A LOGARITHMIC METHOD OF SUMMABILITY

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[*Extracted from the Journal of the London Mathematical Society, Vol.* 33, 1958.]

1. *Introduction.*

Suppose throughout that $\{s_n\}$ is a sequence of complex numbers and let $\{s_n{}^\lambda\}$ be the sequence of associated (C, λ), means, *i.e.*

$$s_n{}^\lambda = \binom{n+\lambda}{n}^{-1} \sum_{\nu=0}^{n} \binom{\nu+\lambda-1}{\nu} s_{n-\nu} \quad (\lambda > -1).$$

We shall be concerned with methods of summability L and (A, λ) defined as follows:

If
$$\frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}$$

tends to a finite limit s as $x \to 1$ in the open interval $(0, 1)$, we say that $\{s_n\}$ is L-convergent to s and write $s_n \to s$ (L).

If
$$(1-x) \sum_{n=0}^{\infty} s_n{}^\lambda x^n \to s$$

as $x \to 1$ in $(0, 1)$, we say that $\{s_n\}$ is (A, λ)-convergent to s and write

* Received 11 September, 1957; read 21 November, 1957.

$s_n \to s\,(A, \lambda)$. $(A, 0)$ is then the ordinary Abel method which we denote by A.

In this note we investigate some of the properties of the L method. In particular, we consider its relationship to the (A, λ) method and also establish a result about the iteration product of L with any regular Hausdorff method.

2. *Translativity.*

In this section we prove

THEOREM 1. *The L method is translative.*

By this we mean that $s_{n+1} \to s\,(L)$ if and only if $s_n \to s\,(L)$. We require

LEMMA 1. *If α is a real number and $\{s_n\}$ is an L-convergent sequence, and if $(n+\alpha)\,u_n = s_n$ for $n = 0, 1, \ldots$, then $u_n \to 0\,(L)$.*

Proof. Let

$$\phi(x) = \sum_{n=m}^{\infty} \frac{s_n}{n+1} x^{n+\alpha-1} \quad (|x| < 1),$$

where $m > |\alpha|+2$. Then $\{\log(1-x)\}^{-1}\phi(x)$ tends to a finite limit as $x \to 1$ in $(0, 1)$ and $x^{-1}\phi(x) \to 0$ as $x \to 0$. Hence $\phi(x) = O\{|\log(1-x)|\}$ for $0 \leqslant x < 1$, and so, as $x \to 1$ in $(0, 1)$,

$$\sum_{n=m}^{\infty} \frac{u_n}{n+1} x^{n+1} = x^{1-\alpha} \int_0^x \phi(t)\,dt$$

$$= O\left\{x^{1-\alpha}\int_0^x |\log(1-t)|\,dt\right\} = o\{|\log(1-x)|\}.$$

The lemma follows.

Proof of Theorem 1. Suppose that $s_n \to s\,(L)$, and note that, for $0 \leqslant x < 1$,

$$\sum_{n=0}^{\infty} \frac{s_{n+1}}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{s_n}{n+1} x^n + \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)} x^n,$$

$$\sum_{n=1}^{\infty} \frac{s_{n-1}}{n+1} x^n = \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} - \sum_{n=0}^{\infty} \frac{s_n}{(n+2)(n+1)} x^{n+1}.$$

Applying Lemma 1, we deduce from the first identity that $s_{n+1} \to s\,(L)$ and from the second that $s_{n-1} \to s\,(L)$. The theorem follows.

3. *Relationship between L and (A, λ) methods.*

We commence this section with some results concerning the (A, λ) method. We use the notation

$$\sigma_\lambda(y) = \frac{1}{\Gamma(\lambda+1)} \frac{y^\lambda}{1+y} \sum_{n=0}^{\infty} s_n{}^\lambda \left(\frac{y}{1+y}\right)^n \quad (\lambda > -1,\ y > 0),$$

so that $s_n \to s\,(A, \lambda)$ if and only if $\Gamma(\lambda+1)\,y^{-\lambda}\sigma_\lambda(y) \to s$ as $y \to \infty$.

