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MATRICES THAT COMMUTE WITH CERTAIN HAUSDORFF MATRICES

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Dedicated to the memory of Alexander Peyerimhoff

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ABSTRACT. We prove that if an infinite matrix A satisfies $AC = CA$ where C is either a Cesàro matrix C_α or a Hölder matrix H_α each of order $\alpha = 1, 2, 3$, or 4, then the matrix A is triangular, and hence is a Hausdorff matrix. We prove also that corresponding to each real $\alpha > 4$ there exists a non-triangular matrix A with rows in ℓ_1 that commutes with H_α, and that corresponding to each integer $\alpha > 4$ there exists a non-triangular matrix A with rows in ℓ_1 that commutes with C_α. In addition, we prove results concerning infinite matrices that commute with H_α when α is a fraction of a certain kind or with the Euler matrix (E, q).

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1. Introduction and basic definitions.

Given an infinite matrix (a_{nk}) we assume implicitly that $n, k = 0, 1, \dots$, and that $a_{jk} = 0$ whenever $j < 0$. The matrix is called *triangular* if $a_{nk} = 0$ whenever $k > n$, and it is said to be *normal* if it is triangular and $a_{nn} \neq 0$ for $n = 0, 1, \dots$. Given a function $\mu(k)$ defined for $k = 0, 1, \dots$, the *Hausdorff* matrix generated by the sequence $(\mu(k))$ is the triangular matrix (a_{nk}) with $a_{nk} := \binom{n}{k}\Delta^{n-k}\mu(k)$ where $\Delta\mu(k) := \mu(k) - \mu(k+1)$. The *Cesàro* matrix C_α, $\alpha > -1$,

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is the Hausdorff matrix generated by the sequence $(\mu(k))$ with $\mu(k) := 1/\binom{k+\alpha}{k}$. The *Hölder* matrix H_α, $\alpha > -1$, is the Hausdorff matrix generated by the sequence $(\mu(k))$ with $\mu(k) := 1/(k+1)^\alpha$.

The matrix $C \equiv C_\alpha$, $\alpha > -1$. In this case $C = (c_{nk})$ is a triangular matrix given by $c_{nk} = \binom{n-k+\alpha-1}{n-k}/\binom{n+\alpha}{n}$. In particular we have $c_{n,0} = \alpha/(n+\alpha)$. The inverse $C' = (c'_{nm})$ of C is also a triangular matrix given by $c'_{nk} = \binom{n-k-\alpha-1}{n-k}\binom{n+\alpha}{n}$. Note that C' is row finite and column finite when α is a positive integer.

The matrix $C \equiv H_\alpha$, $\alpha > -1$. In this case $C = (c_{nk})$ is a again triangular matrix given by $c_{nk} = \binom{n}{k}\Delta^{n-k}\mu(k)$ with $\mu(k) := (k+1)^{-\alpha}$. In particular we have, for $\alpha > 0$, $c_{n,0} = \Delta^n\mu(0) = \int_0^1 (1-t)^n \left(\log\frac{1}{t}\right)^{\alpha-1} dt$. The inverse $C' = (c'_{nk})$ of C is also a triangular matrix given by $c'_{nk} = \binom{n}{k}\Delta^{n-k}\mu(k)$ with $\mu(k) := (k+1)^\alpha$. Note that C' is row finite and column finite when α is a positive integer, since then $\Delta^r\mu(k) = 0$ for $r = \alpha+1, \alpha+2, \ldots$.

It is familiar (see Hardy [1, Theorem 198]) that if a triangular matrix T commutes with a Hausdorff matrix generated by a sequence $(\mu(n))$ with all terms different, then T must be a Hausdorff matrix. The late Professor Brian Kuttner raised the following question: If an infinite matrix A commutes with a given Hausdorff matrix is A necessarily triangular, and hence a Hausdorff matrix? Rhoades [3] considered this question in the case that the given Hausdorff matrix is either a Cesàro matrix of integer order or a Hölder matrix of an integer order. In the next section we state four theorems which show, *inter alia*, that a very restrictive condition imposed by Rhoades [3] on the matrix A can be removed. In Section 8 we consider the case when A commutes with certain Hölder matrices of fractional order, and in Section 9 we prove a result (Theorem 10) concerning A commuting with the Euler matrix (E,q). Rhoades' Theorem 3 in [3] is similar to our Theorem 10 but includes the above mentioned restrictive condition.

2. Infinite matrices that commute with Cesàro or Hölder matrices.

THEOREM 1. *Suppose* $\alpha \in \{1,2,3,4\}$ *and* $A = (a_{nk})$ *is an infinite matrix such that the product* AC_α *exists and* $AC_\alpha = C_\alpha A$. *Then* A *is triangular (and hence is a Hausdorff matrix).*

Though the case $\alpha = 1$ of Theorem 1 was proved implicitly in Jakimovski [2], we give another proof of this case in Section 5.

THEOREM 2. *Suppose* $\alpha \in \{1,2,3,4\}$, *and* $A = (a_{nk})$ *is an infinite matrix such that the product* AH_α *exists and* $AH_\alpha = H_\alpha A$. *Then* A *is triangular (and hence it is a Hausdorff matrix).*

Both Theorems 1 and 2 with the additional assumption that the matrix A satisfies the bounded norm condition $\sup_n \sum_{k=0}^\infty |a_{nk}| < \infty$ were proved in Rhoades [3].

We prove Theorems 1 and 2 in Section 5.

THEOREM 3. *Corresponding to each real* $\alpha > 4$ *there exists a non-triangular matrix* A *with every row in* ℓ_1 *that satisfies* $AH_\alpha = H_\alpha A$.

Theorem 3 with the additional assumption that α is an integer was proved in Rhoades [3].

THEOREM 4. *Corresponding to each integer* $\alpha > 4$ *there exists a non-triangular matrix* A *with every row in* ℓ_1 *that satisfies* $AC_\alpha = C_\alpha A$.

Rhoades [3] dealt with the case $\alpha = 5$ of Theorem 4 using a very complicated method.

We prove Theorems 3 and 4 in Section 7.

3. Some results about commutative infinite matrices.

We prove two general theorems in this section. The first theorem concerns the associativity of certain products, and the second is the key result used in the proofs of Section 5.

THEOREM 5. *Assume that* $A = (a_{nm})$ *is an infinite matrix,* $T = (t_{mk})$ *is a triangular matrix, and* $B = (b_{kr})$ *is a row finite and column finite matrix. Assume also that the product* AT *exists. Then* $(AT)B = A(TB)$ *and* $B(AT) = (BA)T$.

