ON PRODUCTS OF SEQUENCES

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1. *Introduction.*

It is well known (Hardy [3], 288) that if $\kappa > -1$, $\lambda > -1$ and

$$\sum_{n=0}^{\infty} u_n = \sigma\ (C, \kappa), \quad \sum_{n=0}^{\infty} v_n = \tau\ (C, \lambda),$$

and if the sequence $\{w_n\}$ is the Cauchy product of the sequences $\{u_n\}$, $\{v_n\}$, *i.e.* $w_n = \sum_{\nu=0}^{n} u_{n-\nu} v_\nu$, then

$$\sum_{n=0}^{\infty} w_n = \sigma\tau\ (C, \kappa+\lambda+1).$$

Now put $$s_n = \sum_{\nu=0}^{n} u_\nu, \quad t_n = \sum_{\nu=0}^{n} v_\nu, \quad W_n = \sum_{\nu=0}^{n} w_\nu.$$

Then $$\frac{1}{n+1} \sum_{\nu=0}^{n} s_{n-\nu} t_\nu = \frac{1}{n+1} \sum_{\nu=0}^{n} W_\nu;$$

and, in consequence of this and a well-known property of Cesàro means, we obtain

THEOREM A. *If* $\kappa > -1$, $\lambda > -1$, $\kappa+\lambda > -1$, *and* $s_n \to \sigma$ (C, κ), $t_n \to \tau$ (C, λ), *then*

$$\frac{1}{n+1} \sum_{\nu=0}^{n} s_{n-\nu} t_\nu \to \sigma\tau\ (C, \kappa+\lambda).$$

This theorem is concerned with the Cesàro method of summability and the Cauchy product of the sequences $\{s_n\}$ and $\{t_n\}$. The object of this paper is to obtain results of this type involving other methods of summability and products more general than the Cauchy product.

2. *Notation, definitions and preliminary results.*

Suppose throughout that σ, τ are arbitrary complex numbers and that $\{s_n\}$, $\{t_n\}$ $(n = 0, 1, \ldots)$ are arbitrary sequences of complex numbers.

Given two summability methods P and Q, P is said to *include* Q and we write $P \supseteq Q$ if $s_n \to \sigma$ (P) whenever $s_n \to \sigma$ (Q). If $P \supseteq Q$ and $Q \supseteq P$, P and Q are said to be *equivalent*, and we write $P \simeq Q$. The method P is said to be *regular* if $s_n \to \sigma$ (P) whenever $s_n \to \sigma$.

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Given sequences $a = \{a_n\}$, $b = \{b_n\}$ $(n = 0, 1, \ldots)$, we denote by $a * b$ the sequence $\{c_n\}$, where

$$c_n = \sum_{\nu=0}^{n} a_{n-\nu} b_\nu,$$

and we write $(a*b)_n$ for c_n. Further, we write

$$a(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and denote the radius of convergence of the power series by ρ_a; we also use this notation with b and c in place of a.

We define next two methods of summability of which the second is new.

The power series method (J, a) (see Hardy [3], 79-81). Let $a = \{a_n\}$ be a sequence of real non-negative numbers, not all zero, such that $\rho_a > 0$. We write $s_n \to \sigma$ (J, a) if, as $x \to \rho_a$ in the open interval $(0, \rho_a)$,

$$\frac{1}{a(x)} \sum_{n=0}^{\infty} a_n s_n x^n \to \sigma.$$

The generalized Nörlund method (N, a, b). Let $a = \{a_n\}$, $b = \{b_n\}$ $c = \{c_n\}$ be sequences of real numbers such that $c_n = (a*b)_n \neq 0$. Given a sequence $\{s_n\}$, we define the sequence $\{s_n'\}$ of its (N, a, b) means by the relation

$$s_n' = \frac{1}{c_n} \sum_{\nu=0}^{n} a_{n-\nu} b_\nu s_\nu;$$

and we write $s_n \to \sigma$ (N, a, b) if $s_n' \to \sigma$.