From the known identity

$$\frac{y^{\lambda+\delta+n}}{\Gamma(\lambda+\delta+1)} s_n^{\lambda+\delta} = \frac{1}{\Gamma(\lambda+1)\,\Gamma(\delta)} \sum_{\nu=0}^{n} \binom{n}{\nu} s_\nu{}^\lambda \int_0^y (y-t)^{n-\nu+\delta-1} t^{\nu+\lambda}\,dt$$

$$(y > 0,\ \lambda > -1,\ \delta > 0),$$

it is easily deduced that

$$\sigma_{\lambda+\delta}(y) = \frac{1}{\Gamma(\delta)} \int_0^y (y-t)^{\delta-1} \sigma_\lambda(t)\,dt \quad (y > 0,\ \lambda > -1,\ \delta > 0), \tag{1}$$

provided only that $\sigma_\lambda(t)$ is defined for all positive t. For $\lambda \geqslant 0$ this result is due essentially to Kogbetliantz ([7], 37) (see also Lord [8], 243).

In virtue of a familiar theorem on Cesàro limits of functions, an immediate consequence of (1) is that

$$(A,\ \lambda+\delta) \supseteq (A,\ \lambda) \quad (\lambda > -1,\ \delta > 0),$$

[*i.e.* $s_n \to s\,(A, \lambda+\delta)$ whenever $s_n \to s\,(A, \lambda)$].

For $\lambda \geqslant 0$ this result was given by Lord ([8], 243) and for $\lambda > -1$, by Amir [1].

Little additional difficulty is involved in proving the stronger inclusion result

$$(A,\ \lambda+\delta) \supset (A,\ \lambda) \quad (\lambda > -1,\ \delta > 0), \tag{2}$$

the notation signifying that $(A, \lambda+\delta) \supseteq (A, \lambda)$ and that at least one $(A, \lambda+\delta)$-convergent sequence is not (A, λ)-convergent.

Suppose $\lambda > -1$ and let $\{s_n{}^\lambda\}$ be the sequence such that

$$\frac{1}{1-x} \sin\frac{1}{1-x} = \sum_{n=0}^{\infty} s_n{}^\lambda x^n \quad (|x| < 1).$$

The sequence $\{s_n\}$ of which $\{s_n{}^\lambda\}$ is the sequence of (C, λ) means is given by the relation

$$s_n = \sum_{\nu=0}^{n} \binom{n-\nu-\lambda-1}{n-\nu} \binom{\nu+\lambda}{\nu} s_\nu{}^\lambda.$$

For this sequence $\{s_n\}$, $\Gamma(\lambda+1)\,\sigma_\lambda(y) = y^\lambda \sin(1+y)$, so that, for $y > 0$, $\delta > 0$, we have in virtue of (1),

$$\frac{\Gamma(\delta)}{\Gamma(\lambda+1)} y^{-\lambda-\delta} \sigma_{\lambda+\delta}(y) = y^{-\lambda-\delta} \int_0^y (y-t)^{\delta-1} t^\lambda \sin(1+t)\,dt$$

$$= \int_0^1 (1-u)^{\delta-1} u^\lambda \sin(1+uy)\,du.$$

Hence, by the Riemann-Lebesgue theorem, $y^{-\lambda-\delta}\sigma_{\lambda+\delta}(y) \to 0$ as $y \to \infty$; and so $s_n \to 0\,(A, \lambda+\sigma)$. On the other hand $\{s_n\}$ is not (A, λ)-convergent, since $\sin \frac{1}{1-x}$ does not tend to a limit when $x \to 1$ in (0, 1). This establishes (2).

The next theorem extends the known result that $L \supseteq A$ (Hardy [5], 81; see also Borwein [4], 347-8).

THEOREM 2. *For* $1 \geqslant \lambda > -1$, $L \supset (A, \lambda)$.

Proof. Suppose that $s_n \to s$ $(A, 1)$ and let $t_n = s_n^1$. Then $t_n \to s$ (A) and consequently $t_n \to s$ (L). Further $s_{n+1} = t_{n+1} + (n+1)(t_{n+1} - t_n)$, so that for $0 < x < 1$

$$\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_{n+1}}{n+1} x^{n+1}$$

$$= \frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{t_{n+1}}{n+1} x^{n+1} + \frac{1-x}{\log(1-x)} \sum_{n=0}^{\infty} t_n x^n - \frac{t_0}{\log(1-x)}.$$

In view of Theorem 1, it follows that $s_{n+1} \to s(L)$ and hence that $s_n \to s(L)$.