Proof. (i) Since in the following sums, for a given r, only a finitely many of the numbers b_{kr} differ from zero, we get

$$((AT)B)_{nr} = \sum_{k=0}^\infty \Big(\sum_{m=0}^\infty a_{nm}t_{mk}\Big)b_{kr} = \sum_{m=0}^\infty a_{nm} \sum_{k=0}^\infty t_{mk}b_{kr} = (A(TB))_{nr}.$$

Hence $(AT)B = A(TB)$.
(ii) The above argument also yields $(B(AT))_{nr} = \sum_{m=0}^\infty b_{nm} \sum_{k=0}^\infty a_{mk}t_{kr}$ $= ((BA)T)_{nr}$. □

THEOREM 6. *Assume that* $A = (a_{nm})$ *is an infinite matrix, that* $T = (t_{mk})$ *is a normal matrix, and that its inverse matrix* $T' = (t'_{mk})$ *is row finite and column finite. Suppose that the product matrix* AT *exists and* $AT = TA$. *Then* $AT' = T'A$.

Proof. From $AT = TA$ we get $T'(AT) = T'(TA)$. Applying Theorem 5 with $B := T'$ we get $(T'A)T = (T'T)A = A$. Multiplying both sides of the above equality on the right by T' and applying Theorem 5 again we get $T'A = AT'$. □

4. Infinite matrices that commute with certain Hausdorff matrices.

THEOREM 7. *Suppose $A = (a_{nm})$ is an infinite matrix and α is a positive integer. If C is a normal Hausdorff matrix whose inverse $C' = (c'_{nk})$ is given by $c'_{nk} = \binom{n}{k}\Delta^{n-k}\mu(k)$ with $\mu(x)$ a polynomial of degree α in x. Then $(AC')_{nk} = \sum_{r=0}^{\alpha} a_{n,k+r}c'_{r+k,k}$, and $(C'A)_{nk} = \sum_{r=n-\alpha}^{n} c'_{nr}a_{rk}$.*

Proof. We have $(AC')_{nk} = \sum_{r=0}^{\infty} a_{nr}c'_{rk} = \sum_{r=k}^{k+\alpha} a_{nr}c'_{rk} = \sum_{r=0}^{\alpha} a_{n,k+r}c'_{r+k,k}$. This proves the first conclusion of the the theorem. To prove the second conclusion we note that $(C'A)_{nk} = \sum_{r=0}^{\infty} c'_{nr}a_{rk} = \sum_{r=n-\alpha}^{n} c'_{nr}a_{rk}$. □

THEOREM 8. *Suppose α is positive integer and $C = (c_{nk})$ is a normal Hausdorff matrix satisfying*

(1) $\quad (n+1)c_{n,0}$ *is ultimately monotonic and* $\lim_{n\to\infty}(n+1)c_{n,0} \neq 0.$

Suppose its inverse $C' = (c'_{nk})$ is given by

(2) $\quad c'_{nk} = \binom{n}{k}\Delta^{n-k}\mu(k)$ *with $\mu(x)$ a real polynomial of degree α in x.*

Suppose $A = (a_{nk})$ is an infinite matrix such that the product AC exists and $AC = CA$. Suppose that there is an integer $n \geq 0$ such that

(3) $\quad a_{mk} = 0 \quad$ *for* $k = m+1, m+2, \ldots \quad$ *and* $m = -1, 0, 1, \ldots, n-1.$

Write $b_k := a_{nk}$ and $f(z) := \sum_{k=n}^{\infty} b_k z^k$. Then $f(z)$ is finite for $|z| < 1$ and

$$\lim_{t\to 1-}\int_0^t f(u)\,du = \int_0^{1-} f(u)\,du = L, \tag{4}$$

where L is finite.
Further, if $x := e^t$, $\theta \equiv d/dt$, $y = f(1-x)$, then, for $0 < x < 2$, y satisfies the linear differential equation of order α with constant coefficients

$$F(\theta)y := (\mu(\theta) - \mu(n))\,y = R(1-e^t), \tag{5}$$

where $R(z)$ is a polynomial of degree at most $n-1$. If the associated auxiliary equation $F(r) = 0$ has simple roots $r_1, r_2, \ldots, r_\alpha \notin \{0, 1, \ldots, n-1\}$, then

$$f(x) = c_1(1-x)^{r_1} + c_2(1-x)^{r_2} + \cdots + c_\alpha(1-x)^{r_\alpha} + P(1-x), \tag{6}$$

where $c_1, c_2, \ldots, c_\alpha$ are constants and $P(z)$ is a polynomial of degree at most $n-1$.

Proof. Observe that, in view of (1), $\frac{1}{(n+1)c_{n,0}}$ is ultimately monotonic and bounded, and that $(AC)_{n,0} = \sum_{k=0}^{\infty} a_{nk}c_{k,0} = \sum_{k=0}^{\infty} b_k c_{k,0}$ with the series converging by hypothesis. It follows, by Abel's test, that $\sum_{k=0}^{\infty}\frac{b_k}{k+1}$ is convergent, and hence that $f(u)$ is finite for $|u| < 1$. By Abel's theorem we have that $\lim_{t\to 1-}\int_0^t f(u)\,du = \int_0^{1-} f(u)\,du = \sum_{k=n}^{\infty}\frac{b_k}{k+1}$. This establishes (4). Since the matrix C' is row finite and column finite, we have, by Theorem 5, that $AC' = C'A$. Hence, by Theorem 6 and condition (2), we have, for $k = n, n+1, \ldots$, that $\sum_{r=0}^{\alpha} a_{n,k+r}c'_{k+r,k} = \sum_{r=n-\alpha}^{n} c'_{nr}a_{rk} = c'_{nn}a_{nk} = \mu(n)a_{nk}$. In other words we have that

$$\sum_{r=0}^{\alpha} c'_{k+r,k}b_{k+r} = \mu(n)b_k \quad \text{for } k = n, n+1, \ldots. \tag{7}$$

Suppose now that $0 < x < 2$ so that, for $x = e^t$, $t < \log 2$. It follows now from (7) that

$$\begin{aligned}\mu(n)f(1-e^t) &= \sum_{k=n}^{\infty}(1-e^t)^k\sum_{r=0}^{\alpha} c'_{k+r,k}b_{k+r}\\ &= \sum_{k=0}^{\infty}(1-e^t)^k\sum_{r=0}^{\alpha} c'_{k+r,k}b_{k+r} - p(1-e^t),\end{aligned} \tag{8}$$

where $p(z)$ is a polynomial of degree at most $n-1$. Next, we observe that by applying the relation $\mu(\theta)e^{jt} = \mu(j)e^{jt}$, Lemma 2, (2), and absolute convergence we get

$$\begin{aligned}\mu(\theta)\sum_{k=0}^{\infty} b_k(1-e^t)^k &= \sum_{k=0}^{\infty} b_k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\mu(j)e^{jt}\\ &= \sum_{k=0}^{\infty} b_k\sum_{r=0}^{k}(1-e^t)^{k-r}\binom{k}{r}\Delta^r\mu(k-r)\\ &= \sum_{r=0}^{\alpha}\sum_{k=r}^{\infty} b_k(1-e^t)^{k-r}\binom{k}{r}\Delta^r\mu(k-r)\\ &= \sum_{r=0}^{\alpha}\sum_{k=0}^{\infty} b_{k+r}(1-e^t)^k\binom{k+r}{r}\Delta^r\mu(k)\\ &= \sum_{r=0}^{\alpha}\sum_{k=0}^{\infty}(1-e^t)^k c'_{k+r,k}b_{k+r}.\end{aligned}$$

