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Two Tauberian theorems for Dirichlet series methods of summability

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Abstract. We extend known Tauberian results concerning the power series method of summability to results concerning the more general summability method $D_{\lambda,a}$ based on the Dirichlet series $\sum a_n e^{-\lambda_n x}$.

1. Introduction

Suppose throughout that $\{\lambda_n\}$ is an unbounded and strictly increasing sequence of positive numbers, that $\{a_n\}$ is a sequence of non-negative numbers, and that the Dirichlet series

$$a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}$$

has abscissa of convergence $\sigma < \infty$. Let $\{s_n\}$ be a sequence of complex numbers. The Dirichlet series method $D_{\lambda,a}$ is defined as follows:

$$s_n \to s\,(D_{\lambda,a}) \quad \text{if} \quad \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \text{ is convergent for } x > \sigma, \text{ and}$$

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$$\sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \to s$$

as $x \to \sigma+$ through real values. It is well known (and easy to verify) that if $a(x) \to \infty$ as $x \to \sigma+$ and if $a_n \neq 0$ for infinitely many n, then $D_{\lambda,a}$ is regular, i. e. $s_n \to s$ implies $s_n \to s\,(D_{\lambda,a})$. For $\lambda_n = n$ the method $D_{\lambda,a}$ reduces to the power series method J_a.

Our primary purpose is to prove two theorems (Theorems 1 and 2 below) concerning Tauberian conditions on $\{s_n\}$ under which $s_n \to s\,(D_{\lambda,a})$ implies $s_n \to s$. These theorems generalize Theorems 1 and 2 in [2] which deal with power series methods of summability. The main tools for their proofs are results on the asymptotic behaviour of the Dirichlet series and related integrals.

2. The first Tauberian theorem

In this section we prove the following Tauberian result.

Theorem 1. *Suppose that the real functions g and λ satisfy the following conditions:*

$$\text{(C)}\quad \begin{cases} g, \lambda \in C_2[x_0, \infty) \text{ for some } x_0 \in \mathbb{N}, \\ \lambda(x), \dfrac{x\lambda'(x)}{\lambda(x)} \text{ and } G(x) := \dfrac{\lambda^2(x)}{\lambda'(x)} \left(\dfrac{g'(x)}{\lambda'(x)}\right)' \\ \text{are positive and non-decreasing on } [x_0, \infty), \text{ while} \\ \dfrac{\lambda'(x)}{\lambda(x)} \text{ and } L(x) := \lambda'(x) \left(\dfrac{g'(x)}{\lambda'(x)}\right)' \text{ are non-increasing on } [x_0, \infty). \end{cases}$$

Moreover, assume that

$$\text{(1)}\qquad G(x) \to \infty \text{ and } L(x) \to 0 \text{ as } x \to \infty.$$

Let $a_n \sim e^{-g(n)}$ as $n \to \infty$, $\lambda_n = \lambda(n)$ for $n \geq x_0$, and let $\ell(x) := \frac{1}{\sqrt{L(x)}}$. Then

$$\text{(2)}\qquad \lim_{\epsilon \to 0+} \omega(\epsilon) = 0 \text{ with } \omega(\epsilon) := \limsup_{n\to\infty} \max_{n \leq m \leq n+\epsilon\ell(n)} |s_{m+1} - s_n|,$$

and $s_n \to s\,(D_{\lambda,a})$ imply that $s_n \to s$.

Corollary . *Condition (2) in Theorem 1 can be replaced by the (stronger) Tauberian condition $s_n - s_{n-1} = O\left(\sqrt{L(n)}\right)$.*

Proof. Suppose $|s_n - s_{n-1}| \leq M\sqrt{L(n)}$, where M is a positive constant. Then, for $\epsilon > 0$,

$$\max_{n \leq m \leq n+\epsilon\ell(n)} |s_{m+1} - s_n| \leq \sum_{n+1 \leq j \leq n+\epsilon\ell(n)} |s_j - s_{j-1}| \leq M \sum_{n+1 \leq j \leq n+\epsilon\ell(n)} \sqrt{L(j)}$$
$$\leq M\epsilon\ell(n)\sqrt{L(n)} = \epsilon M \to 0 \text{ as } \epsilon \to 0+,$$

and this implies (2). ∎

Proof of Theorem 1. Let $\tilde{a}(x) := \sum_{k=x_0}^{\infty} e^{-g(k)} e^{-\lambda(k)x}$. By Lemma 3 (in Section 4 below), the abscissa of convergence of this Dirichlet series is $\sigma := \lim_{x\to\infty} -\frac{g'(x)}{\lambda'(x)} < \infty$ and $\lim_{x\to\sigma+} \tilde{a}(x) = \infty$. Since $a_n \sim e^{-g(n)}$ and $\lambda_n = \lambda(n)$ for $n \geq x_0$, the series $a(x) := \sum_{k=1}^{\infty} a_k e^{-\lambda_k x}$ has the same abscissa of convergence and $\lim_{x\to\sigma+} a(x) = \infty$.

We introduce a complex parameter α, and consider the following expressions:

$$\tilde{a}_\alpha(x) := \sum_{k=x_0}^{\infty} e^{-\alpha g(k)} e^{-\lambda(k)x} = \sum_{k=x_0}^{\infty} e^{-g(k)} \mu_k(\alpha) e^{-\lambda(k)x} \text{ with } \mu_k(\alpha) := e^{(1-\alpha)g(k)};$$

$$\tau_n(\alpha) := -\alpha\frac{g'(n)}{\lambda'(n)}, \ \tau_n := \tau_n(1); \quad \tilde{\Delta}_n(\alpha) := \tilde{a}_\alpha(\tau_n(\alpha)) e^{\lambda(n)\tau_n(\alpha)}; \text{ and}$$

$$\text{(3)}\qquad f_n(\alpha) := \frac{1}{\tilde{a}_\alpha(\tau_n(\alpha))} \sum_{k=x_0}^{\infty} s_k e^{-\alpha g(k)} e^{-\lambda(k)\tau_n(\alpha)}.$$

Suppose now that the given sequence of complex numbers $\{s_n\}$ satisfies condition (2) and $s_n \to s\,(D_{\lambda,a})$. By Lemma 3 (below) with $\alpha g(x)$ in place of $g(x)$, we have that $\tilde{a}_\alpha$ has abscissa of convergence $\alpha\sigma$ when $\alpha > 0$, and, for such an α, $D_{\lambda,\tilde{a}_\alpha}$ is a regular method of summability. Moreover, by Theorem A (below),

$$\tilde{\Delta}_n := a(\tau_n) e^{\lambda(n)\tau_n} \sim \sqrt{2\pi}\ell(n) e^{-g(n)} \sim \sqrt{2\pi}\ell(n) a_n \text{ as } n \to \infty,$$

whence, for $\epsilon > 0$,

$$\frac{1}{\epsilon} \sum_{n \leq k \leq n+\epsilon\ell(n)} \frac{a_k}{\tilde{\Delta}_k} \sim \frac{1}{\epsilon\sqrt{2\pi}} \sum_{n \leq k \leq n+\epsilon\ell(n)} \frac{1}{\ell(k)} \to \frac{1}{\sqrt{2\pi}} \text{ as } n \to \infty.$$

This is because, for $n \le k \le n + \epsilon\ell(n)$,

$$\ell(n) \le \ell(k) \le \ell(n + \epsilon\ell(n)) \le \frac{n + \epsilon\ell(n)}{n}\ell(n) \sim \ell(n) \text{ as } n \to \infty,$$

since

$$\frac{\ell(x)}{x} = \frac{\lambda(x)}{x\lambda'(x)} \frac{1}{\sqrt{G(x)}} \searrow 0 \text{ as } x \to \infty,$$

by (C) and (1). We first establish the following basic inequalities:

$$(4) \qquad \begin{cases} \limsup\limits_{n\to\infty} |s_n - f_n(\alpha)| \le \omega(\epsilon) + \dfrac{\omega(\epsilon)}{\epsilon}\sqrt{\dfrac{2\pi}{\alpha}} \text{ for all } \epsilon > 0,\ \alpha > 0; \\ \limsup\limits_{n\to\infty} |s_n - \sigma(\tau_n)| \le \omega(1)(1 + \sqrt{2\pi}). \end{cases}$$

Note that the second inequality in (4) follows from the first by setting $\epsilon = \alpha = 1$ and with a_n instead of $e^{-g(n)}$ by using that $a_n \sim e^{-g(n)}$. By Theorem A (below) we have that

$$\phi := \lim_{n\to\infty} \frac{\tilde{\Delta}_n(\alpha)}{\tilde{\Delta}_n \mu_n(\alpha)} = \lim_{n\to\infty} \sqrt{\frac{2\pi}{\alpha}}\ell(n)e^{-\alpha g(n)} \frac{e^{g(n)}}{\sqrt{2\pi}\ell(n)} e^{(\alpha-1)g(n)} = \frac{1}{\sqrt{\alpha}}.$$

Moreover, if $\tilde{\ell}(n) := [\epsilon\ell(n)] \in \mathbb{N}_0$, $\tilde{a}_n := a_n$ or $e^{-g(n)}$, then

$$\lim_{n\to\infty} \frac{1}{\epsilon} \sum_{k=n+1}^{n+\tilde{\ell}(n)+1} \frac{\tilde{a}_{k-1}}{\tilde{\Delta}_{k-1}} = \frac{1}{\sqrt{2\pi}} = \frac{1}{c} \text{ for } c := \sqrt{2\pi} > 0.$$

To continue the proof of (4) we proceed now as in the proof of Lemma 1 in [4]. First we show that

$$(5) \qquad \limsup_{n\to\infty} \sup_{m>n} \frac{|s_m - s_n|}{1 + \frac{c}{\epsilon}\sum_{k=n+1}^{m} \frac{\tilde{a}_{k-1}}{\tilde{\Delta}_{k-1}}} \le \omega(\epsilon).$$

Let $0 < \delta < 1$, $\alpha > 0$. Then there exists a positive integer N such that

$$\tilde{\Delta}_n(\alpha) \le (\phi + \delta)\tilde{\Delta}_n\mu_n(\alpha), \qquad \max_{n \le m \le n+\tilde{\ell}(n)} |s_{m+1} - s_n| \le \omega(\epsilon) + \delta,$$

$$\text{and } c \sum_{k=n+1}^{n+\tilde{\ell}(n)+1} \frac{\tilde{a}_{k-1}}{\tilde{\Delta}_{k-1}} \ge (1 - \delta)\epsilon \text{ for all } n \ge N.$$

Now let $m \ge n \ge N$, and define integers $r, n_0, n_1, \ldots, n_r$ by setting

$$n_0 := n < n_1 := n_0 + \tilde{\ell}(n_0) + 1 < \ldots < n_{r-1} := n_{r-2} + \tilde{\ell}(n_{r-2}) + 1$$
$$< m \le n_r := n_{r-1} + \tilde{\ell}(n_{r-1}) + 1.$$

It follows that

$$(6) \qquad \begin{aligned} |s_m - s_n| &\le \sum_{\nu=1}^{r-1} |s_{n_\nu} - s_{n_{\nu-1}}| + |s_m - s_{n_{r-1}}| \\ &\le (\omega(\epsilon) + \delta)\left(1 + \frac{c}{(1-\delta)\epsilon} \sum_{k=n+1}^{m} \frac{\tilde{a}_{k-1}}{\tilde{\Delta}_{k-1}}\right), \end{aligned}$$

and (5) is an immediate conseqence. For $n \ge N$, we consider the following inequality:

$$|f_n(\alpha) - s_n| \le \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,$$

where

$$\Sigma_1 := \frac{1}{\tilde{a}_\alpha(\tau_n(\alpha))} \sum_{k=x_0}^{N-1} e^{-\alpha g(k) - \lambda(k)\tau_n(\alpha)} |s_k - s_N|,$$

$$\Sigma_2 := \frac{1}{\tilde{a}_\alpha(\tau_n(\alpha))} \sum_{k=x_0}^{N-1} e^{-\alpha g(k) - \lambda(k)\tau_n(\alpha)} |s_N - s_n|,$$

$$\Sigma_3 := \frac{1}{\tilde{a}_\alpha(\tau_n(\alpha))} \sum_{k=N}^{n-1} e^{-\alpha g(k) - \lambda(k)\tau_n(\alpha)} |s_k - s_n|, \text{ and}$$

$$\Sigma_4 := \frac{1}{\tilde{a}_\alpha(\tau_n(\alpha))} \sum_{k=n+1}^{\infty} e^{-\alpha g(k) - \lambda(k)\tau_n(\alpha)} |s_k - s_n|.$$

