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Multi-Variable Sinc Integrals and Volumes of Polyhedra*

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Abstract. We investigate multi-variable integrals of products of sinc functions and show how they may be interpreted as volumes of symmetric convex polyhedra. We then derive an explicit formula for computing such sinc integrals and so equivalently volumes of polyhedra.

Key words: multi-variable sinc integrals, Fourier transforms, convolution, Parseval's theorem, volumes of polyhedra

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1. Introduction

Stimulated by our recent prior work with one dimensional sinc integrals we study a class of multivariable sinc integrals. In Sections 2 through 5 we obtain results concerning the relationship between such a multi-variable sinc integral $\sigma(S)$ (defined in Section 2 below) and the volume of an associated symmetric convex polyhedron. Section 6 is devoted to establishing a partial fraction decomposition to be used in Section 7. In Section 7, we derive (Theorem 4) an explicit algebraic (determinant) formula for the computation of $\sigma(S)$. This formula entirely generalizes that given in [1], wherein more motivation and references may also be found.

2. Sinc and polyhedron spaces

As is quite usual we set

$$\operatorname{sinc}(t) := \frac{\sin t}{t}.$$

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Given $x := (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$ in $\mathbb{R}^m$, we use the notation $xy := x_1y_1 + x_2y_2 + \cdots + x_my_m$ to denote the dot product.

We first define the classes of sinc integrals and of polyhedra we will study and identify them with certain spaces of matrices. We define the *sinc space* $\mathcal{S}^{m,n}$ to be the set of $m \times (m+n)$ matrices $S = (s_1\ s_2\ \dots\ s_{m+n})$ of column vectors in $\mathbb{R}^m$ such that

$$\int_{\mathbb{R}^m} \left| \prod_{k=1}^{m+n} \operatorname{sinc}(s_k y) \right| dy < \infty,$$

and a function $\sigma : \mathcal{S}^{m,n} \to \mathbb{R}$ by

$$\sigma(S) := \int_{\mathbb{R}^m} \prod_{k=1}^{m+n} \operatorname{sinc}(s_k y)\, dy.$$

Note that $\mathcal{S}^{m,n} \subset \mathbb{R}^{m\times(m+n)}$ and that (by Lemma 2 below) when $n \ge m \ge 1$ a sufficient condition for $S \in \mathcal{S}^{m,n}$ is that two completely disjoint $m \times m$ submatrices of S be non-singular. In fact, a little more work shows that the condition is necessary and sufficient for the integrand to be Lebesgue integrable. Hence a typical non-absolute example is

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{sinc}^3(x-y)\operatorname{sinc}(x+y)\, dy\, dx.$$

Note that when $n \ge m = 1$, the condition is satisfied as soon as all entries are non-zero.

We correspondingly define the *polyhedron space* $\mathcal{P}^{m,n}$ to be $\mathbb{R}^{n\times(m+n)}$. Thus an element $P \in \mathcal{P}^{m,n}$ is a matrix $(p_1\ p_2\ \dots\ p_{m+n})$ of column vectors in $\mathbb{R}^n$. Next, we define a function $\nu : \mathcal{P}^{m,n} \to \mathbb{R}$ by

$$\nu(P) := \operatorname{Vol}\{x \in \mathbb{R}^n : |p_k x| \le 1 \text{ for } k = 1, 2, \dots, m+n\}.$$

The integral $\sigma(S)$ will be our primary object of study. Observe that $\nu(P)$ is the volume of a convex polyhedron with the symmetry $x \to -x$. The polyhedron has dimension n and represents the region between $m+n$ pairs of parallel n-planes. Note that any such symmetric convex polyhedron may be represented as an element of $\mathcal{P}^{m,n}$ and vice-versa. But to evaluate *all* symmetric volumes we would need to consider improper integrals and we choose not to do so.

In the case studied in [1], we have $m = 1$ and are ultimately making the following association:

$$\int_0^\infty \prod_{k=0}^{n} \operatorname{sinc}(a_k x)\, dx = \frac{\pi}{2a_0} \frac{\operatorname{Vol}(P_n)}{2^n a_1 a_2 \dots a_n} = \frac{\pi}{2a_0} \frac{\operatorname{Vol}(Q_n)}{\operatorname{Vol}(H_n)}.$$

Here $a_0, a_1, \dots, a_n > 0$, and we denote the polyhedra $P_n := \{(x_1, x_2, \dots, x_n) \mid -a_0 \le \sum_{k=1}^n x_k \le a_0, -a_k \le x_k \le a_k \text{ for } k = 1, 2, \dots, n\}$, $Q_n := \{(x_1, x_2, \dots, x_n) \mid -a_0 \le \sum_{k=1}^n a_k x_k \le a_0, -1 \le x_k \le 1 \text{ for } k = 1, 2, \dots, n\}$, and the cube $H_n := \{(x_1, x_2, \dots, x_n) \mid -1 \le x_k \le 1 \text{ for } k = 1, 2, \dots, n\}$.

3. Two duality theorems

Of the two theorems stated in this section, the first is elementary while the proof of the second requires some results about Fourier transforms.