Whence, by (8),

$$\mu(n)f(1-e^t) + p(1-e^t) = \mu(\theta)\sum_{k=0}^{\infty} b_k(1-e^t)^k = \mu(\theta)f(1-e^t) + q(1-e^t),$$

where $q(z)$ is a polynomial of degree at most $n-1$. Thus y satisfies the differential equation (5) a particular integral of which is

$$\frac{1}{F(\theta)}R(1-e^t) =: \frac{1}{F(\theta)}\sum_{k=0}^{n-1} a_k e^{kt} = \sum_{k=0}^{n-1}\frac{a_k e^{kt}}{F(k)} = \sum_{k=0}^{n-1}\frac{a_k x^k}{F(k)} =: P(x),$$

where $P(x)$ is a polynomial of degree at most $n-1$. It follows that the general solution of (5) is given by

$$\begin{aligned} y &= c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_\alpha e^{r_\alpha t} + P(e^t) \\ &= c_1 x^{r_1} + c_2 x^{r_2} + \cdots + c_\alpha x^{r_\alpha} + P(x) = f(1-x), \end{aligned}$$

so that $f(x) = c_1(1-x)^{r_1} + c_2(1-x)^{r_2} + \cdots + c_\alpha(1-x)^{r_\alpha} + P(1-x)$. □

5. Proofs of Theorems 1 and 2.

We shall use the following two useful lemmas.

LEMMA 1. *Suppose that $a \le -1$, $b > 0$ and $c < a$.*
(i) *If c_1 and c_2 are fixed complex numbers and*

$$c_1\int_0^t (1-x)^{a+ib}\,dx + c_2\int_0^t (1-x)^{a-ib}\,dx \to L \text{ as } t \to 1-,$$

where L is finite, then $c_1 = c_2 = 0$.
(ii) *If c_0, c_1 and c_2 are fixed complex numbers and*

$$c_0\int_0^t (1-x)^c\,dx + c_1\int_0^t (1-x)^{a+ib}\,dx + c_2\int_0^t (1-x)^{a-ib}\,dx \to L \text{ as } t \to 1-,$$

where L is finite, then $c_0 = c_1 = c_2 = 0$.

Proof. (i) Let $0 < t < 1$, $v = 1-t$. Then $\int_0^t (1-x)^{a+ib}\,dx = K(1-v^{a+1+ib})$, where $K = 1/(1+a+ib)$. Hence

$$c_1\int_0^t (1-x)^{a+ib}\,dx + c_2\int_0^t (1-x)^{a-ib}\,dx = c_1 K + c_2\bar{K} - v^{a+1-ib}\left(Kc_1 v^{2ib} + \bar{K}c_2\right).$$

Suppose first that $a < -1$. Since in this case $|v^{a+1-ib}| \to \infty$ as $v \to 0+$, we must have that $Kc_1 v^{2ib} \to -\bar{K}c_2$ as $v \to 0+$, which is only possible if $c_1 = c_2 = 0$. Suppose now that $a = -1$. This case is more delicate. We have that

$$Kc_1 v^{ib} + \bar{K}c_2 v^{-ib} \to -L - (Kc_1 + \bar{K}c_2) \text{ as } v \to 0+.$$

If we take $v = e^{-\frac{k\pi}{b}}$ for $k = 1, 2, \ldots$, we obtain that

$$(-1)^k(Kc_1 + \bar{K}c_2) \to -L - (Kc_1 + \bar{K}c_2) \text{ as } k \to \infty$$

which implies that $Kc_1 + \bar{K}c_2 = 0$. Likewise, if we take $v = e^{-\frac{(2k+1)\pi}{2b}}$ for $k = 1, 2, \ldots$, we obtain that

$$(-1)^{k+1}i(Kc_1 - \bar{K}c_2) \to -L - (Kc_1 + \bar{K}c_2) \text{ as } k \to \infty$$

which implies that $Kc_1 - \bar{K}c_2 = 0$, and hence that $c_1 = c_2 = 0$.
(ii) From $(1-x)^{-c}\{c_0(1-x)^c + c_1(1-x)^{a+ib} + c_2(1-x)a^{a-ib}\} \to c_0$ as $x \to 1-$ we get $c_0 = 0$. Applying now part (i) of the lemma we get $c_0 = c_1 = c_2 = 0$. □

LEMMA 2. $\displaystyle\sum_{j=0}^{k}\binom{k}{j}\mu(j)x^j = \sum_{s=0}^{k}(1+x)^{k-s}\binom{k}{s}\Delta^s\mu(k-s).$

Proof. Denote by $\mathbf{E}$ the shift operator given by $\mathbf{E}\mu(k) = \mu(k+1)$. Then

$$\begin{aligned} \sum_{s=0}^{k}(1+x)^{k-s}\binom{k}{s}\Delta^s\mu(k-s) &= \sum_{s=0}^{k}\binom{k}{s}(1-\mathbf{E})^s((1+x)\mathbf{E})^{k-s}\mu(0) \\ &= (1+x\mathbf{E})^k\mu(0) = \sum_{j=0}^{k}\binom{k}{j}\mu(j)x^j. \end{aligned}$$ □

Proof of Theorem 1. Suppose that $C \equiv C_\alpha$ with $\alpha \in \{1,2,3,4\}$. We use the notation of Theorem 8. The matrix C satisfies condition (1) of Theorem 8 since $c_{n,0} = \frac{\alpha}{n+\alpha}$, and the inverse matrix C' satisfies condition (2) of Theorem 8 with $\mu(k) = \binom{k+\alpha}{k}$. We shall prove by induction that $a_{nk} = 0$ for $k > n = 0, 1, 2, \ldots$. Assume now the inductive hypothesis that for a fixed integer $n \ge 0$ condition (3) of Theorem 8 holds. The differential equation (5) now becomes

$$F(\theta)y =: \left(\binom{\theta+\alpha}{\alpha} - \binom{n+\alpha}{\alpha}\right)y = R(1-e^t),$$

where $R(x)$ is a polynomial of degree at most $n-1$. Recall that $y = f(x-1)$, $x = e^t$, $\theta \equiv d/dt$, $b_k = a_{nk}$ and $f(z) = \sum_{k=n}^{\infty} b_k z^k$. Observe that the associated auxiliary equation is

$$\text{(9)} \qquad \alpha! F(r) = \prod_{k=1}^{\alpha}(r+k) - \prod_{k=1}^{\alpha}(n+k) = 0,$$

and that $F(r) < 0$ for $0 \le r < n$.
We consider the cases $\alpha = 1, 2, 3$ and 4 separately.