Since, by Lemma 3, $\tau_n(\alpha) \searrow \alpha\sigma$ and $\tilde{a}_\alpha(\tau_n(\alpha)) \to \infty$ as $n \to \infty$, it is immediate that $\Sigma_1 \to 0$ as $n \to \infty$ when $\sigma > -\infty$. If $\sigma = -\infty$, then $\tilde{a}_\alpha(\tau_n(\alpha)) \ge e^{-\alpha g(N)}e^{-\lambda(N)\tau_n(\alpha)}$, so that $\Sigma_1 \le Ce^{-(\lambda(N-1)-\lambda(N))\tau_n(\alpha)} \to 0$ as $n \to \infty$, since C is independent of n, $\lambda(N-1) < \lambda(N)$, and $\tau_n(\alpha) \to -\infty$ as $n \to \infty$. Thus in either case $\Sigma_1 \to 0$ as $n \to \infty$. Next, it follows from (6) that

$$\Sigma_2 + \Sigma_3 \le \frac{\omega(\epsilon) + \delta}{\tilde{a}_\alpha(\tau_n(\alpha))} \left\{ \sum_{k=x_0}^{n-1} e^{-\alpha g(k) - \lambda(k)\tau_n(\alpha)} \left(1 + \frac{c}{(1-\delta)\epsilon} \sum_{\nu=k+1}^{n} \frac{\tilde{a}_{\nu-1}}{\tilde{\Delta}_{\nu-1}}\right) \right\}$$

and

$$\Sigma_4 \le \frac{\omega(\epsilon) + \delta}{\tilde{a}_\alpha(\tau_n(\alpha))} \left\{ \sum_{k=n+1}^{\infty} e^{-\alpha g(k) - \lambda(k)\tau_n(\alpha)} \left(1 + \frac{c}{(1-\delta)\epsilon} \sum_{\nu=n+1}^{k} \frac{\tilde{a}_{\nu-1}}{\tilde{\Delta}_{\nu-1}}\right) \right\}.$$

Hence, for large enough n, we get

$$|f_n(\alpha) - s_n| \le \delta + (\omega(\epsilon) + \delta) + \frac{c(\omega(\epsilon)+\delta)}{\tilde{a}_\alpha(\tau_n(\alpha))}(\Sigma_5 + \Sigma_6), \text{ where}$$

$$\Sigma_5 := \sum_{k=x_0}^{n-1} e^{-\alpha g(k) - \lambda(k)\tau_n(\alpha)} \sum_{\nu=k+1}^{n} \frac{\tilde{a}_{\nu-1}}{\tilde{\Delta}_{\nu-1}}, \text{ and}$$

$$\Sigma_6 := \sum_{k=n+1}^{\infty} e^{-\alpha g(k) - \lambda(k)\tau_n(\alpha)} \sum_{\nu=n+1}^{k} \frac{\tilde{a}_{\nu-1}}{\tilde{\Delta}_{\nu-1}}.$$

Since $\tau_n(\alpha) \le \tau_\nu(\alpha)$ for $\nu \le n$, we have

$$\begin{aligned}
\Sigma_5 &= \sum_{\nu=x_0}^{n-1} \frac{\tilde{a}_\nu}{\tilde{\Delta}_\nu} \frac{\tilde{\Delta}_\nu(\alpha)}{\tilde{\Delta}_\nu(\alpha)} e^{\lambda(\nu)(\tau_\nu(\alpha) - \tau_n(\alpha))} \sum_{k=x_0}^{\nu} e^{-\alpha g(k) - \lambda(k)\tau_\nu(\alpha)} \times \\
&\qquad \times e^{(\lambda(\nu)-\lambda(k))(\tau_n(\alpha)-\tau_\nu(\alpha))} \\
&\le \sum_{\nu=x_0}^{N-1} \tilde{a}_\nu e^{-\lambda(\nu)\tau_n(\alpha)} \frac{\tilde{\Delta}_\nu(\alpha)}{\tilde{\Delta}_\nu} + \sum_{\nu=N}^{n-1} e^{-\alpha g(\nu) - \lambda(\nu)\tau_n(\alpha)} \frac{\tilde{\Delta}_\nu(\alpha)}{\mu_\nu(\alpha)\tilde{\Delta}_\nu}(1+\delta) \\
&\le \delta \tilde{a}(\tau_n(\alpha)) + (\phi + \delta) \sum_{\nu=N}^{n-1} e^{-\alpha g(\nu) - \lambda(\nu)\tau_n(\alpha)}(1+\delta),
\end{aligned}$$

and similarly, since $\tau_n(\alpha) \ge \tau_\nu(\alpha)$ for $\nu \ge n$, we have

$$\begin{aligned}
\Sigma_6 &= \sum_{\nu=n}^{\infty} \frac{\tilde{a}_\nu}{\tilde{\Delta}_\nu} \frac{\tilde{\Delta}_\nu(\alpha)}{\tilde{\Delta}_\nu(\alpha)} e^{\lambda(\nu)(\tau_\nu(\alpha) - \tau_n(\alpha))} \sum_{k=\nu+1}^{\infty} e^{-\alpha g(k) - \lambda(k)\tau_\nu(\alpha)} \times \\
&\qquad \times e^{(\lambda(\nu)-\lambda(k))(\tau_n(\alpha)-\tau_\nu(\alpha))} \\
&\le \sum_{\nu=n}^{\infty} e^{-\alpha g(\nu) - \lambda(\nu)\tau_n(\alpha)} \frac{\tilde{\Delta}_\nu(\alpha)}{\mu_\nu(\alpha)\tilde{\Delta}_\nu}(1+\delta) \\
&\le (\phi + \delta) \sum_{\nu=n}^{\infty} e^{-\alpha g(\nu) - \lambda(\nu)\tau_n(\alpha)}(1+\delta).
\end{aligned}$$

It follows from the above estimates that, for large n,

$$|f_n(\alpha) - s_n| \le 2\delta + \omega(\epsilon) + \frac{c(\omega(\epsilon)+\delta)}{\epsilon(1-\delta)}((\phi+\delta)(1+\delta) + \delta),$$

and (4) is an immediate consequence.

Since $s_n \to s\,(D_{\lambda,a})$ and $\tau_n \to \sigma+$, we have that $\sigma(\tau_n) \to s$. Hence, by (4), the sequence $\{s_n\}$ is bounded, and therefore, since $a_n \sim e^{-g(n)}$, we have that

$$\tilde{\sigma}(x) := \frac{1}{\tilde{a}_1(x)} \sum_{k=x_0}^{\infty} e^{-g(k)} s_k e^{-\lambda(k)x} \to s \text{ as } x \to \infty.$$

Suppose next that $0 < \alpha < 1$. Then, by Theorem A (below), we have that, as $n \to \infty$,

$$\begin{aligned}
\mu_n(\alpha) \sim \mu_n^*(\alpha) &:= \sqrt{\frac{1-\alpha}{2\pi}} \int_{x_0}^{\infty} \frac{1}{\ell(t)} e^{(1-\alpha)\left\{g(t) + (\lambda(n)-\lambda(t))\frac{g'(t)}{\lambda'(t)}\right\}}\,dt \\
&= \int_{u_0}^{(\alpha-1)\sigma} e^{u\lambda(n)}\,d\chi_\alpha(u), \quad u = (1-\alpha)\frac{g'(t)}{\lambda'(t)},\ u_0 = (1-\alpha)\frac{g'(x_0)}{\lambda'(x_0)},
\end{aligned}$$

where $\chi_\alpha(u)$ is non-decreasing and absolutely continuous in $[u_0, (\alpha-1)\sigma)$. Let

$$a^*(x) := \sum_{k=x_0}^{\infty} e^{-g(k)} \mu_k^*(\alpha) e^{-\lambda(k)x},$$

and note that the abscissa of convergence of this Dirichlet series is $\alpha\sigma$ and that $a^*(x) \to \infty$ as $x \to \alpha\sigma+$. Let δ be an arbitrary positive number. Since $\tilde{\sigma}(x) \to s$ as $x \to \sigma+$, there exists $x_1 > \sigma$ such that $|\tilde{\sigma}(x) - s| < \delta$ for $\sigma < x < x_1$. Hence, for some $C > 0$, $\alpha\sigma < x < \alpha x_1$, and

$$\sigma^*(x) := \frac{1}{a^*(x)} \sum_{k=x_0}^{\infty} e^{-g(k)} \mu_k^*(\alpha) s_k e^{-\lambda(k)x},$$

we have that

$$\begin{aligned}
|\sigma^*(x) - s| &= \left| \frac{1}{a^*(x)} \sum_{k=x_0}^{\infty} e^{-g(k)} \mu_k^*(\alpha)(s_k - s) e^{-\lambda(k)x} \right| \\
&\le \frac{1}{a^*(x)} \int_{u_0}^{(\alpha-1)\sigma} \left| \sum_{k=x_0}^{\infty} e^{-g(k)}(s_k - s) e^{-\lambda(k)(x-u)} \right| d\chi_\alpha(u) \\
&= \frac{1}{a^*(x)} \int_{u_0}^{(\alpha-1)\sigma} |\tilde{\sigma}(x-u) - s| \tilde{a}(x-u)\,d\chi_\alpha(u) \\
&\le \delta + \frac{1}{a^*(x)} \int_{u_0}^{x-x_1} |\tilde{\sigma}(x-u) - s| \tilde{a}(x-u)\,d\chi_\alpha(u) \\
&\le \delta + C\frac{\tilde{a}(x_1)}{a^*(x)} \to \delta \text{ as } x \to \alpha\sigma+,
\end{aligned}$$

the change of order of summation and integration implicit in the first inequality being justified because $s_n - s = O(1)$ implies that

$$\sum_{k=x_0}^{\infty} e^{-g(k)}\mu_k^*(\alpha)|s_k - s|e^{-\lambda(k)x} < \infty.$$

Hence, $\sigma^*(\tau_n(\alpha)) \to s$ for $0 < \alpha < 1$. Since $\mu_n^*(\alpha) \sim \mu_n(\alpha)$, $s_n = O(1)$, and $\tau_n(\alpha) \to \alpha\sigma+$ as $n \to \infty$, it follows that

(7) $$\lim_{n\to\infty} f_n(\alpha) = s \text{ for } 0 < \alpha < 1.$$

Finally, for $M := \sup_{n\ge 1}|s_n| < \infty$, and α complex with $\beta = \Re\alpha > 0$, we have, by Theorem A (below), that

$$|f_n(\alpha)| \le M\left|\frac{a_\beta(\tau_n(\beta))}{a_\alpha(\tau_n(\alpha))}\right| = M\sqrt{\left|\frac{\alpha}{\beta}\right|}\left|\frac{1+R_2(n;\alpha)+R_3(n;\alpha)}{1+R_2(n;\beta)+R_3(n;\beta)}\right|,$$

where, in view of assumption (1), $R_2(n;\alpha)$, $R_3(n;\alpha)$, $R_2(n;\beta)$, $R_3(n;\beta)$ tend uniformly to 0 on compact subsets of the half-plane $\Re\alpha > 0$ as $n \to \infty$. Hence, the functions in the sequence $\{f_n(\alpha)\}$ are holomorphic on compact subsets of some region $U \supset (0,\infty)$. Therefore, by (7), and Vitali's theorem (see [4, Theorem 5·2·1]),

$$\lim_{n\to\infty} f_n(\alpha) = s \text{ for all } \alpha > 0,$$

and it follows from (4) that

$$\limsup_{n\to\infty} |s_n - s| \le \omega(\epsilon) + \frac{\omega(\epsilon)}{\epsilon}\sqrt{\frac{2\pi}{\alpha}} \text{ for all } \alpha > 0,\ \epsilon > 0.$$

Letting $\alpha \to \infty$ and then $\epsilon \to 0+$, we deduce from this and assumption (2) that $s_n \to s$. ∎

Observe that in the preceding proof we also established the following result.

