e^{-\delta b^2/a^2} + b \operatorname{erfc}(\sqrt{\delta b/a})$$
$$- \sum_{m,n} \left(\frac{1}{r_\varepsilon} - \frac{1}{\sqrt{r_\varepsilon^2 + a^2/4\pi^2}}\right) + o(1) \quad \text{as } \delta \to 0+.$$

It follows that

$$E_\varepsilon(a, \delta) - E_\varepsilon(b, \delta b^2/a^2) = \frac{a}{\sqrt{\pi\delta}} (e^{-\delta b^2/a^2} - e^{-\delta})$$
$$+ a \operatorname{erfc}(\sqrt{\delta}) - b \operatorname{erfc}(\sqrt{\delta b/2})$$
$$- \sum_{m,n} \left(\frac{1}{\sqrt{r_\varepsilon^2 + b^2/4\pi^2}} - \frac{1}{\sqrt{r_\varepsilon^2 + a^2/4\pi^2}}\right)$$
$$+ o(1) \quad \text{as } \delta \to 0+,$$

and hence that

$$E_\varepsilon(a) = E_\varepsilon(b) + a - b - \sum_{m,n}\left(\frac{1}{\sqrt{r_\varepsilon^2 + b^2/4\pi^2}} - \frac{1}{\sqrt{r_\varepsilon^2 + a^2/4\pi^2}}\right). \tag{5}$$

Now, for $a > 0$, $y > 1$, let

$$E(a) := \lim_{\varepsilon \to (0+,0+)} E_\varepsilon(a) \quad \text{and} \quad F(y) := \sum_{n=1}^{\infty} \frac{c_n}{n^{(1+y)/2}} = \sum_{n=1}^{\infty} d_n \lambda_n^{-y},$$

where $d_n := c_n/\sqrt{n}$, $\lambda_n := \sqrt{n}$, and, as before, c_n is the number of ways of expressing n as the sum of two integer squares. Observe that $E(a)$ exists because of the uniform convergence of the series defining $E_\varepsilon(a)$, and that

$$E(a) = \sum_{m,n}{}' \frac{e^{-ar}}{r} = \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n}} e^{-a\sqrt{n}} = \sum_{n=1}^{\infty} d_n e^{-a\lambda_n}.$$

Hence

$$F(y) = \frac{1}{\Gamma(y)} \sum_{n=1}^{\infty} d_n \int_0^\infty a^{y-1} e^{-a\lambda_n}\, da = \frac{1}{\Gamma(y)} \int_0^\infty a^{y-1} E(a)\, da,$$

the change of order of summation and integration being legitimate since all the terms are positive. Next, letting $\varepsilon \to (0+, 0+)$ in (5), we get, for $a > 0$, $b > 0$, $a \neq b$,

$$E(a) - \frac{2\pi}{a} = E(b) - \frac{2\pi}{b} + a - b - \sum_{m,n}{}' \left(\frac{1}{\sqrt{r^2 + b^2/4\pi^2}} - \frac{1}{\sqrt{r^2 + a^2/4\pi^2}} \right). \tag{6}$$

It follows that $E(a)$ is analytic for $a > 0$, that

$$\gamma := \lim_{a \to 0+} \left(E(a) - \frac{2\pi}{a} \right)$$

exists, and hence that

$$\lim_{a \to 0+} \left(E(a) - \frac{2\pi}{a} e^{-a} \right) = \gamma + 2\pi.$$

Further, for $y > 1$,

$$\begin{aligned} \frac{1}{\Gamma(y)} \int_0^\infty a^{y-1} \left(E(a) - \frac{2\pi}{a} e^{-a} \right) da &= F(y) - \frac{2\pi}{y-1} \\ &= 4\zeta\left(\frac{1+y}{2}\right) \beta\left(\frac{1+y}{2}\right) - \frac{2\pi}{y-1}, \end{aligned}$$

by Hardy's decomposition of the two-dimensional zeta function (see [1, Section 9.2]). Both the left-hand and right-hand terms of this equation are analytic for $y > 0$. Hence, by analytic continuation, equality holds in this range. Also, the limit of the left-hand term as $y \to 0+$ is $\gamma + 2\pi$, and of the right-hand term is $4\zeta(1/2)\beta(1/2) + 2\pi$. We have thus proved

$$\lim_{a \to 0+} \left(E(a) - \frac{2\pi}{a} \right) = 4\zeta\left(\frac{1}{2}\right) \beta\left(\frac{1}{2}\right) = \delta\left(\frac{1}{2}\right). \tag{7}$$