The case $\alpha = 1$. In this case equation (9) reduces to $r - n = 0$ of which the root is $r = n$. Hence, by conclusion (6) of Theorem 8, we have, for $0 < x < 2$, that $f(x) = c_1(1-x)^n + P(1-x)$ where $P(x)$ is a polynomial of degree at most $n-1$. Clearly (4) holds. Thus, subject to the inductive hypothesis (3) we must have $b_k = a_{nk} = 0$ for $k > n$. But (3) is clearly satisfied for $n = 0$, and so the proof is complete by induction in this case.

The case $\alpha = 2$. In this case equation (9) reduces to $r^2 + 3r - n^2 - 3n = 0$ of which the roots are $r = n, -n-3$. Hence, by conclusion (6) of Theorem 8, we have, for $0 < x < 2$, that $f(x) = c_1(1-x)^n + c_2(1-x)^{-n-3} + P(1-x)$, where $P(z)$ is a polynomial of degree at most $n-1$. In order for (4) to hold we must have that $\int_0^{1-} f(x)\,dx$ exists as a finite number which is only possible if $c_2 = 0$. Thus, subject to the inductive hypothesis (3), we must have $b_k = a_{nk} = 0$ for $k > n$. But (3) is automatically satisfied for $n = 0$, and so the proof is complete by induction in this case.

The case $\alpha = 3$. Equation (9) now becomes $r^3 + 6r^2 + 11r - n^3 - 6n^2 - 11n = 0$ of which the roots are $r = n, -\frac{n}{2} - 3 \pm \frac{i}{2}\sqrt{3n^2+12n+8}$. It follows as in the case $\alpha = 2$ that, for $0 < x < 2$,

$$\begin{aligned} f(x) &= c_1(1-x)^n + c_2(1-x)^{-\frac{n}{2}-3+\frac{i}{2}\sqrt{3n^2+12n+8}} \\ &\quad + c_3(1-x)^{-\frac{n}{2}-3-\frac{i}{2}\sqrt{3n^2+12n+8}} + P(1-x), \end{aligned}$$

where $P(z)$ is a polynomial of degree at most $n-1$. In order for (4) to hold we must have that $\int_0^{1-} f(x)\,dx$ exists as a finite number which, by Lemma 1, is only possible if $c_2 = c_3 = 0$. The proof of this case can now be completed by induction as in the previous case.

The case $\alpha = 4$. In this final case (9) becomes $r^4 + 10r^3 + 35r^2 + 50r - n^4 - 10n^3 - 35n^2 - 50n = 0$ of which the roots are $r = n, -n-5, -\frac{5}{2} \pm \frac{i}{2}\sqrt{4n^2+20n+15}$. As before it follows that, for $0 < x < 2$,

$$\begin{aligned} f(x) &= c_1(1-x)^n + c_2(1-x)^{-n-5} + c_3(1-x)^{-\frac{5}{2}+\frac{i}{2}\sqrt{4n^2+20n+15}} \\ &\quad + c_4(1-x)^{-\frac{5}{2}-\frac{i}{2}\sqrt{4n^2+20n+15}} + P(1-x), \end{aligned}$$

where $P(z)$ is a polynomial of deree at most $n-1$. In order for (4) to hold we must have that $\int_0^{1-} f(x)\,dx$ exists as a finite number. Lemma 1 now shows that $c_2 = c_3 = c_4 = 0$, and the proof can be completed by induction as in the previous cases. □

Proof of Theorem 2. Since $H_1 = C_1$, the case $\alpha = 1$ is known (and a proof was given above). So suppose that $C \equiv H_\alpha$ with $\alpha \in \{2,3,4\}$. We use the notation of Theorem 8. Since α is a positive integer, we have that, for $0 \le t < 1$,

$\left(\log\frac{1}{1-t}\right)^{\alpha-1} = \left(\sum_{k=1}^\infty \frac{t^k}{k}\right)^{\alpha-1} = \sum_{k=1}^\infty c_k t^k$, where $c_k > 0$ for $k = 1,2,\ldots$. It follows that

$$(n+1)c_{n,0} = (n+1)\int_0^1 t^n \left(\log\frac{1}{1-t}\right)^{\alpha-1} dt = \sum_{k=1}^\infty c_k \frac{n+1}{n+k+1}$$

is positive and increasing, and hence that the matrix C satisfies condition (1) of Theorem 8. Further, the inverse matrix C' satisfies the condition (2) of Theorem 8 with $\mu(k) = (k+1)^\alpha$. We shall prove by induction that $a_{nk} = 0$ for $k > n = 0,1,2,\ldots$. Assume now the inductive hypothesis that, for a fixed integer $n \ge 0$, condition (3) of Theorem 8 holds. The differential equation (5) now becomes $F(\theta)y := ((\theta+1)^\alpha - (n+1)^\alpha)y = R(1-e^t)$, where $R(z)$ is a polynomial of degree at most $n-1$. Recall that $y = f(x-1)$, $x = e^t$, $\theta \equiv d/dt$, $b_k = a_{nk}$ and $f(z) = \sum_{k=n}^\infty b_k z^k$. Observe that the associated auxiliary equation is

$$F(r) = (r+1)^\alpha - (n+1)^\alpha = 0, \tag{10}$$

and that $F(r) < 0$ for $0 \le r < n$.

We consider the cases $\alpha = 2, 3$ and 4 separately.

The case $\alpha = 2$. In this case equation (10) reduces to $(r+1)^2 - (n+1)^2 = 0$ of which the roots are $r = n, -n-2$. Hence, by conclusion (6) of Theorem 8, we have, for $0 < x < 2$, that

$$f(x) = c_1(1-x)^n + c_2(1-x)^{-n-2} + P(1-x),$$

where $P(z)$ is a polynomial of degree at most $n-1$. In order for (4) to hold we must have that $\int_0^{1-} f(x)\,dx$ exists as a finite number which is only possible if $c_2 = 0$. Thus subject to the inductive hypothesis (3) we must have $b_k = a_{nk} = 0$ for $k > n$. But (3) is automatically satisfied for $n = 0$, and so the proof is complete by induction in this case.