Proposition. *Suppose that the conditions* (C) *and* (1) *of Theorem* 1 *hold, and let* $a_n \sim e^{-g(n)}$ *as* $n \to \infty$, $\lambda_n = \lambda(n)$ *for* $n \ge x_0$. *Then* $s_n = O(1)$ *and* $s_n \to s(D_{\lambda,a})$ *imply that* $\lim_{n\to\infty} f_n(\alpha) = s$ *for all* $\alpha > 0$, *where* $f_n(\alpha)$ *is defined by* (3).

Remark. Theorem 1 remains valid if the condition $L(x) \to 0$ as $x \to \infty$ in (1) is replaced by $L(x) \ge \delta > 0$ on $[x_0,\infty)$, for then condition (2) reduces to $s_{n+1} - s_n \to 0$, and Theorem 2 (below) becomes applicable. Theorem 2, however, establishes a lot more than this in that it does not require $L(x)$ to be monotonic or $G(x)$ to be unbounded on $[x_0,\infty)$.

3. The second Tauberian theorem

In this section we prove the following result to supplement Theorem 1.

Theorem 2. *Suppose that the real functions* g *and* λ *satisfy the following conditions:*

(H) $$\begin{cases} g,\lambda \in C_2[x_0,\infty) \text{ for some } x_0 \in \mathbb{N}, \\ \lambda(x) \text{ and } \dfrac{x\lambda'(x)}{\lambda(x)} \text{ are positive and non-decreasing on } [x_0,\infty), \\ \dfrac{\lambda'(x)}{\lambda(x)} \text{ is non-increasing on } [x_0,\infty), \\ \text{and } L(x) := \lambda'(x)\left(\dfrac{g'(x)}{\lambda'(x)}\right)' \ge \delta > 0 \text{ on } [x_0,\infty). \end{cases}$$

Let $a_n \sim e^{-g(n)}$ *as* $n \to \infty$, $\lambda_n = \lambda(n)$ *and* $A_n := \min\left(e^{\alpha_n}, e^{\beta_n}\right)$ *for* $n > x_0$, *where*

$$\alpha_n := g(n) - g(n-1) - (\lambda(n) - \lambda(n-1))\frac{g'(n-1)}{\lambda'(n-1)}, \text{ and}$$
$$\beta_n := g(n-1) - g(n) + (\lambda(n) - \lambda(n-1))\frac{g'(n)}{\lambda'(n)}.$$

Then $s_n - s_{n-1} = O(A_n)$ *and* $s_n \to s\,(D_{\lambda,a})$ *imply that* $s_n \to s$.

We require two lemmas.

Lemma 1. *Suppose that* (H) *holds, that* $L(x) \ge \eta \ge \delta > 0$ *on* $[x_0,\infty)$, *and that*

$$I_n := \sum_{k=x_0}^{\infty} |s_k - s_n|e^{h(k,n)}, \text{ where } h(k,n) := g(n) - g(k) - (\lambda(n) - \lambda(k))\frac{g'(n)}{\lambda'(n)}.$$

Then, for $n > x_0$, $I_n \le C$ *if* $s_n - s_{n-1} = O(A_n)$, *and* $I_n \le Ce^{-\frac{1}{2}\gamma\eta}$ *if* $s_n = O(1)$, *where* C *is a positive constant depending on* δ *but not on* η, *and* $\gamma := \min\left(\frac{1}{3}, e^{-2\lambda'(x_0)/\lambda(x_0)}\right)$.

Proof. Observe that

$$\alpha_n = -h(n, n-1) \text{ and } \beta_n = -h(n-1, n). \tag{8}$$

Let $x_0 \le \zeta \le \xi \le \zeta + 2$. Then, by (H),

$$\frac{\lambda'(\zeta)}{\lambda'(\xi)} \le \frac{1}{\lambda(\xi)} \frac{\lambda(\zeta+2)}{\lambda'(\zeta+2)} \lambda'(\zeta) \le \frac{\lambda'(\zeta)}{\lambda(\zeta)} \frac{\lambda(\zeta+2)}{\lambda'(\zeta+2)} \le \frac{\zeta+2}{\zeta} \le 3, \text{ and}$$

$$\frac{\lambda'(\zeta)}{\lambda'(\xi)} \ge \lambda(\zeta) \frac{\lambda'(\xi)}{\lambda(\xi)} \frac{1}{\lambda'(\xi)} \ge \frac{\lambda(\zeta)}{\lambda(\zeta+2)} = \exp\left(-\int_\zeta^{\zeta+2} \frac{\lambda'(t)}{\lambda(t)}\, dt\right)$$
$$\ge \exp\left(-2\frac{\lambda'(x_0)}{\lambda(x_0)}\right).$$

We have thus shown, by the definition of γ, that

$$\frac{\lambda'(\zeta)}{\lambda'(\xi)} \ge \gamma > 0 \text{ whenever } \zeta, \xi \ge x_0 \text{ and } |\zeta - \xi| \le 2. \tag{9}$$

By C and $C(\delta)$ we denote constants which may depend on δ but not on η, and which may be different on different occasions. We have $I_n = \Sigma_1 + \Sigma_2$, where

$$\Sigma_1 := \sum_{k=x_0}^{n-1} |s_k - s_n| e^{h(k,n)}, \quad \Sigma_2 := \sum_{k=n+1}^{\infty} |s_k - s_n| e^{h(k,n)}.$$

Suppose first that $a_n = O(A_n)$. Then, for $x_0 \le k \le n$,

$$|s_k - s_n| \le \sum_{j=k+1}^{n} |s_j - s_{j-1}| \le C(n-k) \max_{k+1 \le j \le n} e^{-h(j-1,j)}, \text{ and}$$

$$h(k,n) + \max_{k+1 \le j \le n} (-h(j-1,j))$$
$$= -\int_k^n \int_u^n \frac{\lambda'(u)}{\lambda'(t)} L(t)\, dt\, du + \max_{k+1 \le j \le n} \int_{j-1}^j \int_u^j \frac{\lambda'(u)}{\lambda'(t)} L(t)\, dt\, du$$
$$\le -\sum_{j=k+1}^{n} \int_{j-1}^j \frac{L(t)}{\lambda'(t)} \left(\int_k^t \lambda'(u)\, du - \int_{j-1}^t \lambda'(u)\, du \right) dt$$
$$\le -\eta \sum_{j=k+1}^{n} \int_{j-1}^j \int_k^{j-1} \frac{\lambda'(u)}{\lambda'(t)}\, du\, dt$$
$$\le -\eta \sum_{j=k+2}^{n} \int_{j-1}^j \int_{j-2}^{j-1} \frac{\lambda'(u)}{\lambda'(t)}\, du\, dt \le -\delta\gamma(n-k-1), \text{ by (9)}.$$

Hence

$$\Sigma_1 \le C \sum_{k=x_0}^{n-1} (n-k) e^{-\delta\gamma(n-k-1)} \le C \sum_{\nu=0}^{\infty} (\nu+1) e^{-\delta\gamma\nu} = C(\delta).$$

Next, for $k > n$,

$$|s_k - s_n| \le \sum_{j=n+1}^{k} |s_j - s_{j-1}| \le C(k-n) \max_{n+1 \le j \le k} e^{-h(j,j-1)}, \text{ and}$$

$$h(k,n) + \max_{n+1 \le j \le k} (-h(j,j-1))$$
$$= -\int_n^k \int_n^u \frac{\lambda'(u)}{\lambda'(t)} L(t)\, dt\, du + \max_{n+1 \le j \le k} \int_{j-1}^j \int_j^u \frac{\lambda'(u)}{\lambda'(t)} L(t)\, dt\, du$$
$$\le -\sum_{j=n+1}^{k} \int_{j-1}^j \frac{L(t)}{\lambda'(t)} \left(\int_t^k \lambda'(u)\, du - \int_t^j \lambda'(u)\, du \right) dt$$
$$\le -\eta \sum_{j=n+1}^{k} \int_{j-1}^j \int_j^k \frac{\lambda'(u)}{\lambda'(t)}\, du\, dt$$
$$\le -\eta \sum_{j=n+1}^{k-1} \int_{j-1}^j \int_j^{j+1} \frac{\lambda'(u)}{\lambda'(t)}\, du\, dt \le -\delta\gamma(k-n-1), \text{ by (9)}.$$

Hence

$$\Sigma_2 \le C \sum_{k=n+1}^{\infty} (k-n) e^{-\delta\gamma(k-n-1)} = C \sum_{\nu=0}^{\infty} (\nu+1) e^{-\delta\gamma\nu} = C(\delta).$$

The first part of Lemma 1 follows from the above inequalities. To prove the second part suppose that $s_n = O(1)$. If $x_0 \le k < n$, then, as above,

$$h(k,n) = -\sum_{j=k+1}^{n} \int_{j-1}^j \int_k^t \frac{\lambda'(u)}{\lambda'(t)} L(t)\, du\, dt$$
$$\le -\eta \sum_{j=k+1}^{n} \int_{j-1}^j \int_{j-1}^t \frac{\lambda'(u)}{\lambda'(t)}\, du\, dt \le -\gamma\eta \frac{1}{2}(n-k).$$

That is

$$h(k,n) \le -\frac{1}{2}\gamma\eta(n-k) \text{ for } x_0 \le k < n, \tag{10}$$

and so

$$\Sigma_1 \le C \sum_{k=x_0}^{n-1} e^{-\frac{1}{2}\gamma\eta(n-k-1)-\frac{1}{2}\gamma\eta} \le Ce^{-\frac{1}{2}\gamma\eta}\sum_{\nu=0}^{\infty} e^{-\frac{1}{2}\gamma\delta\nu} = C(\delta)e^{-\frac{1}{2}\gamma\eta}.$$

Finally, if $k > n$, then

$$\begin{aligned} h(k,n) &= -\sum_{j=n+1}^{k} \int_{j-1}^{j}\int_t^k \frac{\lambda'(u)}{\lambda'(t)} L(t)\, du\, dt \\ &\le -\eta \sum_{j=n+1}^{k} \int_{j-1}^{j}\int_t^k \frac{\lambda'(u)}{\lambda'(t)}\, du\, dt \\ &\le -\eta \sum_{j=n+1}^{k} \int_{j-1}^{j}\int_t^j \frac{\lambda'(u)}{\lambda'(t)}\, du\, dt \le -\gamma\eta\frac{1}{2}(k-n). \end{aligned}$$

That is

(11) $$h(k,n) \le -\frac{1}{2}\gamma\eta(k-n) \text{ for } k > n,$$

and so

$$\Sigma_2 \le C \sum_{k=n+1}^{\infty} e^{-\frac{1}{2}\gamma\eta(k-n-1)-\frac{1}{2}\gamma\eta} \le Ce^{-\frac{1}{2}\gamma\eta}\sum_{\nu=0}^{\infty} e^{-\frac{1}{2}\gamma\delta\nu} = C(\delta)e^{-\frac{1}{2}\gamma\eta}.$$

This completes the proof of Lemma 1. ∎

Remark. Observe that $I_n \ge |s_n - s_{n-1}|e^{h(n-1,n)} = |s_n - s_{n-1}|e^{-\beta_n}$ and $I_{n-1} \ge |s_n - s_{n-1}|e^{h(n,n-1)} = |s_n - s_{n-1}|e^{-\alpha_n}$. This shows that the condition $s_n - s_{n-1} = O(A_n)$ of the first part of Lemma 1 cannot be weakened, and throws light on the form of the Tauberian condition in Theorem 2.