Theorem 1. *Let M be a non-singular $m \times m$ matrix and $S \in \mathcal{S}^{m,n}$. Then*

$$\sigma(S) = |\det(M)|\sigma(MS).$$

Similarly, if N is a non-singular $n \times n$ matrix and $P \in \mathcal{P}^{m,n}$, then

$$\nu(P) = |\det(N)|\nu(NP).$$

Proof: Both of these statements follow from the change of basis theorem for Lebesgue integrals [3, p. 391]. □

Theorem 2. *Suppose that $n \geq m$ and that the matrix $A := (a_1\ a_2\ \dots\ a_n) \in \mathbb{R}^{m\times n}$ has at least one non-singular $m \times m$ submatrix. Then*

$$\sigma(I^m \mid A) = \frac{\pi^m}{2^n}\nu(I^n \mid A^T) \leq \pi^m$$

with equality if and only if $\max_{1\leq j\leq m} \sum_{i=1}^n |a_{i,j}| \leq 1$ where $a_{i,j}$ is the j-th component of the vector a_i.

Here and subsequently I^r denotes the $r \times r$ identity matrix, A^T the transpose of A, and $(C \mid D)$ the appropriate concatenation of matrices C and D. We defer the proof of Theorem 2 until the end of Section 5. Observe that Theorem 2 shows also that the integral is positive. As an immediate consequence of Theorems 1 and 2 we have more generally:

Corollary 1. *If $n \geq m$, A is a non-singular $m \times m$ matrix, and B is any $m \times n$ matrix having m of its columns linearly independent, then*

$$\sigma(A \mid B) = \frac{\sigma(I^m \mid A^{-1}B)}{|\det(A)|} = \frac{\pi^m}{2^n}\frac{\nu(I^n \mid (A^{-1}B)^T)}{|\det(A)|}.$$

Further, if $n \geq m$, C is a non-singular $n \times n$ matrix, and D is any $n \times m$ matrix such that $C^{-1}D$ has m linearly independent rows, then

$$\nu(C \mid D) = \frac{\nu(I^n \mid C^{-1}D)}{|\det(C)|} = \frac{2^n}{\pi^m}\frac{\sigma(I^m \mid (C^{-1}D)^T)}{|\det(C)|}.$$

4. Further definitions and basic Fourier results

For $a \in \mathbb{R}^m$, define the Borel measure δ_a to be the linear Lebesgue measure restricted to the set $\iota_a := \{x \in \mathbb{R}^m : x = ta, -1 \leq t \leq 1\}$, i.e., for any Borel set $B \subset \mathbb{R}^m$, $\delta_a(B) = \delta_a(B \cap \iota_a)$, and for any locally L_1-integrable complex Borel measurable function f on $\mathbb{R}^m$,

$$\int_{\mathbb{R}^m} f(x)\delta_a(dx) = \int_{-1}^{1} f(ta)\,dt.$$

Hence

$$\int_{\mathbb{R}^m} e^{ixy}\delta_a(dx) = \int_{-1}^{1} e^{itay}\,dt = 2\operatorname{sinc}(ay).$$

Further, for $b \in \mathbb{R}^m$, and the convolution measure $\lambda := \delta_a * \delta_b$, we have, by definition and application of Fubini's theorem [3, p. 352], that

$$\int_{\mathbb{R}^m} f(x)\,\lambda(dx) = \int_{\mathbb{R}^m} \delta_a(dx) \int_{\mathbb{R}^m} f(x+w)\delta_b(dw),$$

so that

$$\begin{aligned}\int_{\mathbb{R}^m} e^{ixy}\lambda(dx) &= \int_{\mathbb{R}^m} \delta_a(dx) \int_{\mathbb{R}^m} e^{iy(x+w)}\delta_b(dw)\\ &= \int_{\mathbb{R}^m} e^{ixy}\delta_a(dx) \int_{\mathbb{R}^m} e^{iwy}\delta_b(dw) = 4\operatorname{sinc}(ay)\operatorname{sinc}(by),\end{aligned}$$

and in general, if $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ and

$$\mu := \delta_{a_1} * \delta_{a_2} * \cdots * \delta_{a_n},$$

then

$$\int_{\mathbb{R}^m} e^{ixy}\mu(dx) = 2^n \prod_{k=1}^{n} \operatorname{sinc}(a_k y). \tag{1}$$

Also, for any Borel set B in $\mathbb{R}^m$ we have that

$$\begin{aligned}\int_B \lambda(dx) &= \int_{\mathbb{R}^m} \chi_B(x)\,\lambda(dx) = \int_{\mathbb{R}^m} \delta_a(dx) \int_{\mathbb{R}^m} \chi_B(x+w)\delta_b(dw)\\ &= \int_{\mathbb{R}^m} \delta_a(dx) \int_{-1}^{1} \chi_B(x+t_2 b)\,dt_2 = \int_{-1}^{1} dt_1 \int_{-1}^{1} \chi_B(t_1 a + t_2 b)\,dt_2\\ &= \int_{H^2} \chi_B(t_1 a + t_2 b)\,dt\end{aligned}$$

and so if $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$, $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$ and $\mu = \delta_{a_1} * \delta_{a_2} * \cdots * \delta_{a_n}$, then

$$\int_B \mu(dx) = \int_{H^n} \chi_B(t_1 a_1 + t_2 a_2 + \cdots + t_n a_n)\,dt, \tag{2}$$

and so, for the hypercube $H^m := [-1, 1]^m$ in place of B, we have that

$$\int_{H^m} \mu(dx) = \int_{H^n} \chi_{H^m}(t_1 a_1 + t_2 a_2 + \cdots + t_n a_n)\, dt = \nu(I^n \mid A^T), \tag{3}$$

in the notation of Section 2 with A the $m \times n$ matrix $(a_1\ a_2\ \ldots\ a_n)$.