The method (N, a, b) reduces to the ordinary Nörlund method (N, a) (Hardy [3], 64) when $b_n = 1$, and to the method $(\bar{N}, b)$ ([3], 57) when $a_n = 1$. Further, when $a_n = \alpha^n/n!$ and $b_n = \beta^n/n!$ $(\alpha > 0, \beta > 0)$ we find that (N, a, b) is equivalent to the Euler-Knopp method $(E, \alpha/\beta)$ ([3], 180).

We say that the method (N, a, b) is *bi-regular* if both (N, a, b) and (N, b, a) are regular.

The following two results are consequences respectively of Toeplitz' theorem (Hardy [3], 43) and a simple extension of part of this theorem (Knopp [4], 73).

I. *Necessary and sufficient conditions for the method* (N, a, b) *to be regular are*

(i) $\sum_{\nu=0}^{n} |a_{n-\nu} b_\nu| < H |c_n|$, *where* H *is a positive number independent of* n;

(ii) *for each integer* $\nu \geqslant 0$, $a_{n-\nu} b_\nu / c_n \to 0$ *as* $n \to \infty$.

II. *If* $s_n \to \sigma$, $t_n \to \tau$ *and the method* (N, a, b) *is bi-regular, then, as* $n \to \infty$,

$$\frac{1}{c_n} \sum_{\nu=0}^{n} a_{n-\nu} b_\nu s_{n-\nu} t_\nu \to \sigma\tau.$$

3. *The main theorems.*

Theorem† 1. *If*

(i) a, b, c, k, l, m *are sequences of real numbers such that* $c = a * b$, $m = k * l$,

(ii) $c_n \neq 0$, $(k*a)_n \neq 0$, $(l*b)_n \neq 0$, $(m*c)_n \neq 0$,

(iii) *the method* $(N, k*a, l*b)$ *is bi-regular,*

(iv) $s_n \to \sigma$ (N, k, a), $t_n \to \tau$ (N, l, b), *then*

$$u_n = \frac{1}{c_n} \sum_{\nu=0}^{n} a_{n-\nu} b_\nu s_{n-\nu} t_\nu \to \sigma\tau \ (N, m, c).$$

Proof. Let

$$f = k*a, \quad g = l*b, \quad h = f*g,$$

$$s_n' = \frac{1}{f_n} \sum_{\nu=0}^{n} k_{n-\nu} a_\nu s_\nu, \quad t_n' = \frac{1}{g_n} \sum_{\nu=0}^{n} l_{n-\nu} b_\nu t_\nu, \quad v_n = \sum_{\nu=0}^{n} f_{n-\nu} g_\nu s_{n-\nu}' t_\nu'.$$

We now have the formal identities

$$\sum_{n=0}^{\infty} v_n x^n = \sum_{n=0}^{\infty} f_n s_n' x^n . \sum_{n=0}^{\infty} g_n t_n' x^n$$

$$= \sum_{n=0}^{\infty} k_n x^n . \sum_{n=0}^{\infty} a_n s_n x^n . \sum_{n=0}^{\infty} l_n x^n . \sum_{n=0}^{\infty} b_n t_n x^n$$

$$= \sum_{n=0}^{\infty} m_n x^n . \sum_{n=0}^{\infty} c_n u_n x^n,$$

from which we deduce that

$$v_n = \sum_{\nu=0}^{n} m_{n-\nu} c_\nu u_\nu.$$

Similarly we find that

$$h_n = (m*c)_n \neq 0.$$

Now, by hypothesis (iv), $s_n' \to \sigma$, $t_n' \to \tau$ and consequently, in virtue of II and hypothesis (iii),

$$v_n / h_n \to \sigma\tau.$$

Hence, $u_n \to \sigma\tau$ (N, m, c), and the proof is complete.

† *Cf.* Mears ([5], Theorem 2).

Theorem 2. *If* (i) a, b, c *are sequences of real non-negative numbers such that* $c_n = (a*b)_n > 0$ *and* $\rho_a = \rho_b = \rho > 0$, (ii) $s_n \to \sigma$ (J, a), $t_n \to \tau$ (J, b), *then*

$$u_n = \frac{1}{c_n} \sum_{\nu=0}^{n} a_{n-\nu} b_\nu s_{n-\nu} t_\nu \to \sigma\tau \ (J, c).$$

Proof. Note that, for $|x| < \rho$,

$$c(x) = a(x) b(x),$$

so that, by a familiar result concerning the singularities of a power series on its circle of convergence, $\rho_c = \rho$.