Lemma 2. *Assume* (H) *concerning only the function* λ. *Let* $\rho > 0$, *and define*

$$\chi(x) := \rho \int_{x_0}^{x}\int_{x_0}^{t} \frac{\lambda'(t)}{\lambda'(u)}\, du\, dt \text{ for } x \ge x_0.$$

Then

$$\int_{x_0}^{\infty} \exp\left(\chi(t) - \chi(x) + (\lambda(x) - \lambda(t))\frac{\chi'(t)}{\lambda'(t)}\right) dt \to C = C(\rho) > 0 \text{ as } x \to \infty.$$

Proof. Assume first that $\frac{\lambda'(x)}{\lambda(x)} \to 0$ as $x \to \infty$. Since

$$\frac{\chi'(x)}{\lambda'(x)} = \rho \int_{x_0}^{x} \frac{du}{\lambda'(u)},$$

we have that

$$\lambda'(x)\left(\frac{\chi'(x)}{\lambda'(x)}\right)' \equiv \rho \text{ and } \frac{\lambda^2(x)}{\lambda'(x)}\left(\frac{\chi'(x)}{\lambda'(x)}\right)' = \rho\left(\frac{\lambda(x)}{\lambda'(x)}\right)^2 \to \infty \text{ as } x \to \infty.$$

Hence, by Theorem A (in Section 4 below),

$$\int_{x_0}^{\infty} \exp\left(\chi(t) - \chi(x) + (\lambda(x) - \lambda(t))\frac{\chi'(t)}{\lambda'(t)}\right) dt \to C = \sqrt{\frac{2\pi}{\rho}} \text{ as } x \to \infty,$$

which is the desired result in this case.

Now suppose that $\frac{\lambda'(x)}{\lambda(x)} \searrow \delta > 0$ as $x \to \infty$. Then, for $x > x_0$,

$$\log\frac{\lambda(x)}{\lambda(x_0)} = \int_{x_0}^{x} \frac{\lambda'(t)}{\lambda(t)}\, dt \ge \delta(x - x_0), \text{ and so } \lambda(x) \ge \lambda(x_0)e^{\delta(x-x_0)}.$$

Moreover

$$\lambda'(x) \sim \delta\lambda(x) \ge \delta\lambda(x_0)e^{\delta(x-x_0)} \text{ as } x \to \infty.$$

Let

$$\begin{aligned} h(t,x) :&= \chi(t) - \chi(x) + (\lambda(x) - \lambda(t))\frac{\chi'(t)}{\lambda'(t)} \\ &= \rho \int_x^t \int_u^x \frac{\lambda'(v)}{\lambda'(u)}\, dv\, du \text{ for } x, t \ge x_0. \end{aligned}$$

Note that, for $x, t \ge x_0$,

$$\frac{\lambda'(t)}{\lambda'(x)} = \frac{\lambda'(t)}{\lambda(t)}\frac{\lambda(x)}{\lambda'(x)} \exp\left(\int_x^t \frac{\lambda'(v)}{\lambda(v)}\, dv\right) =: \gamma(t,x)e^{(\delta+\epsilon(t,x))(t-x)},$$

where $0 < \delta\frac{\lambda(x_0)}{\lambda'(x_0)} \le \gamma(t,x) \to 1$ and $0 \le \epsilon(t,x) \to 0$ as $\min(t,x) \to \infty$. Hence

$$\begin{aligned} h(t,x) &= -\rho \int_x^t \int_x^u \frac{\lambda'(v)}{\lambda'(u)}\, dv\, du = -\rho \int_x^t \int_x^u \gamma(v,u)e^{(\delta+\epsilon(v,u))(v-u)}\, dv\, du \\ &= -\rho \int_0^{t-x}\int_0^u \gamma e^{-(\delta+\epsilon)v}\, dv\, du, \end{aligned}$$

where $\gamma := \gamma(u + x - v, u + x) \to 1$ and $\epsilon := \epsilon(u + x - v, u + x) \to 0$ as $x \to \infty$.

It follows, using dominated convergence, that

$$\begin{aligned}\int_x^\infty e^{h(t,x)}\,dt &= \int_0^\infty e^{h(x+t,x)}\,dt\\ &= \int_0^\infty \exp\left(-\rho\int_0^t\int_0^u \gamma e^{-(\delta+\epsilon)v}\,dv\,du\right)dt\\ &\to \int_0^\infty \exp\left(-\rho\int_0^t\int_0^u e^{-\delta v}\,dv\,du\right)dt\\ &= \int_0^\infty \exp\left(-\frac{\rho t}{\delta}+\frac{\rho}{\delta^2}-\frac{\rho}{\delta^2}e^{-\delta t}\right)dt =: C_1 < \infty \text{ as } x\to\infty,\end{aligned}$$

and

$$\begin{aligned}\int_{x_0}^x e^{h(t,x)}\,dt &= \int_0^{x-x_0} e^{h(x-t,x)}\,dt\\ &= \int_0^{x-x_0} \exp\left(-\rho\int_0^{-t}\int_0^u \gamma e^{-(\delta+\epsilon)v}\,dv\,du\right)dt\\ &\to \int_0^\infty \exp\left(-\rho\int_0^{-t}\int_0^u e^{-\delta v}\,dv\,du\right)dt\\ &= \int_0^\infty \exp\left(\frac{\rho t}{\delta}+\frac{\rho}{\delta^2}-\frac{\rho}{\delta^2}e^{\delta t}\right)dt =: C_2 < \infty \text{ as } x\to\infty.\end{aligned}$$

Thus the assertion holds with $C = C_1 + C_2$ in this case. ∎

Proof of Theorem 2. Suppose that $s_n \to s\,(D_{\lambda,a})$ and $s_n - s_{n-1} = O(A_n)$. Let x_1 be a sufficiently large but fixed integer in $[x_0,\infty)$. Again C will denote possibly different positive constants. Since $x_n := -g'(n)/\lambda'(n) \to \sigma+$ and $a_n \sim e^{-g(n)}$, we have that

$$\limsup_{n\to\infty}|s_n - s| = \limsup_{n\to\infty}|s_n - \sigma(x_n)| \le T_1 + T_2,$$

where

$$T_1 := \limsup_{n\to\infty}\frac{1}{a(x_n)}\sum_{k=1}^{x_1} a_k|s_k - s_n|e^{-\lambda_k x_n},$$

and

$$T_2 := \limsup_{n\to\infty}\frac{1}{a(x_n)}\sum_{k=x_1}^{\infty} a_k|s_k - s_n|e^{-\lambda(k)x_n}.$$

Now

$$a(x_n) = \sum_{k=1}^\infty a_k e^{-\lambda_k x_n} \ge a_n e^{-\lambda(n)x_n}.$$

Hence, by Lemma 1,

$$T_2 \le C\limsup_{n\to\infty}\sum_{k=x_0}^\infty |s_k - s_n|e^{h(k,n)} \le C.$$

Also, by the proof of Lemma 1, we have that

$$|s_k - s_n|e^{h(k,n)} \le C(n-k)e^{-\delta\gamma(n-k-1)} \quad \text{for } n > k,$$

whence

$$\begin{aligned}T_1 &\le C\limsup_{n\to\infty}\sum_{k=1}^{x_1}|s_k - s_n|e^{h(k,n)}\\ &\le C\limsup_{n\to\infty}\sum_{k=1}^{x_1}(n-k)e^{-\delta\gamma(n-k-1)} = 0.\end{aligned}$$

Hence $T_1 = 0, T_2 \le C$, and, since $a_n \sim e^{-g(n)}$, we get that

$$(12)\qquad s_n = O(1);\ \text{and } s_n \to s\,(D_{\lambda,\tilde a}),\ \text{where } \tilde a(x) := \sum_{k=x_1}^\infty e^{-g(k)}e^{-\lambda(k)x}.$$

We introduce a complex parameter α, and consider the functions:

$$\begin{aligned}\chi(x) &:= \int_{x_0}^x\int_{x_0}^t \frac{\lambda'(t)}{\lambda'(u)}\,du\,dt, \quad \mu_k(\alpha) := e^{-\alpha\chi(k)};\\ \tilde a_\alpha(x) &:= \sum_{k=x_1}^\infty e^{-g(k)-\alpha\chi(k)}e^{-\lambda(k)x} = \sum_{k=x_1}^\infty e^{-g(k)}\mu_k(\alpha)e^{-\lambda(k)x};\\ z_n(\alpha) &:= -\frac{g'(n)}{\lambda'(n)} - \alpha\int_{x_0}^n \frac{dt}{\lambda'(t)};\\ f_n(\alpha) &:= \frac{1}{\tilde a_\alpha(z_n(\alpha))}\sum_{k=x_1}^\infty s_k e^{-g(k)-\alpha\chi(k)}e^{-\lambda(k)z_n(\alpha)};\ \text{and}\\ \tilde\sigma(x) &:= \frac{1}{\tilde a(x)}\sum_{k=x_1}^\infty s_k e^{-g(k)}e^{-\lambda(k)x}.\end{aligned}$$

Then, by (12), $f_n(0) = \tilde\sigma(z_n(0)) \to s$ as $n\to\infty$.

Suppose first that $\alpha < 0$. By Lemma 2, we have that

$$\begin{aligned}\mu_n(\alpha) \sim \mu_n^*(\alpha) &: = C(\alpha)\int_{x_0}^\infty \exp\left\{-\alpha\left(\chi(t) + (\lambda(n)-\lambda(t))\frac{\chi'(t)}{\lambda'(t)}\right)\right\}dt\\ &= \int_{u_0}^{\sigma^*} e^{u\lambda(n)}\,d\chi_\alpha(u), \quad u = -\alpha\frac{\chi'(t)}{\lambda'(t)},\ u_0 = -\alpha\frac{\chi'(x_0)}{\lambda'(x_0)},\end{aligned}$$

where $\chi_\alpha(u)$ is non-decreasing and absolutely continuous in $[u_0, \sigma^*)$, with

$$\sigma^* := \lim_{n\to\infty} -\alpha\frac{\chi'(n)}{\lambda'(n)} = -\alpha\int_{x_0}^{\infty}\frac{dt}{\lambda'(t)} \in (0,\infty].$$