The case $\alpha = 3$. Equation (10) now becomes $(r+1)^3 - (n+1)^3 = 0$ of which the roots are $r = n, -\frac{n+3}{2} \pm \frac{i\sqrt{3}(n+1)}{2}$. It follows as in the case $\alpha = 2$ that, for $0 < x < 2$,

$$\begin{aligned} f(x) &= c_1(1-x)^n + c_2(1-x)^{-\frac{n+3}{2}+\frac{i\sqrt{3}(n+1)}{2}} \\ &\quad + c_3(1-x)^{-\frac{n+3}{2}-\frac{i\sqrt{3}(n+1)}{2}} + P(1-x), \end{aligned}$$

where $P(z)$ is a polynomial of degree at most $n-1$. In order for (4) to hold we must have that $\int_0^{1-} f(x)\,dx$ exists as a finite number which, by Lemma 1, is only possible if $c_2 = c_3 = 0$. The proof of this case can now be completed by induction as in the previous case.

The case $\alpha = 4$. In this final case (10) becomes $(r+1)^4 - (n+1)^4 = 0$ of which the roots are $r = n, -n-2, -1 \pm i(n+1)$. As before it follows that, for $0 < x < 2$,

$$f(x) = c_1(1-x)^n + c_2(1-x)^{-n-2} + c_3(1-x)^{-1+i(n+1)} + c_4(1-x)^{-1-i(n+1)} + P(1-x),$$

where $P(z)$ is a polynomial of degree at most $n-1$. In order for (4) to hold we must have that $\int_0^{1-} f(x)\,dx$ exists as a finite number. Lemma 1 now shows that $c_2 = c_3 = c_4 = 0$, and the proof can be completed by induction as in the previous cases. □

Remarks. Note that the key formulae in the proofs of Theorems 1 and 2 are (9) and (10), respectively. Though these formulae are essentially the same as (10) and (15), respectively, in Rhoades' paper [3], we derived them by a method entirely different from Rhoades'. His proofs use the restrictive assumption that $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$ and cannot work without some such restriction. Our approach, based on Theorems 5, 6, 7, and 8, allows us to dispense with the restriction.

6. Matrices with rows in ℓ_1 that commute with Hausdorff matrices.

In the following $\gamma(t)$ will denote a function of bounded variation over $[0, 1]$.

For complex s with $Re\ s \geq 0$, let $\mu(s) := \int_0^1 t^s d\gamma(t)$, and denote by (H, γ) the Hausdorff matrix generated by the moment sequence $(\mu(k))$, i.e., $(H, \gamma) = (h_{nk})$ where

$$h_{nk} := \begin{cases} \binom{n}{k} \Delta^{n-k}\mu(k) = \int_0^1 \binom{n}{k} t^k (1-t)^{n-k} d\gamma(t) & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k > n. \end{cases}$$

We have

$$\sup_{n \geq 0} \sum_{k=0}^{n} |h_{nk}| \leq \int_0^1 |d\gamma(t)| < \infty.$$

LEMMA 3. *Let $A = (a_{nk})$ be an infinite matrix with each of its rows in ℓ_1, i.e., $\sum_{k=0}^{\infty} |a_{nk}| < \infty$. Let $g_n(t) := \sum_{k=0}^{\infty} a_{nk}(1-t)^k$ for $0 \leq t \leq 1$, $n = 0, 1, \ldots$, and let $H = (H, \gamma) = (h_{nk})$ be a Hausdorff matrix. Then $AH = HA$ if and only if*

$$\int_0^1 g_n(tu)\,d\gamma(u) = \mu(n) g_n(t) + \sum_{k=0}^{n-1} h_{nk} g_k(t) \quad \text{for } 0 \leq t \leq 1,\ n = 0, 1, \ldots.$$

Proof. Let $n, j = 0, 1, 2, \ldots$, and $0 \leq t \leq 1$. Suppose $AH = HA$ which means explicitly that

$$(11) \qquad (AH)_{nj} = \sum_{k=0}^{\infty} a_{nk} h_{kj} = \sum_{k=0}^{n} h_{nk} a_{kj} = (HA)_{nj},$$

the convergence of the infinite series being guaranteed because

$$\sum_{k=0}^{\infty} |a_{nk}||h_{kj}| \leq \int_0^{\infty} |d\gamma(t)| \sum_{k=0}^{\infty} |a_{nk}| < \infty.$$

Also

$$\sum_{k=0}^{\infty} |a_{nk}| \sum_{j=0}^{k} |h_{kj}|(1-t)^j \leq \int_0^1 \sum_{k=0}^{\infty} |a_{nk}|(1-tu)^k |d\gamma(u)| \leq \int_0^1 |d\gamma(u)| \sum_{k=0}^{\infty} |a_{nk}| < \infty.$$

This justifies all the following interchanges of orders of summation and integration. We have $\sum_{j=0}^{\infty}(1-t)^j \sum_{k=0}^{\infty} a_{nk}h_{kj} = \sum_{j=0}^{\infty}(1-t)^j \sum_{k=0}^{n} h_{nk}a_{kj}$, and so

$$\sum_{k=0}^{\infty} a_{nk} \sum_{j=0}^{k} h_{kj}(1-t)^j = \sum_{k=0}^{n} h_{nk} \sum_{j=0}^{\infty} a_{kj}(1-t)^j = \mu(n)g_n(t) + \sum_{k=0}^{n-1} h_{nk} g_k(t).$$

Now

$$\sum_{j=0}^{\infty} h_{kj}(1-t)^j = \int_0^1 \sum_{j=0}^{k} \binom{k}{j} u^j (1-t)^j (1-u)^{k-j} d\gamma(u) = \int_0^1 (1-ut)^k d\gamma(u).$$

Thus

$$\sum_{k=0}^{\infty} a_{nk} \sum_{j=0}^{k} h_{kj}(1-t)^j = \int_0^1 \sum_{k=0}^{\infty} a_{nk}(1-tu)^k d\gamma(u) = \int_0^1 g_n(tu)\,d\gamma(u).$$

Hence

$$(12) \qquad \int_0^1 g_n(tu)\,d\gamma(u) = \mu(n)g_n(t) + \sum_{k=0}^{n-1} h_{nk} g_k(t).$$

Since all the steps leading to (12) are reversible we get that (12) is equivalent to (11) when $(a_{nk})_{k \geq 0} \in \ell_1$ for $n = 0, 1, \ldots$. □

LEMMA 4. *Let $H = (H, \gamma) = (h_{nk})$ be a given Hausdorff matrix with $\gamma(t)$ increasing on $[0, 1]$ and $\gamma(1) - \gamma(0) = 1$. Suppose there is a complex or real sequence (s_r) satisfying $s_0 = 0$, $Re\ s_r > 0$ for $r \geq 1$, and $\mu(s_r) = \mu(r)$ for $r \geq 1$. Then there exists a triangular matrix (c_{nr}) such that $c_{nn} = 1$ for $n \geq 0$ and the function $g_n(t) := \sum_{r=0}^{n} c_{nr} t^{s_r}$ satisfies (12) for $n = 0, 1, \ldots$, and $0 \leq t \leq 1$.*

Remark. Of course for any given value of r the choice $s_r = r$ satisfies the assumption of Lemma 4 for that value.