We have thus proved that $L \supseteq (A, 1)$. The full result is now a consequence of (2) and the following theorem.

THEOREM 3. *There is an L-convergent sequence which is not* (A, λ)-*convergent for any* $\lambda > -1$.

Proof. Let $\{s_n\}$ be the sequence such that

$$(1-x)^{-1-i} = \sum_{n=0}^{\infty} s_n x^n \quad (|x| < 1),$$

in which case $\sigma_0(y) = (1+y)^i$. Hence, by (1) we have for $\lambda > 0$, $y > 0$,

$$\Gamma(\lambda) y^{-\lambda} \sigma_\lambda(y) = y^{-\lambda} \int_0^y (y-t)^{\lambda-1} (1+t)^i \, dt$$

$$= \frac{\Gamma(\lambda)\Gamma(1+i)}{\Gamma(\lambda+i+1)} y^{-\lambda} (1+y)^{\lambda+i} - y^{-\lambda} \int_{-1}^0 (y-t)^{\lambda-1} (1+t)^i \, dt.$$

Since the final term tends to zero as $y \to \infty$, it follows that $y^{-\lambda}\sigma_\lambda(y)$ does not tend to a limit as $y \to \infty$. Consequently $\{s_n\}$ is not (A, λ)-convergent for any $\lambda > 0$ and *a fortiori* for any $\lambda > -1$.

On the other hand, as $x \to 1$ in (0, 1),

$$\sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} = \int_0^x (1-t)^{-1-i} \, dt = o\{|\log(1-x)|\},$$

so that $s_n \to 0$ (L). This completes the proof.

In order to prove the final theorem in this section we require a lemma.

LEMMA 2. *If* $\lambda > 0$, *then, as* $y \to \infty$,

(i) $y^{-\lambda} \int_0^y (y-t)^{\lambda-1} \cos(1+t) \, dt \to 0$,

(ii) $y^{-\lambda} \int_0^y (y-t)^{\lambda-1} \cos(1+t) \log(1+t) \, dt \to 0$,

(iii) $y^{-\lambda-1} \int_0^y (y-t)^{\lambda} (1+t) \sin(1+t) \log(1+t) \, dt \to 0$.

Proof of (i). The result is well-known and is an immediate consequence of the Riemann-Lebesgue theorem.

Proof of (ii). Suppose $0 < \lambda \leqslant 1$. In virtue of the inclusion theorem for Cesàro limits of functions no loss of generality is involved in so restricting λ. Let

$$I(y) = y^{-\lambda} \int_0^y (y-t)^{\lambda-1} \cos t \log t \, dt \quad (y > 0).$$

Now, as $y \to \infty$,

$$I(1+y) - (1+y)^{-\lambda} \int_0^y (y-t)^{\lambda-1} \cos(1+t) \log(1+t) \, dt$$

$$= (1+y)^{-\lambda} \int_{-1}^0 (y-t)^{\lambda-1} \cos(1+t) \log(1+t) \, dt = o(1),$$

so that it is enough to prove that $I(y) \to 0$ as $y \to \infty$. We have, for $y > 0$,

$$I(y) = \int_0^1 (1-t)^{\lambda-1} \log t \cos ty \, dt + \log y \int_0^1 (1-t)^{\lambda-1} \cos ty \, dt$$

$$= I_1(y) + \log y \, I_2(y),$$

say; and by the Riemann-Lebesgue theorem $I_1(y) \to 0$ as $y \to \infty$.

Further*, for $y > 1$,

$$|I_2(y)| \leqslant \left| \int_0^{1/y} u^{\lambda-1} \cos(1-u) y \, du \right| + \left| \int_{1/y}^1 u^{\lambda-1} \cos(1-u) y \, du \right|$$

$$\leqslant y^{-\lambda} \lambda^{-1} + y^{1-\lambda} \left| \int_\xi^1 \cos(1-u) y \, du \right| \quad (y^{-1} \leqslant \xi < 1)$$

$$\leqslant y^{-\lambda}(\lambda^{-1} + 2).$$

Consequently $\log y \, I_2(y) \to 0$ as $y \to \infty$; and the proof of (ii) is complete.