Lemma 1. *Suppose that $n \geq m$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ with the $m \times m$ matrix $A := (a_1 \ldots a_m)$ non-singular. Let $\mu_r := \delta_{a_1} * \delta_{a_2} * \cdots * \delta_{a_r}$. Then, for any Borel set $B \subset H^m$,*

(i) $\mu_m(B) = \int_B \frac{\chi_{H^m}((A^T)^{-1}x)}{|\det A|} dx$,

(ii) $\mu_n(B) = \int_B \phi_n(x) dx$, *where ϕ_n is bounded real-valued non-negative Borel measurable function supported on a bounded set in $\mathbb{R}^m$.*

Proof: Part (i) follows from (3) with $m = n$ by the change of basis theorem for integrals. For part (ii), observe that the result is true with $\phi_m(x) := \frac{\chi_{H^m}((A^T)^{-1}x)}{|\det A|}$. Now define

$$\phi_{m+1}(x) := \int_{\mathbb{R}^m} \phi_m(x - y)\delta_{a_{m+1}}(dy) = \int_{-1}^{1} \phi_m(x - t a_{m+1})\, dt$$

which is evidently non-negative, bounded and of bounded support. Further, for any Borel set B in $\mathbb{R}^m$,

$$\begin{aligned}\int_B \phi_{m+1}(x)\, dx &= \int_{\mathbb{R}^m} \chi_B(x)\phi_{m+1}(x)\, dx = \int_{\mathbb{R}^m} \delta_{s_{m+1}}(dx) \int_{\mathbb{R}^m} \chi_B(x + w)\phi_m(w)\, dw \\ &= \int_{\mathbb{R}^m} \delta_{s_{m+1}}(dx) \int_{\mathbb{R}^m} \chi_B(x + w)\mu_m(dw) = \int_{\mathbb{R}^m} \chi_B(x)\mu_{m+1}(dx) \\ &= \int_B \mu_{m+1}(dx),\end{aligned}$$

and this establishes (ii) when $n = m + 1$.

Continuing in this way we find that (ii) holds in generality. □

Lemma 2. *Suppose that $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ with $n \geq m$, and that the $m \times m$ matrix $A := (a_1\ a_2\ \ldots\ a_m)$ is non-singular. Then*

$$\int_{\mathbb{R}^m} \prod_{k=1}^{n} \operatorname{sinc}^2(a_k y)\, dy < \infty.$$

Proof: By the change of basis theorem for integrals, we get that

$$\int_{\mathbb{R}^m} \prod_{k=1}^{n} \operatorname{sinc}^2(a_k y)\, dy \leq \int_{\mathbb{R}^m} \prod_{k=1}^{m} \operatorname{sinc}^2(a_k y)\, dy = \int_{\mathbb{R}^m} \frac{1}{|\det A|} \prod_{k=1}^{m} \operatorname{sinc}^2(x_k)\, dx < \infty.$$

□

5. Fourier transforms and sinc integrals in $\mathbb{R}^m$

We first state some standard results about the *Fourier transform* (FT) which may be found in texts such as [3, pp. 358–362].

The FT of a given function $f \in L_2(\mathbb{R}^m)$ is the function $\hat{f}$ that is the L_2-limit as $\rho \to \infty$ of

$$c_\rho(x) := \frac{1}{(\sqrt{2\pi})^m} \int_{[-\rho,\rho]^m} f(y)\, e^{-ixy}\, dy, \text{ i.e. } \int_{\mathbb{R}^m} |c_\rho(x) - \hat{f}(x)|^2\, dx \to 0 \text{ as } \rho \to \infty.$$

This function $\hat{f}$ exists, is unique apart from sets of zero Lebesgue measure, and $\hat{f} \in L_2(\mathbb{R}^m)$. Further, if $\hat{f}_1, \hat{f}_2$ are FTs of $f_1, f_2 \in L_2(\mathbb{R}^m)$ and $f_1, \hat{f}_1$ are real, then we have the following version of *Parseval's theorem*:

$$\int_{\mathbb{R}^m} f_1(x) f_2(x)\, dx = \int_{\mathbb{R}^m} \hat{f}_1(x) \hat{f}_2(x)\, dx.$$

Lemma 3. *Suppose that* $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ *with* $n \geq m$ *and the* $m \times m$ *matrix* $A := (a_1 \ldots a_m)$ *non-singular. Let*

$$f_1(y) := \prod_{k=1}^{m} \operatorname{sinc}(y_k), \quad f_2(y) := \prod_{k=1}^{n} \operatorname{sinc}(a_k y).$$

Then, for $\mu := \delta_{a_1} * \delta_{a_2} * \cdots * \delta_{a_n}$ *and* $H^m := [-1, 1]^m$,

$$\sigma(I^m \mid A) = \int_{\mathbb{R}^m} f_1(y) f_2(y)\, dy = \frac{\pi^m}{2^n} \int_{\mathbb{H}^m} \mu(dy).$$

Proof: By Lemma 1 and (1) we have that

$$\frac{1}{(\sqrt{2\pi})^m} \int_{\mathbb{R}^m} e^{ixy} \phi(x)\, dx = \frac{2^n}{(\sqrt{2\pi})^m} f_2(y) \text{ where } \phi \in L_1(\mathbb{R}^m) \cap L_2(\mathbb{R}^m),$$

$$\text{i.e., } \frac{2^n}{(\sqrt{2\pi})^m} f_2(y) = \hat{\phi}(-y) \text{ for } y \in \mathbb{R}^m.$$