Proceeding as in the proof of Theorem 1, we consider

$$\sigma^*(x) := \frac{1}{a^*(x)}\sum_{k=x_1}^{\infty} e^{-g(k)}\mu_k^*(\alpha)s_k e^{-\lambda(k)x},$$

where

$$a^*(x) := \sum_{k=x_1}^{\infty} e^{-g(k)}\mu_k^*(\alpha)e^{-\lambda(k)x},$$

and note that the abscissa of convergence of this Dirichlet series is

$$\begin{aligned}\sigma^{**} &= \lim_{x\to\infty}\{-\frac{g'(x)}{\lambda'(x)} - \alpha\frac{\chi'(x)}{\lambda'(x)}\} = -\frac{g'(x_0)}{\lambda'(x_0)} - \lim_{x\to\infty}\int_{x_0}^{x}\frac{L(t)+\alpha}{\lambda'(t)}\,dt\\ &\le -\frac{g'(x_0)}{\lambda'(x_0)} - \lim_{x\to\infty}\int_{x_0}^{x}\frac{\delta}{2\,\lambda'(t)}\,dt \le -\frac{g'(x_0)}{\lambda'(x_0)} < \infty\end{aligned}$$

for $\alpha > -\frac{\delta}{2}$, by (H). Moreover, $\sigma^{**} \ge \sigma$, $a^*(x) \to \infty$ as $x \to \sigma^{**}+$. Let $-\frac{\delta}{2} < \alpha < 0$, and let ϵ be an arbitrary positive number. Since $\tilde\sigma(x) \to s$ as $x \to \sigma+$, there exists $\tilde x_1 > \sigma$ such that $|\tilde\sigma(x) - s| < \epsilon$ for $\sigma < x < \tilde x_1$. Hence, for some $C > 0$, $\sigma^{**} < x < \tilde x_1 + \sigma^*$, we have that

$$\begin{aligned}|\sigma^*(x) - s| &= \left|\frac{1}{a^*(x)}\sum_{k=x_1}^{\infty} e^{-g(k)}\mu_k^*(\alpha)(s_k - s)e^{-\lambda(k)x}\right|\\ &\le \frac{1}{a^*(x)}\int_{u_0}^{\sigma^*}\left|\sum_{k=x_1}^{\infty} e^{-g(k)}(s_k - s)e^{-\lambda(k)(x-u)}\right| d\chi_\alpha(u)\\ &= \frac{1}{a^*(x)}\int_{u_0}^{\sigma^*}|\tilde\sigma(x-u) - s|\tilde a(x-u)\,d\chi_\alpha(u)\\ &\le \epsilon + \frac{1}{a^*(x)}\int_{u_0}^{x-\tilde x_1}|\tilde\sigma(x-u) - s|\tilde a(x-u)\,d\chi_\alpha(u)\\ &\le \epsilon + C\frac{\tilde a(\tilde x_1)}{a^*(x)} \to \epsilon \text{ as } x \to \sigma^{**}+.\end{aligned}$$

Since $\mu_n^*(\alpha) \sim \mu_n(\alpha)$, $s_n = O(1)$, and $z_n(\alpha) \to \sigma^{**}+$ as $n \to \infty$, it follows that $\sigma^*(z_n(\alpha)) \to s$ for $-\frac{1}{2}\delta < \alpha < 0$, and hence that

(13) $$\lim_{n\to\infty} f_n(\alpha) = s \text{ for } -\frac{1}{2}\delta < \alpha < 0.$$

Next we shall show that Vitali's theorem can be applied to the sequence $\{f_n(\alpha)\}$. To this end let

$$h_\alpha(t,x) := g(x) + \alpha\chi(x) - g(t) - \alpha\chi(t) - (\lambda(x) - \lambda(t))\left(\frac{g'(x)}{\lambda'(x)} + \alpha\int_{x_0}^{x}\frac{dt}{\lambda'(t)}\right),$$

so that

(14) $$f_n(\alpha) = \frac{g_n(\alpha)}{h_n(\alpha)}$$

where

$$g_n(\alpha) := \sum_{k=x_1}^{\infty} s_k e^{h_\alpha(k,n)} \text{ and } h_n(\alpha) := \sum_{k=x_1}^{\infty} e^{h_\alpha(k,n)}.$$

Observe first that, for $\alpha > -\frac{1}{2}\delta$, we have

$$\left(\frac{g'(x)}{\lambda'(x)}\right)'\lambda'(x) + \alpha\left(\frac{\chi'(x)}{\lambda'(x)}\right)'\lambda'(x) = L(x) + \alpha \ge \eta(\alpha) := \delta + \alpha \ge \frac{1}{2}\delta.$$

Since $h_n(\alpha) \ge e^{h_\alpha(n,n)} = 1$ for all real α, it follows from (12) and Lemma 1 [with $g(x) + \alpha\chi(x)$ instead of $g(x)$, and $h_\alpha(t,x)$ instead of $h(t,x)$] that

(15) $$\limsup_{n\to\infty}|f_n(\alpha) - s_n| \le Ce^{-\frac{1}{2}\gamma\alpha} \text{ for } \alpha > -\frac{1}{2}\delta,$$

C and γ being positive constants depending on δ but not on α. We proceed as in the proof of Lemma 1. We have that

$$h_\alpha(t,x) = -\int_t^x\int_u^x\frac{\lambda'(u)}{\lambda'(v)}(L(v) + \alpha)\,dv\,du,$$

where $L(v) \ge \delta > 0$ by (H). Hence, by (10) and (11),

$$h_\alpha(k,n) \le -\frac{1}{2}\gamma(\alpha+\delta)|k-n| \le -\frac{1}{4}\gamma\delta|k-n| \le 0,$$

for all integers $k, n \ge x_1$, and all real $\alpha > -\frac{1}{2}\delta$. Then, since $s_n = O(1)$, we obtain that

$$|g_n(\alpha)| \le C\sum_{k=x_1}^{n} e^{-\frac{1}{4}\gamma\delta(n-k)} + C\sum_{k=n+1}^{\infty} e^{-\frac{1}{4}\gamma\delta(k-n)} \le 2C\sum_{\nu=0}^{\infty} e^{-\frac{1}{4}\gamma\delta\nu} = C(\delta)$$

for all $n \geq x_1$, and all complex α with $\Re\alpha > -\frac{1}{2}\delta$. Moreover, $1 = e^{h_\alpha(n,n)} \leq h_n(\alpha) \leq C(\delta)$ for all $n \geq x_1$ and all *real* $\alpha > -\frac{1}{2}\delta$. Next

$$\left|\frac{d}{d\alpha}h_\alpha(k,n)\right| = \left|\int_k^n \int_u^n \frac{\lambda'(u)}{\lambda'(v)}\,dv\,du\right| \leq \frac{2}{\delta}\left|\int_k^n \int_u^n \frac{\lambda'(u)}{\lambda'(v)}(L(u)+\beta)\,dv\,du\right|$$
$$= -\frac{2}{\delta}h_\beta(k,n)$$

for all *complex* α with $\beta = \Re\alpha > -\frac{1}{2}\delta$. Consequently, for such α, we have (since $xe^{-x} \leq e^{-x/2}$ for $x \geq 0$) that

$$|h_n'(\alpha)| \leq -\frac{2}{\delta}\sum_{k=x_1}^{\infty} h_\beta(k,n)e^{h_\beta(k,n)} \leq \frac{2}{\delta}\sum_{k=x_1}^{\infty} e^{\frac{1}{2}h_\beta(k,n)} \leq \frac{4}{\delta}\sum_{\nu=0}^{\infty} e^{-\frac{1}{8}\gamma\delta\nu} =: \frac{1}{2}C_\delta$$

for all $n \geq x_1$. Hence, if

$$G := \{\alpha \in \mathbb{C} \mid \Re\alpha > -\frac{1}{2}\delta,\ |\Im\alpha| < C_\delta^{-1}\},$$

we get that $|h_n(\alpha)| \geq \frac{1}{2}$ for $\alpha \in G$, $n \geq x_1$. Therefore $\{f_n(\alpha)\}$ is a sequence of functions holomorphic and uniformly bounded in G. Hence, by Vitali's theorem, it follows from (13) and (15) that

$$\limsup_{n\to\infty} |s_n - s| \leq Ce^{-\frac{1}{2}\gamma\alpha} \text{ for all real } \alpha > -\frac{1}{2}\delta.$$

Letting $\alpha \to \infty$, we obtain the required result that $s_n \to s$. ∎

Remark. The case $\lambda'(x)/\lambda(x) \geq \delta > 0$ is (at least partially) considered in [1].

4. Asymptotics

The following theorem and lemmas deal with the asymptotic results required in Sections 2 and 3 in the form of strict inequalities which may be useful elsewhere.

Theorem A. *Suppose that the real functions g and λ satisfy the following conditions:*

$$\text{(C)}\quad \begin{cases} g, \lambda \in C_2[x_0,\infty) \text{ for some } x_0 \in \mathbb{N}, \\ \lambda(x),\ \dfrac{x\lambda'(x)}{\lambda(x)} \text{ and } G(x) := \dfrac{\lambda^2(x)}{\lambda'(x)}\left(\dfrac{g'(x)}{\lambda'(x)}\right)' \\ \text{are positive and non-decreasing on } [x_0,\infty), \text{ while} \\ \dfrac{\lambda'(x)}{\lambda(x)} \text{ and } L(x) := \lambda'(x)\left(\dfrac{g'(x)}{\lambda'(x)}\right)' \text{ are non-increasing on } [x_0,\infty). \end{cases}$$

Let $\ell(x) := \frac{1}{\sqrt{L(x)}}$, let α be a complex number with $\beta := \Re\alpha > 0$, and let

$$f_\alpha(z) := \sum_{k=x_0}^{\infty} e^{-\alpha g(k)}e^{-\lambda(k)z} \quad \text{with} \quad z := z(x,\alpha) := -\alpha\frac{g'(x)}{\lambda'(x)}.$$

Then, for all $x \geq 2x_0$,

$$\sqrt{\frac{\alpha}{2\pi}}\int_{x_0}^{\infty} \frac{1}{\ell(t)}\exp\left\{\alpha\left(g(t) + (\lambda(x)-\lambda(t))\frac{g'(t)}{\lambda'(t)}\right)\right\}dt = e^{\alpha g(x)}\Big(1 + R_1(x)\Big),$$

$$f_\alpha(z) = \sqrt{\frac{2\pi}{\alpha}}\ell(x)\exp\left\{\alpha\left(\lambda(x)\frac{g'(x)}{\lambda'(x)} - g(x)\right)\right\}R_2(x) +$$
$$+ \int_{x_0}^{\infty}\exp\left\{\alpha\left(-g(t) + \lambda(t)\frac{g'(x)}{\lambda'(x)}\right)\right\}dt,$$

$$\int_{x_0}^{\infty}\exp\left\{\alpha\left(-g(t) + \lambda(t)\frac{g'(x)}{\lambda'(x)}\right)\right\}dt$$
$$= \sqrt{\frac{2\pi}{\alpha}}\ell(x)\exp\left\{\alpha\left(\lambda(x)\frac{g'(x)}{\lambda'(x)} - g(x)\right)\right\}\Big(1 + R_3(x)\Big),$$

with $|R_1(x)| \leq \frac{C_1}{\sqrt{G(x)}}$, $|R_2(x)| \leq \frac{C_2}{\ell(x)}$, $|R_3(x)| \leq \frac{C_3}{\sqrt{G(x)}}$, where $\sqrt{\alpha}$ denotes the principal branch of the square root, and the constants C_1, C_2, C_3 may be chosen as follows:

$$C_1 = \frac{32}{\sqrt{2\pi}}\frac{|\alpha|\sqrt{|\alpha|}}{\gamma\beta^2} + \frac{16}{\sqrt{2\pi}}\frac{\sqrt{|\alpha|}}{\gamma\beta} + \frac{50\sqrt{5}}{\sqrt{3\pi}}\frac{\sqrt{|\alpha|}}{\gamma^2\beta\sqrt{\beta}\sqrt{G(x_0)}} + \frac{20}{\sqrt{2\pi}}\frac{\sqrt{|\alpha|}}{\gamma\beta},$$

$$C_2 = \frac{3}{\sqrt{2\pi}}\frac{|\alpha|\sqrt{|\alpha|}}{\beta},$$

$$C_3 = \frac{32}{\sqrt{2\pi}}\frac{|\alpha|\sqrt{|\alpha|}}{\gamma\beta^2} + \frac{100\sqrt{5}}{\sqrt{3\pi}}\frac{\sqrt{|\alpha|}}{\gamma^2\beta\sqrt{\beta}\sqrt{G(x_0)}} +$$
$$+ \frac{1600}{\sqrt{2\pi}}\frac{\sqrt{|\alpha|}\Gamma(1/\gamma)}{\beta\gamma^2}\left\{1 + \left(\frac{\beta\gamma^2 G(x_0)}{1600}\right)^{1-\frac{1}{\gamma}}\right\},$$

where $\gamma := \min\left\{1, \frac{x_0\lambda'(x_0)}{\lambda(x_0)}\right\} \in (0,1]$.