* See Hobson ([6], 565) for a similar result.

Proof of (iii). It is readily verified that, for $y>0$,

$$y^{-\lambda-1}\int_0^y (y-t)^\lambda (1+t)\sin(1+t)\log(1+t)\,dt$$

$$=(1+\lambda)y^{-\lambda-1}\int_0^y (y-t)^\lambda\cos(1+t)\log(1+t)\,dt$$

$$+y^{-\lambda-1}\int_0^y (y-t)^\lambda\cos(1+t)\,dt$$

$$+\lambda(1+y)y^{-\lambda-1}\int_0^y (y-t)^{\lambda-1}\cos(1+t)\log(1+t)\,dt;$$

from which, in view of the results (i) and (ii), result (iii) follows.

THEOREM 4. *There is a sequence which is (A,λ)-convergent for every $\lambda>1$ but is not L-convergent.*

Proof. Let $\{s_n\}$ be the sequence such that

$$-\log(1-x)\cos\frac{1}{1-x}=\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}\qquad(|x|<1).$$

This sequence is not L-convergent.

On the other hand, differentiation yields

$$\cos\frac{1}{1-x}+\sin\frac{1}{1-x}\,\frac{\log(1-x)}{1-x}=(1-x)\sum_{n=0}^{\infty}s_nx^n\qquad(|x|<1),$$

so that $\sigma_0(t)=\cos(1+t)-(1+t)\sin(1+t)\log(1+t)$. In view of (1) and Lemma 2, we have for $\lambda>1$,

$$\Gamma(\lambda)y^{-\lambda}\sigma_\lambda(y)=y^{-\lambda}\int_0^y (y-t)^{\lambda-1}\sigma_0(t)\,dt=o(1)\text{ as }y\to\infty.$$

Consequently $\{s_n\}$ is (A,λ)-convergent for every $\lambda>1$, and the proof is complete.

To sum up the main results in this section, we have shown that $L\supset(A,\lambda)$ for $1\geqslant\lambda>-1$, but that, for $\lambda>1$, the methods L and (A,λ) are not comparable.

4. *Product of L and Hausdorff methods.*

In what follows suppose that $\chi(t)$ is a real function of bounded variation in the closed interval $[0,1]$ and let

$$h_n=\sum_{\nu=0}^{n}\binom{n}{\nu}s_\nu\int_0^1 t^\nu(1-t)^{n-\nu}d\chi(t)\qquad(n=0,1,\ldots).$$

If $h_n\to s$ we write $s_n\to s(H_\chi)$. It is known (Hardy [5], §11.8) that the Hausdorff method of summability H_χ so defined is regular [*i.e.* $s_n\to s\,(H_\chi)$ whenever $s_n\to s$] if and only if

$$\chi(0+)=\chi(0),\quad \chi(1)-\chi(0)=1 \tag{3}$$

If $h_n\to s(L)$ we write $s_n\to s(LH_\chi)$; thus defining the product method of summability LH_χ.

The main result in this section is:

THEOREM 5. *If H_χ is a regular Hausdorff method, then $LH_\chi\supseteq L$.*

Similar theorems with other methods of summability in place of L have been obtained by Szász [9] (see also Amir [2], 376, and Borwein [3], 321-2).

We require two lemmas

LEMMA 3. *If*

$$s(t)=\sum_{n=1}^{\infty}\frac{s_n}{n}\left(\frac{t}{1+t}\right)^n$$

and the series is convergent for all $t\geqslant0$, then, for $y\geqslant0$,

$$\sum_{n=1}^{\infty}\frac{h_n}{n}\left(\frac{y}{1+y}\right)^n=\int_0^1\{s(yt)-s_0\log(1+yt)\}d\chi(t)+s_0\log(1+y)\int_0^1 d\chi(t).$$