Further, in view of hypothesis (ii), we have for $|x| < \rho$,

$$\sum_{n=0}^{\infty} c_n u_n x^n = \sum_{n=0}^{\infty} a_n s_n x^n . \sum_{n=0}^{\infty} b_n t_n x^n.$$

The theorem follows.

3. *Special cases.*

We proceed now to obtain corollaries of the main theorems by considering special cases of the methods (N, a, b) and (J, a).

For convenience we denote the binomial coefficient $\binom{n+\delta}{n}$ by ϵ_n^δ. Note that, if $d_n = \epsilon_n^\delta$, $d_n' = \epsilon_n^{\delta'}$, then $(d*d')_n = \epsilon_n^{\delta+\delta'+1}$.

The Cesàro method (C, κ). The definition of this method is standard for the range $\kappa > -1$ and various equivalent definitions have been given for the range $\kappa \leqslant -1$ (see Borwein [2] for references).

For $k_n = \epsilon_n^{\kappa-1}$, $a_n = \epsilon_n^\alpha$, we denote the method (N, k, a) by (C, κ, α). We then have the following result (proved in [2]).

Lemma. *If* $\alpha > -1$, $\kappa+\alpha > -1$, *then* $(C, \kappa) \simeq (C, \kappa, \alpha)$.

The next theorem generalizes Theorem A.

Theorem 3. *If* (i) $\alpha > -1$, $\beta > -1$, $\kappa+\alpha > -1$, $\lambda+\beta > -1$, (ii) $s_n \to \sigma$ (C, κ), $t_n \to \tau$ (C, λ), *then*

$$u_n = \frac{1}{\epsilon_n^{\alpha+\beta+1}} \sum_{\nu=0}^{n} \epsilon_{n-\nu}^{\alpha} \epsilon_\nu^\beta s_{n-\nu} t_\nu \to \sigma\tau \ (C, \kappa+\lambda).$$

Proof. Let $a_n = \epsilon_n^\alpha$, $b_n = \epsilon_n^\beta$, $c_n = \epsilon^{\alpha+\beta+1}$, $k_n = \epsilon_n^{\kappa-1}$, $l_n = \epsilon_n^{\lambda-1}$, $m_n = \epsilon_n^{\kappa+\lambda-1}$, so that

$$c = a*b, \quad m = k*l,$$

and

$$u_n = \frac{1}{c_n} \sum_{\nu=0}^{n} a_{n-\nu} b_\nu s_{n-\nu} t_\nu.$$

Then, in virtue of hypothesis (i),

$$c_n > 0, \quad (k*a)_n = \epsilon_n^{\kappa+\alpha} > 0, \quad (l*b)_n = \epsilon_n^{\lambda+\beta} > 0, \quad (m*c)_n = \epsilon_n^{\alpha+\beta+\kappa+\lambda} > 0,$$

and, by the Lemma,

$$(N, k, a) \simeq (C, \kappa), \quad (N, l, b) \simeq (C, \lambda), \quad (N, m, c) \simeq (C, \kappa+\lambda),$$

$$(N, k*a, l*b) \simeq (C, \kappa+\alpha+1), \quad (N, l*b, k*a) \simeq (C, \lambda+\beta+1).$$

Since $\kappa+\alpha+1>0$, $\lambda+\beta+1>0$, it follows from the final two equivalences that $(N, k*a, l*b)$ is bi-regular.

The theorem is now an immediate consequence of Theorem 1.

The Euler-Knopp method (E, λ). Suppose that $\lambda>0$, $\delta>0$, and recall that the sequence $\{s_n'\}$ of (E, λ) means of a sequence $\{s_n\}$ is given by

$$s_n' = (\lambda+1)^{-n} \sum_{\nu=0}^{n} \binom{n}{\nu} \lambda^{n-\nu} s_\nu$$

$$= (\delta\lambda+\delta)^{-n} n! \sum_{\nu=0}^{n} \frac{(\delta\lambda)^{n-\nu}}{(n-\nu)!} \frac{\delta^\nu}{\nu!} s_\nu;$$

so that, if $l_n = (\delta\lambda)^n/n!$, $b_n = \delta^n/n!$, then $(N, l, b) \simeq (E, \lambda)$.