Proof. We use the inequalities

$$|e^{\alpha t} - e^{\alpha u}| \le \frac{|\alpha|}{\beta}|e^{\beta t} - e^{\beta u}|, \quad |e^t - e^u| \le |t-u|e^{\max(t,u)} \tag{16}$$

for real t, u and $\beta = \Re\alpha > 0$. Next, we define, for $t, x \ge x_0$,

$$h_1(t,x) := g(t) - g(x) + (\lambda(x) - \lambda(t))\frac{g'(t)}{\lambda'(t)},$$
$$h_2(t,x) := g(x) - g(t) - (\lambda(x) - \lambda(t))\frac{g'(x)}{\lambda'(x)},$$

and we write $h(t) = h(t,x)$ for either function whenever the distinction between the use of h_1 or h_2 is immaterial. We have, for $t, x \ge x_0$,

$$R_1(x) = \sqrt{\frac{\alpha}{2\pi}}\int_{x_0}^{\infty}\frac{1}{\ell(t)}e^{\alpha h_1(t,x)}\,dt - 1,$$
$$R_2(x) = \sqrt{\frac{\alpha}{2\pi}}\frac{1}{\ell(x)}\left\{\sum_{k=x_0}^{\infty} e^{\alpha h_2(k,x)} - \int_{x_0}^{\infty} e^{\alpha h_2(t,x)}\,dt\right\},$$
$$R_3(x) = \sqrt{\frac{\alpha}{2\pi}}\frac{1}{\ell(x)}\int_{x_0}^{\infty} e^{\alpha h_2(t,x)}\,dt - 1, \text{ and}$$
$$h_1(t,x) = \int_x^t\int_u^x \lambda'(v)\left(\frac{g'(u)}{\lambda'(u)}\right)'\,dv\,du,$$
$$h_2(t,x) = \int_x^t\int_u^x \lambda'(u)\left(\frac{g'(v)}{\lambda'(v)}\right)'\,dv\,du.$$

From our assumptions and the mean-value theorem for integrals we find that, for $t, x \ge x_0$,

$$\left\{\begin{array}{l} \dfrac{x\lambda'(x)}{\lambda(x)} \ge \gamma > 0, \quad h_1'(t) = (\lambda(x) - \lambda(t))\dfrac{L(t)}{\lambda'(t)}, \\ h_2'(t) = \lambda'(t)\left(\dfrac{g'(x)}{\lambda'(x)} - \dfrac{g'(t)}{\lambda'(t)}\right), \; h(x) = h'(x) = 0, \\ h'(t) < 0,\; h(t) < 0, \text{ if } t > x, \quad h'(t) > 0,\; h(t) < 0, \text{ if } t < x, \\ h(t) = -\dfrac{1}{2}\dfrac{\lambda'(\zeta)}{\lambda'(\xi)}L(\xi)(x-t)^2 \text{ for some } \zeta, \xi \text{ between } t \text{ and } x. \end{array}\right. \tag{17}$$

Next, let $t, x \ge x_0$ with

$$|t - x| \le \delta(x) := \frac{\gamma}{10}\frac{\lambda(x)}{\lambda'(x)} \le \frac{1}{10}\frac{\lambda(x)}{\lambda'(x)}.$$

Then

$$|t-x| \le \frac{x}{10} \text{ and } \frac{9}{10}x \le t \le \frac{11}{10}x.$$

If $t \ge x \ge x_0$, $t - x \le \delta(x)$, then, by our monotonicity conditions, we get:

$$\lambda'(t) \le \lambda(t)\frac{\lambda'(x)}{\lambda(x)},\; \lambda'(t) \ge \frac{\lambda(t)}{t}\frac{x\lambda'(x)}{\lambda(x)} \ge \frac{x}{t}\lambda'(x) \ge \frac{10}{11}\lambda'(x),$$

and

$$0 \le \lambda(t) - \lambda(x) = \int_x^t \lambda'(u)\,du \le (t-x)\lambda(t)\frac{\lambda'(x)}{\lambda(x)} \le \frac{1}{10}\lambda(t),$$

so that

$$\lambda(t) \le \frac{10}{9}\lambda(x),\; \lambda'(t) \le \frac{10}{9}\lambda'(x),\; 0 \le \lambda(t) - \lambda(x) \le (t-x)\frac{10}{9}\lambda'(x),$$
$$\lambda'(t) - \lambda'(x) \le (\lambda(t) - \lambda(x))\frac{\lambda'(x)}{\lambda(x)} \le \frac{10}{9}(t-x)\frac{\lambda'(x)^2}{\lambda(x)},$$

and

$$\lambda'(x) - \lambda'(t) \le \lambda'(x)\left(\frac{t-x}{t}\right) \le \left(\frac{t-x}{x}\right)\lambda'(x) \le \frac{1}{\gamma}(t-x)\frac{\lambda'(x)^2}{\lambda(x)},$$
$$0 \le 1 - \frac{L(t)}{L(x)} = 1 - \frac{G(t)}{G(x)}\left(\frac{\lambda'(t)\lambda(x)}{\lambda'(x)\lambda(t)}\right)^2 \le 1 - \left(\frac{x}{t}\right)^2 \le \left(1 + \frac{x}{t}\right)\left(\frac{t-x}{x}\right)$$
$$\le \frac{2}{\gamma}(t-x)\frac{\lambda'(x)}{\lambda(x)}.$$

Next, if $x \ge t \ge x_0$, $x - t \le \delta(x)$, then we obtain similarly:

$$\lambda'(t) \ge \lambda(t)\frac{\lambda'(x)}{\lambda(x)},\; \lambda'(t) \le \frac{\lambda(t)}{t}\frac{x\lambda'(x)}{\lambda(x)} \le \frac{x}{t}\lambda'(x) \le \frac{10}{9}\lambda'(x),$$
$$0 \le \lambda(x) - \lambda(t) = \int_t^x \lambda'(u)\,du \le \frac{10}{9}\lambda'(x)(x-t) \le \frac{1}{9}\lambda(x),\; \lambda(x) \le \frac{9}{8}\lambda(t),$$
$$\lambda'(t) - \lambda'(x) \le \left(\frac{x}{t} - 1\right)\lambda'(x) \le \frac{10}{9}\frac{(x-t)}{x}\lambda'(x) \le \frac{10}{9\gamma}(x-t)\frac{\lambda'(x)^2}{\lambda(x)},$$
$$\lambda'(x) - \lambda'(t) \le (\lambda(x) - \lambda(t))\frac{\lambda'(x)}{\lambda(x)} \le \frac{10}{9}(x-t)\frac{\lambda'(x)^2}{\lambda(x)},$$
$$0 \le \frac{L(t)}{L(x)} - 1 = \frac{G(t)}{G(x)}\left(\frac{\lambda'(t)\lambda(x)}{\lambda'(x)\lambda(t)}\right)^2 - 1 \le \left(\frac{x}{t}\right)^2 - 1 = \left(1 + \frac{x}{t}\right)\left(\frac{x-t}{t}\right)$$
$$\le \left(1 + \frac{10}{9}\right)\frac{10}{9}\frac{(x-t)}{x} \le \frac{190}{81\gamma}(x-t)\frac{\lambda'(x)}{\lambda(x)}.$$

Thus we have shown that, for $x, t \geq x_0$, $|t-x| \leq \delta(x)$, we have

$$\text{(18)} \qquad \begin{cases} |\lambda(t)-\lambda(x)| \leq \dfrac{10}{9}\lambda'(x)|t-x|, \ |\lambda'(t)-\lambda'(x)| \leq \dfrac{10}{9\gamma}|t-x|\dfrac{\lambda'(x)^2}{\lambda(x)}, \\ \left|\dfrac{L(t)}{L(x)}-1\right| \leq \dfrac{190}{81\gamma}|t-x|\dfrac{\lambda'(x)}{\lambda(x)} \leq \dfrac{19}{81}, \ \lambda'(t) \geq \dfrac{8}{9}\lambda'(x). \end{cases}$$

Moreover, if ζ, ξ are between t and x, then it follows similarly as above that

$$\frac{|\lambda'(\zeta)-\lambda'(\xi)|}{\lambda'(\xi)} \leq \frac{9}{8}\cdot\frac{10}{9\gamma}|t-x|\frac{\lambda'(x)}{\lambda(x)},$$

and this together with (18) implies

$$\begin{aligned} \left|\frac{\lambda'(\zeta)L(\xi)}{\lambda'(\xi)L(x)}-1\right| &\leq \left|\frac{L(\xi)}{L(x)}-1\right| + \frac{L(\xi)}{L(x)}\frac{|\lambda'(\zeta)-\lambda'(\xi)|}{\lambda'(\xi)} \\ &\leq \frac{190}{81\gamma}|t-x|\frac{\lambda'(x)}{\lambda(x)} + \frac{100}{81}\cdot\frac{9}{8}\cdot\frac{10}{9\gamma}|t-x|\frac{\lambda'(x)}{\lambda(x)}, \end{aligned}$$

and hence

$$\text{(19)} \qquad \left|\frac{\lambda'(\zeta)L(\xi)}{\lambda'(\xi)L(x)}-1\right| \leq \frac{315}{81\gamma}|t-x|\frac{\lambda'(x)}{\lambda(x)} \leq \frac{4}{\gamma}|t-x|\frac{\lambda'(x)}{\lambda(x)} < \frac{1}{2}.$$

We will now prove a number of lemmas involving inequalities in which the underlying hypotheses and terminology are those of Theorem A and its proof. Collecting the constants from Lemmas 4 to 11 will yield the desired conclusion in Theorem A. ∎

Lemma 3. *Let* $\sigma := -\lim_{x\to\infty} \frac{g'(x)}{\lambda'(x)}$. *Then the Dirichlet series* $\sum_{k=x_0}^{\infty} e^{-\alpha g(k)}e^{-\lambda(k)z}$ *converges absolutely if* $\Re z > \beta\sigma$ *where* $\beta = \Re\alpha > 0$, *and it diverges if* $\Re z \leq \beta\sigma$.

Remark. The following proof of the lemma does not require the monotonicity of L and G. But this monotonicity is needed to derive the inequality (20) below, which will be used later on.

Proof. Since $\frac{g'(x)}{\lambda'(x)}$ is increasing, we have that $-\frac{g'(x)}{\lambda'(x)} > \sigma \in [-\infty, \infty)$ for $x \geq x_0$.