Proof. For $y\geqslant0$,

$$\sum_{n=1}^{\infty}\frac{h_n}{n}\left(\frac{y}{1+y}\right)^n=\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{y}{1+y}\right)^n\sum_{\nu=1}^{n}\binom{n}{\nu}s_\nu\int_0^1 t^\nu(1-t)^{n-\nu}d\chi(t)$$

$$+s_0\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{y}{1+y}\right)^n\int_0^1(1-t)^n d\chi(t)$$

$$=\int_0^1 d\chi(t)\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}\left(\frac{yt}{1+y}\right)^\nu\sum_{n=\nu}^{\infty}\binom{n-1}{\nu-1}\left(\frac{y-yt}{1+y}\right)^{n-\nu}$$

$$+s_0\int_0^1 d\chi(t)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{y-yt}{1+y}\right)^n$$

$$=\int_0^1 d\chi(t)\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}\left(\frac{yt}{1+yt}\right)^\nu-s_0\int_0^1\log\frac{1+yt}{1+y}\,d\chi(t);$$

all the inversions being legitimate since $\int_0^1|d\chi(t)|<\infty$ and, for $0\leqslant t\leqslant1$, $y\geqslant0$,

$$\sum_{\nu=1}^{\infty}\frac{|s_\nu|}{\nu}\left(\frac{yt}{1+yt}\right)^\nu\leqslant\sum_{\nu=1}^{\infty}\frac{|s_\nu|}{\nu}\left(\frac{y}{1+y}\right)^\nu<\infty.$$

The lemma follows.

LEMMA 4. *If $f(t)$ is a continuous function for $t \geqslant 0$ which tends to a finite limit l as $t \to \infty$, and if χ satisfies* (3), *then, as $y \to \infty$,*

$$F(y) = \frac{1}{\log(1+y)} \int_0^1 f(yt) \log(1+yt)\, d\chi(t) \to l.$$

Proof. Suppose first that $f(t) \to 0$ as $t \to \infty$, and let $m(x) = \sup_{t \geqslant x} |f(t)|$; so that $m(0)$ is finite and $m(x) \to 0$ as $x \to \infty$. Then, for $y > \alpha > 0$,

$$|F(y)| \leqslant \int_0^1 |f(yt)|\,|d\chi(t)| \leqslant \int_0^{\alpha/y} |f(yt)|\,|d\chi(t)| + \int_{\alpha/y}^1 |f(yt)|\,|d\chi(t)|$$

$$\leqslant m(0) \int_0^{\alpha/y} |d\chi(t)| + m(\alpha) \int_0^1 |d\chi(t)|.$$

Since $\int_0^1 |d\chi(t)| < \infty$ and, in virtue of (3), $\int_0^{\alpha/y} |d\chi(t)| \to 0$ as $y \to \infty$, it follows that $F(y) \to 0$ as $y \to \infty$.

To complete the lemma it remains only to prove that, as $y \to \infty$,

$$\frac{1}{\log(1+y)} \int_0^1 \log(1+yt)\, d\chi(t) \to 1.$$

For $0 < \epsilon < 1$, we have

$$\varlimsup_{y \to \infty} \left| \int_0^1 \left\{1 - \frac{\log(1+yt)}{\log(1+y)}\right\} d\chi(t) \right| \leqslant \int_0^\epsilon |d\chi(t)| + \varlimsup_{y \to \infty} \left\{1 - \frac{\log(1+y\epsilon)}{\log(1+y)}\right\} \int_\epsilon^1 |d\chi(t)|$$

$$= \int_0^\epsilon |d\chi(t)|.$$

The required result follows, since, by (3), $\int_0^\epsilon |d\chi(t)| \to 0$ as $\epsilon \to 0$ in (0, 1) and $\int_0^1 d\chi(t) = 1$.

Proof of Theorem 5. Suppose that $s_n \to s(L)$. Then by Theorem 1, $s_{n+1} \to s(L)$ so that, in the notation of Lemma 3, $\{\log(1+t)\}^{-1} s(t) \to s$ as $t \to \infty$.

Recalling that, in virtue of the hypothesis of the theorem, χ satisfies (3), and appealing to Lemma 3 and Lemma 4, with

$$f(t) = \{\log(1+t)\}^{-1} s(t) - s_0 \quad (t > 0), \quad f(0) = s_1 - s_0,$$

we can now prove that $h_{n+1} \to s(L)$ and hence, by Theorem 1, that $h_n \to s(L)$.

Consequently, $LH_\chi \supseteq L$.

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Printed by C. F. Hodgson & Son Ltd., Pakenham Street, London, W.C.1.