Proof. We construct the matrix (c_{jr}) by induction. Suppose throughout the proof that $0 \le t \le 1$. We have $s_0 = 0$ and $\mu(0) := \int_0^1 d\gamma(t) = \gamma(1) - \gamma(0) = 1$. Choose $c_{00} = 1$. Then the function $g_0(t)$ satisfies (12) for $n = 0$. Assume now c_{jr} has been defined for $0 \le r \le j < n$ so that $g_j(t) = \sum_{r=0}^{j} c_{jr} t^{s_r}$ satisfies (12) for $0 \le j < n$. We will now construct a function $g_n(t)$ that also satisfies (12) and thereby extend the definition of c_{jr} to the range $0 \le r \le j \le n$. We have

$$\sum_{k=0}^{n-1} h_{nk} g_k(t) = \sum_{k=0}^{n-1} h_{nk} \sum_{r=0}^{k} c_{kr} t^{s_r} = \sum_{r=0}^{n-1} t^{s_r} \sum_{k=r}^{n-1} h_{nk} c_{kr} = \sum_{r=0}^{n-1} d_{nr} t^{s_r}$$

where $d_{nr} := \sum_{k=r}^{n-1} h_{nk} c_{kr}$ for $0 \le r < n$. For the function

$$h_n(t) := \sum_{r=0}^{n-1} \frac{d_{nr}}{\mu(r) - \mu(n)} t^{s_r}$$

we have

$$\begin{aligned}\int_0^1 h_n(tu) d\gamma(u) &= \sum_{r=0}^{n-1} \frac{d_{nr}}{\mu(r) - \mu(n)} t^{s_r} \int_0^1 u^{s_r} d\gamma(u) \\ &= \sum_{r=0}^{n-1} \frac{d_{nr}\mu(r)}{\mu(r) - \mu(n)} t^{s_r} = \mu(n) \sum_{r=0}^{n-1} \frac{d_{nr} t^{s_r}}{\mu(r) - \mu(n)} + \sum_{r=0}^{n-1} d_{nr} t^{s_r}.\end{aligned}$$

Therefore $\int_0^1 h_n(tu) d\gamma(u) = \mu(n) h_n(t) + \sum_{k=0}^{n-1} h_{nk} g_k(t)$. Now define $g_n(t) := t^{s_n} + h_n(t)$. Then

$$\begin{aligned}\int_0^1 g_n(tu)\, d\gamma(u) &= t^{s_n} \int_0^1 u^{s_n} d\gamma(u) + \int_0^1 h_n(tu) d\gamma(u) \\ &= \mu(s_n) t^{s_n} + \mu(n) h_n(t) + \sum_{k=0}^{n-1} h_{nk} g_k(t) \\ &= \mu(n) g_n(t) + \sum_{k=0}^{n-1} h_{nk} g_k(t).\end{aligned}$$

Hence $g_n(t)$ satisfies (12) and has the representation

$$g_n(t) := t^{s_n} + \sum_{r=0}^{n-1} \frac{d_{nr}}{\mu(r) - \mu(n)} t^{s_r} = \sum_{r=0}^{n} c_{nr} t^{s_r},$$

where $c_{nn} = 1$ and $c_{nr} = \frac{d_{nr}}{\mu(r) - \mu(n)}$ when $0 \le r < n$. The proof can now be completed by induction. □

LEMMA 5. *Let $H = (H, \gamma) = (h_{nk})$ be a given Hausdorff matrix with $\gamma(t)$ increasing on $[0,1]$ and $\gamma(1) - \gamma(0) = 1$. Suppose that to some positive integer m there corresponds some number s_m with the following properties:*
(i) s_m is not an integer;
(ii) $Re\ s_m > 0$ and
(iii) $\mu(s_m) = \mu(m)$.
Then there exists a non-triangular matrix A with every row in ℓ_1 that satisfies $AH = HA$.

Proof. For each non-negative integer r different from m choose $s_r := r$. Then, by Lemma 4, there exists of a triangular matrix $C := (c_{nr})$ such that the function $g_n(t) := \sum_{r=0}^{n} c_{nr} t^{s_r}$ satisfies (12) for $n = 0, 1, \ldots$, and $0 \le t \le 1$. We have, for $0 < t \le 1$,

$$t^{s_r} = (1 - (1-t))^{-1+s_r+1} = \sum_{k=0}^{\infty} \binom{k - s_r - 1}{k} (1-t)^k$$

Hence, for $0 < t \le 1$,

$$g_n(t) = \sum_{k=0}^{\infty} a_{nk} (1-t)^k, \tag{13}$$

where $a_{nk} := \sum_{r=0}^{n} c_{nr} \binom{k - s_r - 1}{k}$. Observe that $\binom{k-s-1}{k} \sim \frac{k^{-s-1}}{\Gamma(-s)}$ as $k \to \infty$ when $s \ne 0, 1, 2, \ldots$, and hence that $\sum_{k=0}^{\infty} \left|\binom{k-s-1}{k}\right| < \infty$ when $Re\ s > 0$. Since $Re\ s_r > 0$, it follows that $\sum_{k=0}^{\infty} |a_{nk}| < \infty$, and therefore, by Abel's theorem, that (13) in fact holds for $0 \le t \le 1$. Thus all the rows of the matrix $A := (a_{nk})$ are in ℓ_1 and, by Lemma 3, $AH = HA$. Further

$$\begin{aligned}a_{mk} &= \sum_{r=0}^{m-1} c_{mr} \binom{k - s_r - 1}{k} + c_{mm} \binom{k - s_m - 1}{k} \\ &= \sum_{r=0}^{m-1} c_{mr} \binom{k - r - 1}{k} + \binom{k - s_m - 1}{k}.\end{aligned}$$

Since $\binom{k-r-1}{k} = 0$ for $k \ge r+1$ and since s_m is not an integer it follows that, for $k \ge m$,

$$a_{mk} = \binom{k - s_m - 1}{k} = \left(1 - \frac{s_m + 1}{1}\right)\left(1 - \frac{s_m + 1}{2}\right) \cdots \left(1 - \frac{s_m + 1}{k}\right) \ne 0.$$

Hence the m'th row of the matrix A has an infinite number of elements different from zero, and so the matrix is not triangular. □

LEMMA 6. *To each integer $r \geq 5$ there corresponds some positive number M_r such that for each integer $m > M_r$ there is a complex number $s_r^{(m)}$ satisfying:*
(i) $s_r^{(m)}$ *is not a real number;*
(ii) $Re\ s_r^{(m)} > 3$
and
(iii) $1/\binom{s_r^{(m)}+r}{r} = 1/\binom{m+r}{r}$.