First, suppose that $\Re z > \beta\sigma$. Then $\Re z > -\beta\frac{g'(x)}{\lambda'(x)}$ for x sufficiently large, and so it suffices to show that $\sum_{k=x_0}^{\infty} e^{-\beta h_2(k,x)} < \infty$ for all $x \geq x_0$. Put

$$\delta := \delta(x) = \frac{\gamma}{10}\frac{\lambda(x)}{\lambda'(x)}, \qquad \tilde{\gamma} := \tilde{\gamma}(x) = \frac{x\lambda'(x)}{\lambda(x)},$$

and let $t \geq x+\delta$. Then, by (C) and (17), $\tilde{\gamma} \geq \gamma > 0$,

$$\begin{aligned} \frac{\lambda(t)}{\lambda(x)} &= \exp\left(\int_x^t \frac{\lambda'(u)}{\lambda(u)}\,du\right) \geq \exp\left(\frac{x\lambda'(x)}{\lambda(x)}\log\left(\frac{t}{x}\right)\right) = \left(\frac{t}{x}\right)^{\tilde{\gamma}} \geq \left(\frac{t}{x}\right)^{\gamma}, \\ \log\frac{\lambda(t)}{\lambda(x+\frac{1}{2}\delta)} &= \int_{x+\frac{1}{2}\delta}^{t} \frac{\lambda'(u)}{\lambda(u)}\,du \geq \frac{x\lambda'(x)}{\lambda(x)}\int_{x+\frac{1}{2}\delta}^{x+\delta}\frac{du}{u} \geq \frac{x}{x+\delta}\frac{\lambda'(x)}{\lambda(x)}\frac{\delta}{2} \geq \frac{\gamma}{30}, \\ \frac{\lambda(t)}{\lambda(t)-\lambda(x+\frac{1}{2}\delta)} &\leq \left(1-e^{-\gamma/30}\right)^{-1} \leq \frac{30}{\gamma}\left(1+\frac{\gamma}{30}\right) \leq \frac{40}{\gamma}, \end{aligned}$$

and, by (19),

$$\frac{g'(x+\frac{1}{2}\delta)}{\lambda'(x+\frac{1}{2}\delta)} - \frac{g'(x)}{\lambda'(x)} = \int_x^{x+\frac{1}{2}\delta}\frac{L(u)}{\lambda'(u)}\,du \geq \frac{1}{2}\frac{L(x)}{\lambda'(x)}\frac{\delta}{2} = \frac{\gamma}{40}\frac{G(x)}{\lambda(x)}.$$

Hence

$$\begin{aligned} h_2(t,x) &= -\int_x^t \lambda'(u)\left(\frac{g'(u)}{\lambda'(u)} - \frac{g'(x)}{\lambda'(x)}\right)du \leq -\frac{\gamma}{40}\frac{G(x)}{\lambda(x)}\int_{x+\frac{1}{2}\delta}^{t}\lambda'(u)\,du \\ &\leq -\left(\frac{\gamma}{40}\right)^2 G(x)\frac{\lambda(t)}{\lambda(x)} \leq -\left(\frac{\gamma}{40}\right)^2 G(x)\left(\frac{t}{x}\right)^{\tilde{\gamma}}, \end{aligned}$$

so that

$$\text{(20)} \qquad h_2(t,x) \leq -\left(\frac{\gamma}{40}\right)^2 G(x)\left(\frac{t}{x}\right)^{\tilde{\gamma}}, \quad \text{for } t \geq x+\delta,$$

and $\lim_{t\to\infty} h_2(t,x) = -\infty$. It follows that the series in question converges absolutely.

Suppose finally that $\Re z \leq \beta\sigma < -\beta\frac{g'(x)}{\lambda'(x)}$ for all $x \geq x_0$. Then $e^{-\alpha g(k)-\lambda(k)z}$ does not tend to zero as $k \to \infty$, because

$$\begin{aligned} -\beta g(x) - \lambda(x)\Re z + \beta g(x_0) + \lambda(x_0)\Re z \\ &\geq -\beta(g(x)-g(x_0)) - \beta\sigma(\lambda(x)-\lambda(x_0)) \\ &= -\beta\int_{x_0}^{x}\lambda'(t)\left(\frac{g'(t)}{\lambda'(t)}+\sigma\right)dt \geq 0 \quad \text{for all } x \geq x_0. \end{aligned}$$

Thus the series cannot converge. ∎

Lemma 4. *For all $x \ge x_0$,*

$$|R_2(x)| \le \sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\ell(x)}\sum_{k=x_0}^{\infty}\left|e^{\alpha h_2(k,x)} - \int_k^{k+1} e^{\alpha h_2(t,x)}\,dt\right| \le \frac{3}{\sqrt{2\pi}}\frac{|\alpha|\sqrt{|\alpha|}}{\beta}\frac{1}{\ell(x)}.$$

Proof. Recall that, by (17), $h_2(t) = h_2(t,x)$ is increasing on $(x_0, x]$, decreasing on $[x_0, \infty)$, and always ≤ 0. Hence, by (16),

$$\begin{aligned}
&\sqrt{\frac{2\pi}{|\alpha|}}\ell(x)|R_2(x)| \\
&\le \sum_{k=x_0}^{\infty}\left|e^{\alpha h_2(k,x)} - \int_k^{k+1} e^{\alpha h_2(t,x)}\,dt\right| \\
&\le \frac{|\alpha|}{\beta}\left\{\sum_{x_0\le k\le x}\left(e^{\beta h_2(k+1)} - e^{\beta h_2(k)}\right) + 1 + \sum_{k>x}\left(e^{\beta h_2(k)} - e^{\beta h_2(k+1)}\right)\right\} \\
&\le 3\frac{|\alpha|}{\beta}.
\end{aligned}$$

■

In the following lemmas we suppose that $x \ge 2x_0$ and use

$$\delta = \delta(x) = \frac{\gamma}{10}\frac{\lambda(x)}{\lambda'(x)} \le \frac{x}{10}.$$

Lemma 5.

$$\sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\ell(x)}\int_{x-\delta}^{x+\delta}\left|e^{\alpha h(t,x)} - e^{-\frac12 L(x)\alpha(t-x)^2}\right|dt \le \frac{32}{\sqrt{2\pi}}\frac{|\alpha|\sqrt{|\alpha|}}{\gamma\beta^2}\frac{1}{\sqrt{G(x)}}.$$

Proof. We obtain from (17) and (19) that, for $|t-x| \le \delta$,

$$\begin{aligned}
\left|h(t) + L(x)\frac{(t-x)^2}{2}\right| &= \left|-\frac{\lambda'(\zeta)}{\lambda'(\xi)}L(\xi) + L(x)\right|\frac{(t-x)^2}{2} \\
&\le \frac{4}{\gamma}\frac{\lambda'(x)}{\lambda(x)}L(x)\frac{|t-x|^3}{2} \le \frac15 L(x)(t-x)^2,
\end{aligned}$$

and so

$$\max\left(h(t), -L(x)\frac{(t-x)^2}{2}\right) \le -L(x)\frac{(t-x)^2}{4}.$$

Hence, by (16),

$$\begin{aligned}
&\sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\ell(x)}\int_{x-\delta}^{x+\delta}\left|e^{\alpha h(t,x)} - e^{-\frac12 L(x)\alpha(t-x)^2}\right|dt \\
&\le \sqrt{\frac{|\alpha|}{2\pi}}\sqrt{L(x)}\int_{x-\delta}^{x+\delta}\frac{|\alpha|}{\beta}\beta\frac{2}{\gamma}\frac{\lambda'(x)}{\lambda(x)}L(x)|t-x|^3e^{-\frac14\beta L(x)(t-x)^2}\,dt \\
&\le \sqrt{\frac{|\alpha|}{2\pi}}\frac{|\alpha|}{\beta}\sqrt{L(x)}\frac{2}{\gamma}\int_{-\infty}^{\infty}\frac{\lambda'(x)}{\lambda(x)}\frac{|u|^3}{\sqrt{\beta L(x)}}e^{-\frac14 u^2}\frac{du}{\sqrt{\beta L(x)}} \qquad \left\{u = \sqrt{\beta L(x)}(t-x)\right\} \\
&= \sqrt{\frac{|\alpha|}{2\pi}}\frac{|\alpha|}{\beta^2}\frac{4}{\gamma\sqrt{G(x)}}\int_0^{\infty}u^3e^{-\frac14 u^2}\,du = \frac{32}{\sqrt{2\pi}}\frac{|\alpha|\sqrt{|\alpha|}}{\gamma\beta^2}\frac{1}{\sqrt{G(x)}}.
\end{aligned}$$

■

Lemma 6.

$$\sqrt{\frac{|\alpha|}{2\pi}}\int_{x-\delta}^{x+\delta}\left|\left(\frac{1}{\ell(x)} - \frac{1}{\ell(t)}\right)e^{\alpha h_1(t,x)}\right|dt \le \frac{16}{\sqrt{2\pi}}\frac{\sqrt{|\alpha|}}{\gamma\beta}\frac{1}{\sqrt{G(x)}}.$$

Proof. By the proof above we have that $h_1(t) \le -\frac14 L(x)(t-x)^2$. Moreover, by (19),

$$\left|\frac{1}{\ell(x)} - \frac{1}{\ell(t)}\right| = \left|\frac{L(x)-L(t)}{\sqrt{L(x)}+\sqrt{L(t)}}\right| \le \frac{4}{\gamma}\frac{|t-x|\lambda'(x)}{\lambda(x)\sqrt{L(x)}}L(x).$$

Again using the substitution $u = \sqrt{\beta L(x)}(t-x)$, we get that

$$\begin{aligned}
&\sqrt{\frac{|\alpha|}{2\pi}}\int_{x-\delta}^{x+\delta}\left|\left(\frac{1}{\ell(x)} - \frac{1}{\ell(t)}\right)e^{\alpha h_1(t,x)}\right|dt \\
&\le \sqrt{\frac{|\alpha|}{2\pi}}\frac{4}{\gamma}\sqrt{L(x)}\frac{\lambda'(x)}{\lambda(x)}\int_{x-\delta}^{x+\delta}|t-x|e^{-\frac14\beta L(x)(t-x)^2}\,dt \\
&\le \sqrt{\frac{|\alpha|}{2\pi}}\frac{4}{\gamma}\sqrt{L(x)}\frac{\lambda'(x)}{\lambda(x)}\int_{-\infty}^{\infty}\frac{|u|}{\sqrt{\beta L(x)}}e^{-\frac14 u^2}\frac{du}{\sqrt{\beta L(x)}} \\
&= \frac{16}{\sqrt{2\pi}}\frac{\sqrt{|\alpha|}}{\gamma\beta}\frac{1}{\sqrt{G(x)}}.
\end{aligned}$$

■

Lemma 7.

$$\left|\sqrt{\frac{\alpha}{2\pi}}\frac{1}{\ell(x)}\int_{x-\delta}^{x+\delta} e^{-\frac{1}{2}\alpha L(x)(x-t)^2}\,dt - 1\right| \le 10\sqrt{2}\frac{\sqrt{|\alpha|}}{\gamma\beta}\frac{1}{\sqrt{G(x)}}.$$

Proof. Since

$$\int_{-\infty}^{\infty} e^{-\frac{\alpha}{2\beta}u^2}\,du = \sqrt{\frac{2\pi\beta}{\alpha}},$$

the above substitution leads to

$$\begin{aligned}
\left|\sqrt{\frac{\alpha}{2\pi}}\frac{1}{\ell(x)}\int_{x-\delta}^{x+\delta} e^{-\frac{1}{2}\alpha L(x)(x-t)^2}\,dt - 1\right| &= \left|2\sqrt{\frac{\alpha}{2\pi}}\frac{1}{\sqrt{\beta}}\int_{\delta\sqrt{\beta L(x)}}^{\infty} e^{-\frac{\alpha}{2\beta}u^2}\,du\right| \\
&\le 2\sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\sqrt{\beta}}\int_{\delta\sqrt{\beta L(x)}}^{\infty} e^{-\frac{1}{2}u^2}\,du \\
&\le 2\sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\sqrt{\beta}}\frac{1}{\delta\sqrt{\beta L(x)}}\int_0^{\infty} e^{-\frac{1}{4}u^2}\,du \qquad \left\{\text{since } e^{-\frac{1}{4}u^2} < \frac{1}{u} \text{ for } u > 0\right\} \\
&= 10\sqrt{2}\frac{\sqrt{|\alpha|}}{\gamma\beta}\frac{1}{\sqrt{G(x)}}.
\end{aligned}$$

■

Lemma 8.

$$\sqrt{\frac{|\alpha|}{2\pi}}\int_{x_0}^{x-\delta}\frac{1}{\ell(t)}e^{\beta h_1(t,x)}\,dt \le \frac{50\sqrt{5}}{\sqrt{3\pi}}\frac{\sqrt{|\alpha|}}{\gamma^2\beta\sqrt{\beta G(x_0)}}\frac{1}{\sqrt{G(x)}}.$$