It follows, by [3, p. 362, Exercise 13(w)], that $\hat{f}_2 = 2^{-n}(\sqrt{2\pi})^m \phi$, and likewise we get that $\hat{f}_1 = 2^{-m}(\sqrt{2\pi})^m \psi$ where $\psi(y) = \chi_{H^m}(y)$. Since $f_1, f_2 \in L_2(\mathbb{R}^m)$ by Lemma 2 or [3, p. 362. Exercise 13(w)], we can apply Parseval's theorem to get that

$$\int_{\mathbb{R}^m} f_1(y) f_2(y)\, dy = \frac{(2\pi)^m}{2^{m+n}} \int_{\mathbb{R}^m} \psi(y)\phi(y)\, dy = \frac{\pi^m}{2^n} \int_{\mathbb{H}^m} \mu(dy).$$

□

Proof of Theorem 2: Combining (3) and Lemma 3 we obtain that

$$\sigma(I^m \mid A) = \frac{\pi^m}{2^n} \nu(I^n \mid A^T).$$

The rest of the theorem follows readily from the definition of $\nu(I^n \mid A^T)$. □

6. A partial fraction decomposition

Before evaluating the sinc integrals, we need to introduce some multilinear algebra so as to derive an appropriate partial fractional decomposition. Our precise goal in this section is to prove Theorem 3 so as to obtain Corollary 3 below. Theorem 3 is a multilinear analogue of *Cramer's rule* that computes a change of basis for tensors. It reduces to the traditional version of Cramer's rule in the case $n = 1$.

Let I denote the set of $\binom{m+n-1}{n}$ integer sequences $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_m\}$ satisfying $\kappa_1 = 1 < \kappa_2 < \cdots < \kappa_m \le m+n$, and let I' denote the set of integer sequences $\kappa' = \{\kappa'_1, \kappa'_2, \ldots, \kappa'_n\}$ satisfying $1 < \kappa'_1 < \kappa'_2 < \cdots < \kappa'_n \le m+n$. Let κ^c denote the complement of κ in $\{1, 2, \ldots, m+n\}$. Note that the complement operator ^{c} is a bijection between I and I'.

For $t_1, t_2, \ldots, t_n \in \mathbb{R}^m$, $S = (s_1\ s_2\ \ldots\ s_{m+n}) \in \mathbb{R}^{m\times(m+n)}$ and $y \in \mathbb{R}^m$, let

$$\beta_\kappa(t_1, t_2, \ldots, t_n) := \prod_{j=1}^{n} \frac{\det\big(t_j\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\big)}{\det\big(s_{\kappa^c_j}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\big)} \quad \text{for } \kappa \in I,$$

$$B(t_1, t_2, \ldots, t_n) := \left\{ \sum_{\kappa\in I} \beta_\kappa(t_1, t_2, \ldots, t_n) \left(\prod_{j=1}^{n} \big(s_{\kappa^c_j} y\big) \right) \right\} - \prod_{j=1}^{n} (t_j y).$$

Observe that B is a symmetric n-linear form.

Lemma 4. *Let k' be a fixed element of I', and let the matrix $S = (s_1\ s_2\ \ldots\ s_{m+n}) \in \mathbb{R}^{m\times(m+n)}$ have every $m \times m$ submatrix non-singular. Then, for all $\kappa \in I$,*

$$\beta_\kappa := \beta_\kappa\big(s_{\kappa'_1}, \ldots, s_{\kappa'_n}\big) = \prod_{j=1}^{n} \frac{\det\big(s_{\kappa'_j}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\big)}{\det\big(s_{\kappa^c_j}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\big)} = \delta_{\kappa',\kappa^c}.$$

Proof: Clearly, $\beta_\kappa = 0$ if $\kappa'_j = \kappa_i$ for some $j \in \{1, 2, \ldots, m\}$ and $i \in \{2, \ldots, m\}$, since this will cause s_{κ_i} to be repeated in some numerator determinant. More precisely, $\beta_\kappa \neq 0$ if and only if $\kappa' = \kappa^c$. This is because κ^c is the only n-element subsequence of $\{1, 2, \ldots, m+n\}$ which is disjoint from κ. Consequently, no vectors in the numerator determinants are repeated, and since each m-element subset of S is linearly independent by hypothesis, the numerator determinant is non-zero. Moreover, when $\kappa' = \kappa^c$, the numerator determinant and the denominator determinant are equal and so $\beta_\kappa = 1$. This establishes that $\beta_\kappa = \delta_{\kappa',\kappa^c}$. □

Corollary 2. *For all $\kappa' \in I'$, $B(s_{\kappa'_1}, s_{\kappa'_2}, \ldots, s_{\kappa'_n}) = 0$.*

Proof: By Lemma 4,

$$\begin{aligned} B\big(s_{\kappa'_1}, s_{\kappa'_2}, \ldots, s_{\kappa'_n}\big) &= \left\{ \sum_{\kappa\in I} \delta_{\kappa',\kappa^c} \left(\prod_{j=1}^{n} \big(s_{\kappa^c_j} y\big) \right) \right\} - \prod_{j=1}^{n} \big(s_{\kappa'_j} y\big) \\ &= \prod_{j=1}^{n} \big(s_{\kappa'_j} y\big) - \prod_{j=1}^{n} \big(s_{\kappa'_j} y\big) = 0. \end{aligned}$$