Proof. For a given integer $r \geq 5$, write $f(w) := w^r - 1$, $p(w) := (w + \frac{1}{m})(w + \frac{2}{m}) \dots (w + \frac{r}{m})$, and $g(w) := p(w) - p(1) - f(w)$. We have

$$g(w) = (\frac{1}{m} + \dots + \frac{r}{m})(w^{r-1} - 1) + (\dots + \frac{j}{m} \cdot \frac{n}{m} + \dots)(w^{r-2} - 1) + \dots .$$

Hence, for $|w| \leq 3$, $m \geq r$,

$$|g(w)| \leq (3^{r-1} + 1)\left(\binom{r}{1}\frac{r}{m} + \binom{r}{2}(\frac{r}{m})^2 + \dots + \binom{r}{r-1}(\frac{r}{m})^{r-1}\right)$$

$$\leq \frac{r}{m}(3^{r-1} + 1)2^r. \tag{14}$$

Denote by γ_j the circle $\{w : |w - w_j| = \epsilon\}$ where $\epsilon := \frac{1}{2}\min\left(\sin\frac{\pi}{r}, \cos\frac{2\pi}{r}\right)$ and $w_j := e^{i\frac{2\pi}{r}j}$. Then γ_j includes exactly one zero of $w^r - 1 = 0$ in its interior. Also

$$\delta := \min_{w \in \gamma_j, 0 \leq j < r} |f(w)| > 0.$$

From (14) we see that there exists a positive number M such that, for all $m > M$ and $j = 0, 1, \dots, r-1$, $\max_{w \in \gamma_j} |g(w)| < \delta$. Hence by Rouché's theorem, for each $m > M$, the functions $w^r - 1$ and $(w + \frac{1}{m}) \dots (w + \frac{r}{m}) - (1 + \frac{1}{m}) \dots (1 + \frac{r}{m})$ have the same number of zeros inside each of the circles γ_j, i.e., exactly one. Hence, for each $m > M$, the solutions $z_0^{(m)}, \dots, z_{r-1}^{(m)}$ of $(z+1)\dots(z+r) = (m+1)\dots(m+r)$ satisfy $\left|z_j^{(m)}/m - e^{i\frac{2\pi}{r}j}\right| < \epsilon$, or $|z_j^{(m)} - me^{i\frac{2\pi}{r}j}| < \epsilon m$ for $j = 0, \dots, r-1$. For $j = 1$ we get

$$s_r^{(m)} := z_1^{(m)} = m(e^{i\frac{2\pi}{r}} + \eta\epsilon e^{i\theta}) \quad \text{for some} \quad \eta \in [0, 1).$$

Now

$$Re\ z_1^{(m)} = m(\cos\frac{2\pi}{r} + \eta\epsilon\cos\theta) > m(\cos\frac{2\pi}{r} - \epsilon) \geq \frac{m}{2}\cos\frac{2\pi}{r} \to \infty \quad \text{as} \quad m \to \infty$$

and

$$Im\ z_1^{(m)} = m\big(\sin\frac{2\pi}{r} + \eta\epsilon\sin\theta)\big) > m(\sin\frac{2\pi}{r} - \epsilon)$$
$$\geq 2m\sin\frac{\pi}{r}\left(\cos\frac{\pi}{r} - \frac{1}{4}\right) > 2m\sin\frac{\pi}{r}\left(\cos\frac{\pi}{5} - \frac{1}{4}\right) > 0.$$

Hence $s_r^{(m)}$ is not a real number and so not a positive integer, and there is a positive number $M_r \geq M$ such that $Re\ s_r^{(m)} > 3$ whenever $m > M_r$. □

7. Proofs of Theorems 3 and 4.

Proof of Theorem 3. Recall that, for $\alpha > 0$, the Hölder matrix H_α is the Hausdorff matrix (H, γ) with $\gamma(x) := \frac{1}{\Gamma(\alpha)}\int_0^x \left(\log\frac{1}{t}\right)^{\alpha-1} dt$, and that its generating moment sequence $(\mu(n))$ is given by $\mu(n) := \int_0^1 t^n\, d\gamma(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 t^n \left(\log\frac{1}{t}\right)^{\alpha-1} dt = \frac{1}{(n+1)^\alpha}$. Observe that, for $\alpha > 0$ and complex z with $Re\ z > -1$, we have (by applying Cauchy's theorem) $\mu(z) = \int_0^1 t^z d\gamma(t) = \int_0^\infty u^{\alpha-1}e^{-(z+1)u}du = \frac{1}{(z+1)^\alpha}$. Hence, for $s := (m+1)e^{\frac{2\pi i}{\alpha}} - 1$, $\alpha > 4$, $m > \sec\frac{2\pi}{\alpha} - 1$, we have that s is not a real number and so not an integer, $Re s > 0$, and $\mu(s) = \frac{1}{(m+1)^\alpha e^{2\pi i}} = \mu(m)$. The desired result is now a consequence of Lemma 5. □

Proof of Theorem 4. Recall that, for $\alpha > 0$, the Cesàro matrix C_α is the Hausdorff matrix (H, γ) with $\gamma(x) := \int_0^x (1-t)^{\alpha-1}dt$, and that its generating moment sequence $(\mu(n))$ is given by $\mu(n) := \int_0^1 t^n\, d\gamma(t) = \alpha\int_0^1 t^n(1-t)^{\alpha-1}dt = \frac{1}{\binom{n+\alpha}{n}}$. Observe that, for $\alpha > 0$ and complex z with $Re\ z > -1$, we have $\mu(z) = \int_0^1 t^z d\gamma(t) = \alpha\int_0^1 t^z(1-t)^{\alpha-1}dt = \frac{1}{\binom{z+\alpha}{\alpha}}$. The desired result is now an immediate consequence of Lemmas 5 and 6. □

8. Certain Hölder matrices of fractional order.

Notation. Given an infinite matrix $A = (a_{nk})$ we write $|A| := (|a_{nk}|)$.

We require two additional lemmas. The proof of the first is quite straightforward.