In dealing with analytic continuation we shall suppose that the principal branch of any involved square root is considered. It follows from (5) that $E_\varepsilon(a)$ has an analytic continuation to the half-plane re $a > 0$, and that, for $b > 0$, $|\theta| \neq 2\pi r_\varepsilon = 2\pi \sqrt{(m + \varepsilon_1)^2 + (n + \varepsilon_2)^2}$ $(m, n = 0, \pm 1, \pm 2, \cdots)$,

$$\lim_{x \to 0+} E_\varepsilon(x + i\theta) = i\theta + E_\varepsilon(b) - b - \sum_{m,n} \left(\frac{1}{\sqrt{r_\varepsilon^2 + b^2/4\pi^2}} - \frac{1}{\sqrt{r_\varepsilon^2 - \theta^2/4\pi^2}} \right).$$

Consequently, for $|\theta| \neq 2\pi r_\varepsilon$,

$$\lim_{x\to 0+} E_\varepsilon(x + i\theta) = i\theta + \lim_{b\to 0+} E_\varepsilon(b) - \sum_{m,n}\left(\frac{1}{r_\varepsilon} - \frac{1}{\sqrt{r_\varepsilon^2 - \theta^2/4\pi^2}}\right). \quad (8)$$

Likewise, it follows from (6) that $E(a) - 2\pi/a$ has an analytic continuation to half-plane re $a > 0$, and that, for $b > 0$, $|\theta| \neq 2\pi r = 2\pi\sqrt{m^2 + n^2}$ $(m, n = 0, 1, 2, \cdots)$,

$$\lim_{x\to 0+} E(x + i\theta) = i\left(\theta - \frac{2\pi}{\theta}\right) + E(b) - \frac{2\pi}{b} - b$$
$$- \sum_{m,n}{}' \left(\frac{1}{\sqrt{r^2 + b^2/4\pi^2}} - \frac{1}{\sqrt{r^2 - \theta^2/4\pi^2}}\right).$$

Letting $b \to 0+$, we deduce by means of (7) that, for $|\theta| \neq 2\pi r$,

$$\lim_{x\to 0+} E(x + i\theta) = i\left(\theta - \frac{2\pi}{\theta}\right) + \delta\left(\frac{1}{2}\right) - \sum_{m,n}{}' \left(\frac{1}{r} - \frac{1}{\sqrt{r^2 - \theta^2/4\pi^2}}\right). \quad (9)$$

We now specialize these results to deal with $S_\pm(\theta)$ and $C_\pm(\theta)$. Let $E_-(a) := E_\varepsilon(a)$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$. Then, for $b > 0$,

$$E_-(b) = \sum_{m,n}{}' \frac{(-1)^{m+n}}{\sqrt{m^2 + n^2}} e^{-b\sqrt{m^2+n^2}} = \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{\sqrt{n}} e^{-b\sqrt{n}}$$
$$\to \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{\sqrt{n}} = \gamma\left(\frac{1}{2}\right) \quad \text{as } b \to 0+. \quad (10)$$

It follows from (8) and (10) that, for

$$|\theta| \neq 2\pi\rho := \pi\sqrt{(2m + 1)^2 + (2n + 1)^2} \quad (m, n = 0, \pm 1, \pm 2, \cdots),$$

$$\lim_{x\to 0+} E_-(x + i\theta) = \lim_{x\to 0+} \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{\sqrt{n}} e^{-x\sqrt{n}}(\cos(\sqrt{n}\theta) - i\sin(\sqrt{n}\theta))$$
$$= i\theta + \gamma\left(\frac{1}{2}\right) - \sum_{m,n}\left(\frac{1}{\rho} - \frac{1}{\sqrt{\rho^2 - \theta^2/4\pi^2}}\right) \quad (11)$$
$$= \{C_-(\theta) - iS_-(\theta)\}(\tilde{A}).$$