□

Theorem 3. *Let the matrix $S = (s_1\, s_2\, \ldots\, s_{m+n}) \in \mathbb{R}^{m\times(m+n)}$ have every $m\times m$ submatrix non-singular. Then, for any n vectors $t_1, t_2, \ldots, t_n \in \mathbb{R}^m$ and any $y \in \mathbb{R}^m$,*

$$\prod_{j=1}^{n}(t_j y) = \sum_{\kappa\in I}\left(\prod_{j=1}^{n}\frac{\det\left(t_j\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\right)}{\det\left(s_{\kappa_j^c}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\right)}\right)\prod_{j=1}^{n}\left(s_{\kappa_j^c} y\right).$$

Proof: It suffices to prove that $B(t_1, t_2, \ldots, t_n) = 0$. We do this by expanding each of the n variables sequentially in terms of column vectors of S as follows:

$$\begin{aligned}
B(t_1, t_2, \ldots, t_n) &= B\left(\sum_{i_1=2}^{m+1} c_{1,i_1}s_{i_1}, t_2, \ldots, t_n\right)\\
&= \sum_{i_1=2}^{m+1} c_{1,i_1} B\left(s_{i_1}, t_2, \ldots, t_n\right)\\
&= \sum_{i_1=2}^{m+1} c_{1,i_1} B\left(s_{i_1}, \sum_{\substack{i_2=2\\ i_2\neq i_1}}^{m+2} c_{2,i_2}s_{i_2}, t_3, \ldots, t_n\right)\\
&= \sum_{i_1=2}^{m+1}\sum_{\substack{i_2=2\\ i_2\neq i_1}}^{m+2} c_{1,i_1}c_{2,i_2} B\left(s_{i_1}, s_{i_2}, t_3, \ldots, t_n\right)\\
&\ \ \vdots\\
&= \sum_{i_1=2}^{m+1}\sum_{\substack{i_2=2\\ i_2\neq i_1}}^{m+2}\cdots\sum_{\substack{i_n=2\\ i_n\neq i_1\\ \vdots\\ i_n\neq i_{n-1}}}^{m+n}\left(\prod_{j=1}^{n} c_{j,i_j}\right) B\left(s_{i_1}, s_{i_2}, \ldots, s_{i_n}\right) = 0,
\end{aligned}$$

since each $B(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ vanishes. This is a consequence of the symmetry of B combined with Corollary 2, because the vectors $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ are all distinct and are a permutation of $s_{\kappa'_1}, s_{\kappa'_2}, \ldots, s_{\kappa'_n}$ for some $\kappa' \in I'$. □

We may now specialize this result to obtain the partial fraction decomposition needed in the next section.

Corollary 3. *If every $m \times m$ submatrix of the matrix $S = (s_1\ s_2\ \ldots\ s_{m+n}) \in \mathbb{R}^{m\times(m+n)}$ is non-singular, then, for every $y \in \mathbb{R}^m$,*

$$\prod_{i=1}^{m+n}(s_i y)^{-1} = (s_1 y)^{-n}\sum_{\kappa\in I}\alpha_\kappa\prod_{j=1}^{m}\left(s_{\kappa_j} y\right)^{-1},$$

where

$$\alpha_\kappa := \frac{\det\left(s_{\kappa_1}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\right)^n}{\prod_{j=1}^n \det\left(s_{\kappa_j^c}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\right)}.$$

Proof: Taking $t_1 = t_2 = \cdots = t_n = s_1$ in Theorem 3, we get the identity

$$(s_1 y)^n = \sum_{\kappa \in I} \left(\frac{\det\left(s_{\kappa_1}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\right)^n}{\prod_{j=1}^n \det\left(s_{\kappa_j^C}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m}\right)} \right) \prod_{j=1}^n \left(s_{\kappa_j^c} y\right).$$

Divide both sides by $(s_1 y)^n \prod_{i=1}^{m+n} (s_i y)$ to produce the desired identity. □

7. Evaluating the sinc integrals

In all that follows let $g_{r,s}$ denote the *characteristic function*

$$g_{r,s} := \chi_{(-r,-s]\cup[s,r)}$$

for $0 \le s < r \le \infty$.

Lemma 5. *For $a_1, a_2, \ldots, a_m \in \mathbb{R}$, $0 < \eta < \rho < \infty$, $0 < \nu < \rho < \infty$ and $n \ge 0$,*

$$\begin{aligned}
&\int_{\mathbb{R}^m} \frac{1}{u_1^n} \frac{\cos\left(a_1u_1 + a_2u_2 + \cdots + a_mu_m - \frac{\pi}{2}(m+n)\right)}{u_1u_2\ldots u_m} \\
&\quad \times g_{\rho,\eta}(u_1) g_{\rho,\nu}(u_2) \ldots g_{\rho,\nu}(u_m)\, du_1\, du_2 \ldots du_m \\
&\quad = \left(\int_{\mathbb{R}} \frac{1}{u^{n+1}} \cos\left(a_1 u - \frac{\pi}{2}(1+n)\right) g_{\rho,\eta}(u)\, du \right) \left(\prod_{j=2}^m \int_{\mathbb{R}} g_{\rho,\nu}(u) \frac{\sin(a_j u)}{u}\, du \right).
\end{aligned}$$