LEMMA 7. *Assume A, B and C are three infinite matrices such that $|A||B||C|$ exists (the order of mulplication is immaterial). Then $(AB)C = A(BC)$.*

LEMMA 8. *Suppose that each row of a given infinite matrix A is in ℓ_1, and that H is a positive and conservative Hausdorff matrix such that $HA = AH$. Then, for each positive integer m, we have $H^mA = AH^m$.*

Proof. For each positive integer m the matrix H^m is a positive and conservative Hausdorff matrix. In particular H^m is triangular and each of its columns is bounded. The proof is by induction on m. Assume that for a positive integer m we have $H^mA = AH^m$. Of course this is true for $m = 1$ by the lemma's hypothesis. Since the products $H(H^m|A|)$, $H(|A|H^m)$ and $|A|(HH^m)$ all exist, it follows from the inductive hypothesis and Lemma 7 that

$$H^{m+1}A = (HH^m)A = H(H^mA) = H(AH^m)$$
$$= (HA)H^m = (AH)H^m = A(HH^m) = AH^{m+1}.$$

The proof can now be completed by induction. □

THEOREM 9. *Suppose that* $\alpha = \frac{p}{q}$ *where* $p \in \{1,2,3,4\}$ *and* $q \in \{1,2,3,\dots\}$, *and that* A *is an infinite matrix with every row in* ℓ_1 *such that* $AH_\alpha = H_\alpha A$, H_α *being the Hölder matrix of order* α. *Then* A *is triangular (and hence is a Hausdorff matrix).*

Proof. Apply Lemma 8 with $H := H_\alpha$ to obtain that $H^q A = AH^q$. But $H^q = H_p$, and, since $p \in \{1,2,3,4\}$, it follows from Theorem 2 that A is triangular. □

9. Matrices that commute with Euler matrices.

The *Euler* matrix (E,q), $q > 0$, is the triangular matrix (c_{nk}) given by

$$c_{nk} = \binom{n}{k} a^k (1-a)^{n-k} \quad \text{with } a := \frac{1}{1+q}.$$

It is a regular Hausdorff matrix generated by the moment sequence (a^n).

THEOREM 10. *Let* $C = (c_{nk})$ *be the Euler matrix* $(E,q), q > 0$. *Suppose* $A = (a_{nk})$ *is an infinite matrix such that* $AC = CA$. *Suppose, in addition, that*

$$(15) \qquad \sum_{k=n}^{\infty} a_{nk} k^n \quad \text{converges for } n = 0,1,\dots.$$

Then A *is triangular (and hence is a Hausdorff matrix).*

Proof. Note that condition (15) is equivalent to

$$(16) \qquad \sum_{k=n}^{\infty} a_{nk} \left(k(k-1)\cdots(k-n+1)\right) \quad \text{converges for } n = 0,1,\dots.$$

This is a consequence of Abel's test since, for fixed n, $k(k-1)\cdots(k-n+1)k^{-n}$tends monotonically to 1 as $k \to \infty$. Observe that the product CA automatically exists since C is triangular. Further

$$(AC)_{nk} = \sum_{r=0}^{\infty} a_{nr} c_{rk} = \frac{1}{k!}\left(\frac{a}{1-a}\right)^k \sum_{r=k}^{\infty} a_{nr} \left(r(r-1)\cdots(r-k+1)\right)(1-a)^r,$$

the final series being convergent by (16). Thus the product AC also exists.

We prove the conclusion of the theorem by induction. Suppose that there is a an integer $n \ge 0$ such that

$$(17) \qquad a_{mk} = 0 \quad \text{for } k = m+1, m+2, \dots \quad \text{and } m = -1, 0, 1, \dots, n-1.$$

Let

$$b_k := a_{nk} \quad \text{and} \quad f(z) := \sum_{k=n}^{\infty} b_k z^k.$$

Then, by (16) and Abel's theorem, $f(z)$ is holomorphic for $|z| < 1$ and

$$(18) \qquad \lim_{t \to 1-} f^{(n)}(t) = \sum_{k=n}^{\infty} b_k \left(k(k-1)\cdots(k-n+1)\right) = L,$$

where L is finite.

Next we have

$$(AC)_{nk} = \sum_{r=0}^{\infty} a_{nr} c_{rk} = \sum_{r=0}^{\infty} b_r c_{rk} \quad \text{and} \quad (CA)_{nk} = \sum_{r=0}^{n} c_{nr} a_{rk}.$$

In view of (17), it follows that, for $k = n, n+1, \dots$,

$$(19) \qquad \sum_{r=0}^{\infty} b_r c_{rk} = c_{nn} a_{nk} = a^n b_k.$$

Suppose now that $0 < x < 1$. It follows from (19) that

$$a^n f(x) = \sum_{k=n}^{\infty} x^k \sum_{r=0}^{\infty} b_r c_{rk} = \sum_{k=0}^{\infty} x^k \sum_{r=k}^{\infty} b_r c_{rk} - p(x),$$

where $p(x)$ is a polynomial of degree at most $n-1$. Consequently

$$\begin{aligned} a^n f(x) + p(x) &= \sum_{r=0}^{\infty} b_r \sum_{k=0}^{r} c_{rk} x^k = \sum_{r=0}^{\infty} b_r \sum_{k=0}^{r} \binom{r}{k} a^k (1-a)^{r-k} x^k \\ &= \sum_{r=0}^{\infty} b_r (1-a+ax)^r = f(1-a+ax) + q(x), \end{aligned}$$

where $q(x)$ is a polynomial of degree at most $n-1$. Differentiating n times with respect to x, we obtain $f^{(n)}(x) = f^{(n)}(1-a+ax)$, whence, for $u := 1-x$, $f^{(n)}(1-u) = f^{(n)}(1-au)$. Replacing u by au successively $j-1$ times in this equation yields $f^n(x) = f^{(n)}(1-u) = f^{(n)}(1-a^j u)$. Since $1-a^j \to 1-$ as $j \to \infty$, we get, by (18), that $f^{(n)}(x) = L$ for all $x \in (0,1)$, and consequently that $f(x)$ is a polynomial of degree at most n. Hence

$$(20) \qquad b_k = a_{nk} = 0 \text{ for } k > n.$$

Since (17) is automatically satisfied for $n = 0$, it follows by induction that (20) holds for $n = 0,1,\dots$. In other words, A is triangular. □

Remarks. The proof of Theorem 10 is similar to Rhoades' proof of his Theorem 3 in [3] but he has the condition $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$, which is neither necessary nor sufficient for condition (15) (above) to hold. Rhoades' proof of his Theorem 3 fails on the line 3 up from the end of that proof where he states that $g_{m0}^{(m)}(0) = \sum_{k=m}^{\infty} a_{mk}$. This is false on two scores. The first is that his condition is not enough to ensure the existence of the derivative at the origin. The second is more significant. Since, for $|1-u| < 1$, $g_{m0}(u) = \sum_{k=0}^{\infty} a_{mk}(1-u)^k$, what he requires is that

$$\begin{aligned} g_{m0}^{(m)}(u) &= (-1)^m \sum_{k=m}^{\infty} a_{mk} \left(k(k-1)\cdots(k-m+1)\right)(1-u)^{k-m} \\ &\to (-1)^m \sum_{k=m}^{\infty} a_{mk} \left(k(k-1)\cdots(k-m+1)\right) \text{ as } u \to 0+, \end{aligned}$$

which condition (16) (above) ensures.

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