THEOREM 4. *If* (i) $\kappa>0, \lambda>0, \delta>0$, (ii) $s_n \to \sigma\ (E, \kappa)$, $t_n \to \tau\ (E, \lambda)$, *then*

$$u_n = (\delta+1)^{-n} \sum_{\nu=0}^{n} \binom{n}{\nu} \delta^\nu s_{n-\nu} t_\nu \to \sigma\tau \left(E, \frac{\kappa+\delta\lambda}{1+\delta}\right).$$

Proof. Let $a_n = 1/n!$, $b_n = \delta^n/n!$, $c_n = (\delta+1)^n/n!$, $k_n = \kappa^n/n!$, $l_n = (\delta\lambda)^n/n!$, $m_n = (\kappa+\delta\lambda)^n/n!$, so that

$$c = a*b, \quad m = k*l,$$

and

$$u_n = \frac{1}{c_n} \sum_{\nu=0}^{n} a_{n-\nu} b_\nu s_{n-\nu} t_\nu.$$

Then

$$c_n > 0, \quad (k*a)_n = (\kappa+1)^n/n! > 0, \quad (l*b)_n = (\delta\lambda+\delta)^n/n! > 0,$$

$$(m*c)_n = (\kappa+\delta\lambda+\delta+1)^n/n! > 0.$$

Further,

$$(N, k, a) \simeq (E, \kappa), \quad (N, l, b) \simeq (E, \lambda), \quad (N, m, c) \simeq \left(E, \frac{\kappa+\delta\lambda}{1+\delta}\right),$$

and $$(N, k*a, l*b) \simeq \left(E, \frac{\kappa+1}{\delta\lambda+\delta}\right), \quad (N, l*b, k*a) \simeq \left(E, \frac{\delta\lambda+\delta}{\kappa+1}\right).$$

Since (E, γ) is regular for $\gamma>0$, it follows from the final two equivalences that $(N, k*a, l*b)$ is bi-regular.

We now complete the proof by appealing to Theorem 1.

The method A_α. Suppose that $\alpha>-1$ and $a_n = \epsilon_n^\alpha$, so that $\rho_a = 1$ and $a(x) = (1-x)^{\alpha+1}$; and denote the method (J, a) by A_α. Then A_0 is the Abel method. It has been proved elsewhere (Borwein [1]) that if $\beta>\alpha>-1$, $\gamma \geqslant 0$, then $A_\alpha \supseteq A_\beta \supseteq (C, \gamma)$.

The next theorem is a simple corollary of Theorem 2.

THEOREM 5. *If* $\alpha>-1$, $\beta>-1$ *and* $s_n \to \sigma\ (A_\alpha)$, $t_n \to \tau\ (A_\beta)$, *then*

$$\frac{1}{\epsilon_n^{\alpha+\beta+1}} \sum_{\nu=0}^{n} \epsilon_{n-\nu}^{\alpha} \epsilon_\nu^{\beta} s_{n-\nu} t_\nu \to \sigma\tau\ (A_{\alpha+\beta+1}).$$

The Borel exponential method B. Suppose that $\delta>0$, $a_n = 1/n!$, $b_n = \delta^n/n!$, $c_n = (\delta+1)^n/n!$, so that $c = a*b$, $\rho_a = \rho_b = \rho_c = \infty$ and $a(x) = e^x$, $b(x) = e^{\delta x}$, $c(x) = e^{(\delta+1)x}$. We recall that the Borel method B is in fact the method (J, a). Clearly, $B \simeq (J, b) \simeq (J, c)$.

Hence, as a special case of Theorem 2, we obtain

THEOREM 6. *If* $s_n \to \sigma\ (B)$, $t_n \to \tau\ (B)$, *then, for any* $\delta>0$,

$$(\delta+1)^{-n} \sum_{\nu=0}^{n} \binom{n}{\nu} \delta^\nu s_{n-\nu} t_\nu \to \sigma\tau\ (B).$$

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