Proof: Observe that all the integrals are absolutely convergent, and that the result is trivially true for $m = 1$. Further, for $m \ge 2$,

$$\begin{aligned}
&\int_{\mathbb{R}^m} \frac{1}{u_1^n} \frac{\cos\left(a_1u_1 + a_2u_2 + \cdots + a_mu_m - \frac{\pi}{2}(m+n)\right)}{u_1u_2\ldots u_m} \\
&\quad \times g_{\rho,\eta}(u_1) g_{\rho,\nu}(u_2) \ldots g_{\rho,\nu}(u_m)(u_m)\, du_1\, du_2 \ldots du_m \\
&= \int_{\mathbb{R}^m} \frac{1}{u_1^n} \frac{\sin\left(a_mu_m + a_1u_1 + a_2u_2 + \cdots + a_{m-1}u_{m-1} - \frac{\pi}{2}(m-1+n)\right)}{u_1u_2\ldots u_m} \\
&\quad \times g_{\rho,\eta}(u_1) g_{\rho,\nu}(u_2) \ldots g_{\rho,\nu}(u_m)\, du_1\, du_2 \ldots du_m \\
&= \int_{\mathbb{R}^{m-1}} \frac{1}{u_1^n} \frac{g_{\rho,\eta}(u_1) g_{\rho,\nu}(u_2) \ldots g_{\rho,\nu}(u_{m-1})}{u_1u_2\ldots u_{m-1}}\, du_1\, du_2 \ldots du_{m-1} \\
&\quad \times \int_{\mathbb{R}} \frac{\sin\left(a_mu_m + a_1u_1 + a_2u_2 + \cdots + a_{m-1}u_{m-1} - \frac{\pi}{2}(m-1+n)\right)}{u_m} g_{\rho,\nu}(u_m)\, du_m
\end{aligned}$$

$$= \int_{\mathbb{R}^{m-1}} \frac{1}{u_1^n} \frac{\cos\big(a_1u_1 + a_2u_2 + \cdots + a_{m-1}u_{m-1} - \frac{\pi}{2}(m-1+n)\big)}{u_1u_2 \ldots u_{m-1}}$$
$$\times g_{\rho,\eta}(u_1)g_{\rho,\nu}(u_2) \ldots g_{\rho,\nu}(u_{m-1})\, du_1\, du_2 \ldots du_{m-1}$$
$$\times \int_{\mathbb{R}} \frac{\sin(a_m u_m)}{u_m} g_{\rho,\nu}(u_m)\, du_m.$$

since $\cos(a_m u_m)g_{\rho,\nu}(u_m)/u_m$ is an odd function of u_m. Continuing in this way we obtain the desired result. □

For our central result of the section we need some further notation:

Notation. Given a matrix $(s_1\ s_2\ \ldots\ s_{n+m}) \in \mathbb{R}^{m\times(m+n)}$ with all its $m \times m$ submatrices non-singular, we denote $\Gamma := \{-1, 1\}^{\{2,3,\ldots,m+n\}}$, and for each $\gamma \in \Gamma$, we define

$$s_\gamma := s_1 + \sum_{j=2}^{m+n} \gamma_j s_j, \quad \epsilon_\gamma := \prod_{j=2}^{m+n} \gamma_j.$$

For each $\kappa \in I$, denote the matrix $(s_{\kappa_1}\ s_{\kappa_2}\ \ldots\ s_{\kappa_m})$ by S_κ and the j-th component of $S_\kappa^{-1}s_\gamma$ by $s_{\kappa,\gamma;j}$. We then have that, for any $y \in \mathbb{R}^m$,

$$s_\gamma y = \sum_{j=1}^{m} s_{\kappa,\gamma;j}\big(s_{\kappa_j} y\big).$$

and

$$\alpha_\kappa := \frac{\det\big(s_{\kappa_1}\, s_{\kappa_2}\ \ldots\ s_{\kappa_m}\big)^n}{\prod_{j=1}^{n} \det\big(s_{\kappa_j^c}\, s_{\kappa_2}\ \ldots\ s_{\kappa_m}\big)},$$

is as in Corollary 3.

Our aim now is to prove the following surprisingly explicit closed form evaluation in which

$$\operatorname{sgn}(t) := \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0. \end{cases}$$

When $m = 1$ this reduces to the evaluation obtained in [1].

Theorem 4. *Fix notation as immediately above, and suppose that $n \geq m \geq 1$. Suppose that $S = (s_1\, s_2\ \ldots\ s_{n+m})$ is in $\mathbb{R}^{m\times(m+n)}$, and that every $m \times m$ submatrix of S is non-singular. Then*

$$\sigma(S) := \int_{\mathbb{R}^m} \prod_{i=1}^{m+n} \operatorname{sinc}(s_i y)\, dy_1\, dy_2 \ldots dy_m$$
$$= \frac{1}{2^{n-1}n!}\left(\frac{\pi}{2}\right)^m \sum_{\kappa\in I} \frac{\alpha_\kappa}{|\det S_\kappa|} \sum_{\gamma\in\Gamma} \epsilon_\gamma (s_{\kappa,\gamma;1})^n \prod_{j=1}^{m} \operatorname{sgn}(s_{\kappa,\gamma;j}).$$

Proof: Observe that, by Lemma 2, the integral is absolutely convergent, so that $S \in \mathcal{S}^{m\cdot n}$. By Corollary 3, we have that

$$\sigma(S) = \int_{\mathbb{R}^m} \sum_{\kappa \in I} \alpha_\kappa \left(\prod_{i=1}^{m+n} \sin(s_i y) \right) (s_1 y)^{-n} \left(\prod_{j=1}^{m} \left(s_{\kappa_j} y\right)^{-1} \right) dy_1\, dy_2 \ldots dy_m.$$