Proof. Suppose $x_0 \le t \le x-\delta$. Then, by (17) and (19),

$$\begin{aligned}
h(x-\delta) &= -\frac{1}{2}\frac{\lambda'(\zeta)}{\lambda'(\xi)}L(\xi)\delta^2 \le -\frac{1}{2}\delta^2 L(x)\left(1-\frac{4}{\gamma}\frac{\lambda'(x)}{\lambda(x)}\delta\right) \\
&= -\frac{3}{10}\delta^2 L(x) = -\frac{3\gamma^2}{1000}G(x), \\
\log\frac{\lambda(x)}{\lambda(t)} &= \int_t^x \frac{\lambda'(u)}{\lambda(u)}\,du \ge (x-t)\frac{\lambda'(x)}{\lambda(x)} \ge \delta\frac{\lambda'(x)}{\lambda(x)} = \frac{\gamma}{10}, \text{ and by (17),} \\
\frac{1}{\ell(t)h_1'(t)} &= \frac{\sqrt{L(t)}\lambda'(t)}{L(t)(\lambda(x)-\lambda(t))} = \frac{\lambda(t)}{\lambda(x)-\lambda(t)}\frac{1}{\sqrt{G(t)}} \\
&\le \frac{1}{\sqrt{G(x_0)}}\frac{1}{e^{\gamma/10}-1} \le \frac{10}{\gamma}\frac{1}{\sqrt{G(x_0)}}.
\end{aligned}$$

Hence, using the fact that $e^{-t} < \frac{1}{2\sqrt{t}}$ for $t > 0$, we get

$$\begin{aligned}
&\sqrt{\frac{|\alpha|}{2\pi}}\int_{x_0}^{x-\delta}\frac{1}{\ell(t)}e^{\beta h_1(t)}\,dt \\
&\quad\le \sqrt{\frac{|\alpha|}{2\pi}}\frac{10}{\gamma}\frac{1}{\sqrt{G(x_0)}}\int_{x_0}^{x-\delta} h_1'(t)e^{\beta h_1(t)}\,dt \\
&\quad\le \sqrt{\frac{|\alpha|}{2\pi}}\frac{10}{\beta\gamma}\frac{1}{\sqrt{G(x_0)}}e^{\beta h_1(x-\delta)} \le \sqrt{\frac{|\alpha|}{2\pi}}\frac{10}{\beta\gamma}\frac{1}{\sqrt{G(x_0)}}\frac{1}{2\sqrt{-\beta h_1(x-\delta)}} \\
&\quad\le \frac{50\sqrt{5}}{\sqrt{3\pi}}\frac{\sqrt{|\alpha|}}{\gamma^2\beta\sqrt{\beta G(x_0)}}\frac{1}{\sqrt{G(x)}}.
\end{aligned}$$

■

Lemma 9.

$$\sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\ell(x)}\int_{x_0}^{x-\delta} e^{\beta h_2(t,x)}\,dt \le \frac{100\sqrt{5}}{\sqrt{3\pi}}\frac{\sqrt{|\alpha|}}{\gamma^2\beta\sqrt{\beta G(x_0)}}\frac{1}{\sqrt{G(x)}}.$$

Proof. Suppose that $x_0 \le t \le x-\delta$. Then

$$h_2(x-\delta) \le -\frac{3\gamma^2}{1000}G(x) \text{ and } \log\frac{\lambda(x)}{\lambda(t)} \ge \frac{\gamma}{10},$$

by the previous proof. It follows from (17) that

$$\begin{aligned}
h_2'(t) &= \lambda'(t)\int_t^x G(u)\frac{\lambda'(u)}{\lambda^2(u)}\,du \\
&\ge \lambda'(t)G(t)\int_t^x \frac{\lambda'(u)}{\lambda^2(u)}\,du = \lambda'(t)G(t)\left(\frac{1}{\lambda(t)}-\frac{1}{\lambda(x)}\right) \\
&\ge \lambda'(t)G(t)\left(\frac{1}{\lambda(t)}-\frac{e^{-\gamma/10}}{\lambda(t)}\right) \ge \frac{\gamma}{20}\frac{\lambda'(t)}{\lambda(t)}G(t),
\end{aligned}$$

and hence that

$$\begin{aligned}
\frac{1}{\ell(x)h_2'(t)} &\le \frac{20}{\gamma}\frac{\sqrt{L(x)}\lambda(t)}{\lambda'(t)G(t)} \le \frac{20}{\gamma}\frac{\sqrt{L(t)}\lambda(t)}{\lambda'(t)G(t)} = \frac{20}{\gamma}\frac{1}{\sqrt{G(t)}} \\
&\le \frac{20}{\gamma}\frac{1}{\sqrt{G(x_0)}}.
\end{aligned}$$

Consequently

$$\sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\ell(x)}\int_{x_0}^{x-\delta} e^{\beta h_2(t)}\,dt \le \sqrt{\frac{|\alpha|}{2\pi}}\frac{20}{\gamma}\frac{1}{\sqrt{G(x_0)}}\int_{x_0}^{x-\delta} h_2'(t)e^{\beta h_2(t)}\,dt$$
$$\le \sqrt{\frac{|\alpha|}{2\pi}}\frac{20}{\gamma}\frac{1}{\sqrt{G(x_0)}}\frac{1}{\beta}e^{\beta h_2(x-\delta)} \le \sqrt{\frac{|\alpha|}{2\pi}}\frac{20}{\gamma}\frac{1}{\sqrt{G(x_0)}}\frac{1}{\beta}\frac{1}{2\sqrt{-\beta h_2(x-\delta)}}$$
$$\le \frac{100\sqrt{5}}{\sqrt{3\pi}}\frac{\sqrt{|\alpha|}}{\gamma^2\beta\sqrt{\beta G(x_0)}}\frac{1}{\sqrt{G(x)}}.$$

■

Lemma 10.

$$\sqrt{\frac{|\alpha|}{2\pi}}\int_{x+\delta}^{\infty}\frac{1}{\ell(t)}e^{\beta h_1(t,x)}\,dt \le \frac{20}{\sqrt{2\pi}}\frac{\sqrt{|\alpha|}}{\gamma\beta}\frac{1}{\sqrt{G(x)}}.$$

Proof. Suppose $t \ge x+\delta$. Then

$$\log\frac{\lambda(t)}{\lambda(x)} = \int_x^t \frac{\lambda'(u)}{\lambda(u)}\,du \ge x\frac{\lambda'(x)}{\lambda(x)}\int_x^{x+\delta}\frac{1}{u}\,du$$
$$\ge \frac{x}{x+\delta}\frac{\lambda'(x)}{\lambda(x)}\delta \ge \frac{x}{x+\frac{x}{10}}\frac{\lambda'(x)}{\lambda(x)}\frac{\gamma}{10}\frac{\lambda(x)}{\lambda'(x)} = \frac{\gamma}{11},$$

so that

$$\frac{\lambda(t)}{\lambda(t)-\lambda(x)} = \left(1-\frac{\lambda(x)}{\lambda(t)}\right)^{-1} \le \left(1-e^{-\gamma/11}\right)^{-1} \le \frac{20}{\gamma}.$$

Hence, by (17),

$$0 < \frac{-1}{\ell(t)h_1'(t)} = \frac{\sqrt{L(t)}\lambda'(t)}{(\lambda(t)-\lambda(x))L(t)} = \frac{\lambda(t)}{\lambda(t)-\lambda(x)}\frac{1}{\sqrt{G(t)}} \le \frac{20}{\gamma}\frac{1}{\sqrt{G(x)}}.$$

This implies that

$$\sqrt{\frac{|\alpha|}{2\pi}}\int_{x+\delta}^{\infty}\frac{1}{\ell(t)}e^{\beta h_1(t)}\,dt \le \sqrt{\frac{|\alpha|}{2\pi}}\frac{20}{\gamma}\frac{1}{\sqrt{G(x)}}\int_{x+\delta}^{\infty} -h_1'(t)e^{\beta h_1(t)}\,dt$$
$$\le \sqrt{\frac{|\alpha|}{2\pi}}\frac{20}{\beta\gamma}\frac{1}{\sqrt{G(x)}}.$$

■

Lemma 11.

$$\sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\ell(x)}\int_{x+\delta}^{\infty} e^{\beta h_2(t,x)}\,dt \le \frac{1600}{\sqrt{2\pi}}\frac{\sqrt{|\alpha|}\Gamma(1/\gamma)}{\beta\gamma^2}\left\{1+\left(\frac{\beta\gamma^2 G(x_0)}{1600}\right)^{1-\frac{1}{\gamma}}\right\}\frac{1}{\sqrt{G(x)}}.$$

Proof. Recall that

$$\tilde\gamma = \tilde\gamma(x) = \frac{x\lambda'(x)}{\lambda(x)} \ge \gamma > 0.$$

We obtain from (20), via the substitution $u = \beta\left(\frac{\gamma}{40}\right)^2\left(\frac{t}{x}\right)^{\tilde\gamma}G(x)$, that

$$I := \sqrt{\frac{|\alpha|}{2\pi}}\frac{1}{\ell(x)}\int_{x+\delta}^{\infty} e^{\beta h_2(t)}\,dt$$
$$\le \sqrt{\frac{|\alpha|}{2\pi}}\sqrt{G(x)}\frac{\lambda'(x)}{\lambda(x)}\int_x^{\infty}\exp\left\{-\beta\left(\frac{\gamma}{40}\right)^2\left(\frac{t}{x}\right)^{\tilde\gamma}G(x)\right\}dt$$
$$= \sqrt{\frac{|\alpha|}{2\pi}}\sqrt{G(x)}\left\{\beta\left(\frac{\gamma}{40}\right)^2 G(x)\right\}^{-\frac{1}{\tilde\gamma}}\int_{\beta\left(\frac{\gamma}{40}\right)^2 G(x)}^{\infty} e^{-u}u^{\frac{1}{\tilde\gamma}-1}\,du.$$

If $\tilde\gamma \ge 1$, then

$$I \le \sqrt{\frac{|\alpha|}{2\pi}}\sqrt{G(x)}\left\{\beta\left(\frac{\gamma}{40}\right)^2 G(x)\right\}^{-\frac{1}{\tilde\gamma}+\frac{1}{\tilde\gamma}-1}\int_0^{\infty} e^{-u}\,du$$
$$= \sqrt{\frac{|\alpha|}{2\pi}}\frac{1600}{\beta\gamma^2}\frac{1}{\sqrt{G(x)}}.$$

If $\tilde\gamma < 1$ and $\beta\left(\frac{\gamma}{40}\right)^2 G(x) \le 1$, then

$$I \le \sqrt{\frac{|\alpha|}{2\pi}}\sqrt{G(x)}\left\{\beta\left(\frac{\gamma}{40}\right)^2 G(x)\right\}^{-\frac{1}{\gamma}}\Gamma\left(\frac{1}{\gamma}\right)$$
$$\le \sqrt{\frac{|\alpha|}{2\pi}}\left\{\beta\left(\frac{\gamma}{40}\right)^2\right\}^{-\frac{1}{\gamma}}\Gamma\left(\frac{1}{\gamma}\right)\left(G(x_0)\right)^{1-\frac{1}{\gamma}}\frac{1}{\sqrt{G(x)}}.$$

Finally, if $\tilde\gamma < 1$ and $\beta\left(\frac{\gamma}{40}\right)^2 G(x) > 1$, then

$$I \le \sqrt{\frac{|\alpha|}{2\pi}}\sqrt{G(x)}\left\{\beta\left(\frac{\gamma}{40}\right)^2 G(x)\right\}^{-1}\Gamma\left(\frac{1}{\gamma}\right) = \sqrt{\frac{|\alpha|}{2\pi}}\frac{1600}{\beta\gamma^2}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{\sqrt{G(x)}}.$$

Collecting the inequalities from the three cases yields the lemma.

■

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