Thus we can obtain the $\tilde{A}$-values of $C_-(\theta)$ and $S_-(\theta)$ whenever

$$|\theta| \neq 2\pi\rho = \pi\sqrt{(2m + 1)^2 + (2n + 1)^2} \quad (m, n = 0, \pm 1, \pm 2, \cdots)$$

by equating real and imaginary parts of the final equation in (11). In particular, for the principal domain $|\theta| < \sqrt{2}\pi$, we have

$$S_{-}(\theta)(\tilde{A}) = -\theta,$$

and, by the binomial theorem,

$$\begin{aligned} C_{-}(\theta)(\tilde{A}) &= \gamma\left(\frac{1}{2}\right) - \sum_{m,n}\left(\frac{1}{\rho} - \frac{1}{\sqrt{\rho^2 - \theta^2/4\pi^2}}\right) \\ &= \gamma\left(\frac{1}{2}\right) + \sum_{m,n}\sum_{k=1}^{\infty}\binom{2k}{k}\left(\frac{\theta}{4\pi}\right)^{2k}\frac{1}{\rho^{2k+1}} \\ &= \gamma\left(\frac{1}{2}\right) + \sum_{k=1}^{\infty}\binom{2k}{k}\left(\frac{\theta}{4\pi}\right)^{2k}\sum_{m,n}\frac{2^{2k+1}}{((2m+1)^2 + (2n+1)^2)^{k+1/2}} \\ &= \sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{\theta}{4\pi}\right)^{2k}\gamma\left(k + \frac{1}{2}\right). \end{aligned}$$

(See [4] and [1, Section 9.2]). We have thus established (1) and (3), and (2) and (4) follow in like manner from (9).

To describe the behavior of $S_{\pm}(\theta)$ outside the principal domains, we define $o_1, o_2, o_3, \cdots$ to be the numbers $(2n+1)^2 + (2m+1)^2$ ($m, n = 0, \pm 1, \pm 2, \cdots$) in increasing order, and $e_1, e_2, e_3, \cdots$ to be the numbers $4n^2 + 4m^2$ ($m, n = 1, 2, 3, \cdots$) in increasing order. Thus $o_1, o_2, o_3, o_4 \cdots = 2, 10, 18, 26, \cdots$, and $e_1, e_2, e_3, e_4, \cdots = 4, 8, 16, 20, \cdots$. Then it follows from (11) that

$$S_{-}(\theta)(\tilde{A}) = -\theta + \sum_{k=1}^{n}\frac{2\pi}{\sqrt{\theta^2 - \pi^2 o_k}} \qquad \text{for } \pi\sqrt{o_n} < |\theta| < \pi\sqrt{o_{n+1}}, \tag{12}$$

and from (9) that

$$S_{+}(\theta)(\tilde{A}) = \frac{2\pi}{\theta} - \theta + \sum_{k=1}^{n}\frac{2\pi}{\sqrt{\theta^2 - \pi^2 e_k}} \qquad \text{for } \pi\sqrt{e_n} < |\theta| < \pi\sqrt{e_{n+1}}. \tag{13}$$

We can evidently also obtain formulae for $C_{\pm}(\theta)$ appropriate to the above ranges by the method used in deriving (3) and (4). Finally, (8) gives us access to

$$S_{\varepsilon}(\theta) := \sum_{m,n}{}' \frac{\cos(2\pi\varepsilon_1 m)\cos(2\pi\varepsilon_2 n)}{\sqrt{m^2 + n^2}}\sin(\theta\sqrt{m^2 + n^2}).$$

For example, it follows from (8) that, if ν_1 is the smallest of the numbers $(m + \varepsilon_1)^2 + (n + \varepsilon_2)^2$ $(m, n = 0, \pm 1, \pm 2, \cdots)$, then

$$S_\varepsilon(\theta)(\tilde{A}) = -\theta \qquad \text{for } |\theta| < 2\pi\sqrt{\nu_1}.$$

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