We don't deal directly with this integral, but with its better behaved approximant

$$\sigma_{\eta,\nu}(S) := \int_{\mathbb{R}^m} \sum_{\kappa \in I} \alpha_\kappa \left(\prod_{i=1}^{m+n} \sin(s_i y) \right) (s_1 y)^{-n-1} g_{\rho_\kappa,\eta}(s_1 y)$$
$$\times \left(\prod_{j=2}^{m} \left(s_{\kappa_j} y\right)^{-1} g_{\rho_\kappa,\nu}\left(s_{\kappa_j} y\right) \right) dy_1\, dy_2 \ldots dy_m,$$

where $0 < \eta < \rho_\kappa < \infty$, $0 \le \nu < \rho_\kappa < \infty$ and $g_{r,s}$ is the characteristic function defined immediately above Lemma 5. Let

$$f_{\kappa,\eta,\nu}(y) := (s_1 y)^{-n-1} g_{\rho_\kappa,\eta}(s_1 y) \left(\prod_{i=1}^{m+n} \sin(s_i y) \right) \left(\prod_{j=2}^{m} \left(s_{\kappa_j} y\right)^{-1} g_{\rho_\kappa,\nu}\left(s_{\kappa_j} y\right) \right).$$

Since $|f_{\kappa,\eta,\nu}(y)| \le |f_{\kappa,\eta,0}(y)|$, while the latter has bounded support, it follows that

$$\int_{\mathbb{R}^m} |f_{\kappa,\eta,\nu}(y)|\, dy < \infty \text{ for } \nu \ge 0,$$

so that

$$\sigma_{\eta,\nu}(S) := \int_{\mathbb{R}^m} \sum_{\kappa \in I} \alpha_\kappa f_{\kappa,\eta,\nu}(y)\, dy = \sum_{\kappa \in I} \alpha_\kappa \int_{\mathbb{R}^m} f_{\kappa,\eta,\nu}(y)\, dy,$$

and, by dominated convergence,

$$\sigma_{\eta,0}(S) = \lim_{\nu \to 0+} \sigma_{\eta,\nu}(S).$$

By [1, Thm. 2(i)], we have that

$$\prod_{j=1}^{m+n} \sin(s_j y) = 2^{1-m-n} \sum_{\gamma \in \Gamma} \epsilon_\gamma \cos\left(s_\gamma y - \frac{\pi}{2}(m+n) \right),$$

and hence that

$$\int_{\mathbb{R}^m} f_{\kappa,\eta,\nu}(y)\, dy = 2^{1-m-n} \sum_{\gamma \in \Gamma} \epsilon_\gamma \int_{\mathbb{R}^m} (s_1 y)^{-n-1} g_{\rho_\kappa,\eta}(s_1 y) \cos\left(s_\gamma y - \frac{\pi}{2}(m+n) \right)$$
$$\times \left(\prod_{j=2}^{m} \left(s_{\kappa_j} y\right)^{-1} g_{\rho_\kappa,\nu}\left(s_{\kappa_j} y\right) \right) dy_1\, dy_2 \ldots dy_m.$$

Make the change of variables $u_j := s_{\kappa_j} y$. Observing that $s_\gamma y = s_{\kappa,\gamma;1}u_1 + s_{\kappa,\gamma;2}u_2 + \cdots + s_{\kappa,\gamma;m}u_m$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^m} f_{\kappa,\eta,\nu}(y)dy &= \frac{2^{1-m-n}}{|\det S_\kappa|}\sum_{\gamma\in\Gamma}\epsilon_\gamma \int_{\mathbb{R}^m}(u_1)^{-n-1}g_{\rho_\kappa,\eta}(u_1)\\ &\quad\times\cos\left(s_{\kappa,\gamma;1}u_1 + s_{\kappa,\gamma;2}u_2 + \cdots + s_{\kappa,\gamma;m}u_m - \frac{\pi}{2}(m+n)\right)\\ &\quad\times\left(\prod_{j=2}^m (u_j)^{-1}g_{\rho_\kappa,\nu}(u_j)\right)du_1\,du_2\ldots du_m\\ &= \frac{2^{1-m-n}}{|\det S_\kappa|}\sum_{\gamma\in\Gamma}\epsilon_\gamma\left(\int_{\mathbb{R}}\frac{1}{u^{n+1}}\cos\left(s_{\kappa,\gamma;1}u - \frac{\pi}{2}(1+n)\right)g_{\rho_\kappa,\eta}(u)\,du\right)\\ &\quad\times\left(\prod_{j=2}^m\int_{\mathbb{R}} g_{\rho_\kappa,\nu}(u)\frac{\sin(s_{\kappa,\gamma;j}u)}{u}\,du\right),\end{aligned}$$

by Lemma 5. Letting $\nu \to 0+$, we see that

$$\begin{aligned}\int_{\mathbb{R}^m} f_{\kappa,\eta,0}(y)\,dy &= \frac{2^{1-m-n}}{|\det S_\kappa|}\sum_{\gamma\in\Gamma}\epsilon_\gamma\left(\int_{\mathbb{R}}\frac{1}{u^{n+1}}\cos\left(s_{\kappa,\gamma;1}u - \frac{\pi}{2}(1+n)\right)g_{\rho_\kappa,\eta}(u)\,du\right)\\ &\quad\times\left(\prod_{j=2}^m\int_{\mathbb{R}} g_{\rho_\kappa,0}(u)\frac{\sin(s_{\kappa,\gamma;j}u)}{u}\,du\right).\end{aligned}$$

Denote the m-dimensional hypercube $[-\rho,\rho]^m$ by U_ρ. We fix a reference member $\lambda \in I$, and a corresponding parameter ρ_λ, which we will later increase to infinity. Define

$$Y_{\rho_\lambda} := \left(S_\lambda^T\right)^{-1}U_{\rho_\lambda}.$$

Then, by what was proved above, we have that

$$\begin{aligned}\int_{Y_{\rho_\lambda}} f_{\lambda,\eta,\nu}(y)\,dy &= \frac{2^{1-m-n}}{|\det S_\lambda|}\sum_{\gamma\in\Gamma}\epsilon_\gamma\int_{U_{\rho_\lambda}}(u_1)^{-n-1}g_{\infty,\eta}(u_1)\\ &\quad\times\cos\left(s_{\lambda,\gamma;1}u_1 + s_{\lambda,\gamma;2}u_2 + \cdots + s_{\lambda,\gamma;m}u_m - \frac{\pi}{2}(m+n)\right)\\ &\quad\times\left(\prod_{j=2}^m (u_j)^{-1}g_{\infty,\nu}(u_j)\right)du_1\,du_2\ldots du_m.\end{aligned}$$

Now for each $\kappa \in I$, let

$$V_{\kappa,\lambda} := S_\kappa^T Y_{\rho_\lambda}.$$

Observe that $V_{\kappa,\lambda}$ is a parallelopiped with shape that does not vary with ρ_λ. Thus there must be a number $c_{\kappa,\lambda} > 1$ (independent of ρ_λ) such that, for

$$\rho_\kappa := c_{\kappa,\lambda}\rho_\lambda, \quad \tilde{\rho}_\kappa := \frac{\rho_\lambda}{c_{\kappa,\lambda}},$$

we have that

$$U_{\tilde{\rho}_\kappa} \subset V_{\kappa,\lambda} \subset U_{\rho_\kappa},$$

and hence that

$$Y_{\rho_\kappa} := \left(S_\kappa^T\right)^{-1} U_{\rho_\kappa} \supset Y_{\rho_\lambda}.$$

[Here and elsewhere we use the fact that each of the finitely many matrices $\{S_\kappa : \kappa \in I\}$ is invertible.] □

Observe next that

$$\begin{aligned}
&\int_{U_{\rho_\kappa}\setminus U_{\tilde{\rho}_\kappa}} (u_1)^{-n-1} g_{\infty,\eta}(u_1) \left|\cos\left(s_{\kappa,\gamma;1}u_1 + s_{\kappa,\gamma;2}u_2 + \cdots + s_{\kappa,\gamma;m}u_m - \frac{\pi}{2}(m+n)\right)\right| \\
&\quad \times \left(\prod_{j=2}^m (u_j)^{-1} g_{\infty,\nu}(u_j)\right) du_1\, du_2 \ldots du_m \\
&\quad \le \int_{\tilde{\rho}_\kappa}^\infty \frac{du_1}{u_1^{n+1}} \prod_{j=2}^m \int_{\tilde{\rho}_\kappa}^{\rho_\kappa} \frac{du_j}{u_j} = \frac{c_{\kappa,\lambda}^n 2^{m-1} \ln^{m-1}(c_{\kappa,\lambda})}{n\rho_\lambda^n}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{Y_{\rho_\lambda}} f_{\kappa,\eta,\nu}(y)\, dy &= \frac{2^{1-m-n}}{|\det S_\kappa|} \sum_{\gamma\in\Gamma} \epsilon_\gamma \int_{V_{\kappa,\lambda}} (u_1)^{-n-1} g_{\infty,\eta}(u_1) \\
&\quad \times \cos\left(s_{\kappa,\gamma;1}u_1 + s_{\kappa,\gamma;2}u_2 + \cdots + s_{\kappa,\gamma;m}u_m - \frac{\pi}{2}(m+n)\right) \\
&\quad \times \left(\prod_{j=2}^m (u_j)^{-1} g_{\infty,\nu}(u_j)\right) du_1\, du_2 \ldots du_m \\
&= \frac{2^{1-m-n}}{|\det S_\kappa|} \sum_{\gamma\in\Gamma} \epsilon_\gamma \int_{U_{\rho_\kappa}} (u_1)^{-n-1} g_{\infty,\eta}(u_1) \\
&\quad \times \cos\left(s_{\kappa,\gamma;1}u_1 + s_{\kappa,\gamma;2}u_2 + \cdots + s_{\kappa,\gamma;m}u_m - \frac{\pi}{2}(m+n)\right) \\
&\quad \times \left(\prod_{j=2}^m (u_j)^{-1} g_{\infty,\nu}(u_j)\right) du_1\, du_2 \ldots du_m + O\left(\rho_\lambda^{-n}\right) \\
&= \frac{2^{1-m-n}}{|\det S_\kappa|} \sum_{\gamma\in\Gamma} \epsilon_\gamma \left(\int_{\mathbb{R}} \frac{1}{u^{n+1}} \cos\left(s_{\kappa,\gamma;1}u - \frac{\pi}{2}(1+n)\right) g_{\rho_\kappa,\eta}(u)\, du\right) \\
&\quad \times \left(\prod_{j=2}^m \int_{\mathbb{R}} g_{\rho_\kappa,\nu}(u) \frac{\sin(s_{\kappa,\gamma;j}u)}{u}\, du\right) + O\left(\rho_\lambda^{-n}\right),
\end{aligned}$$

and therefore, on letting $\nu \to 0+$, we get

$$\int_{Y_{\rho_\lambda}} f_{\kappa,\eta,0}(y)\,dy = \frac{2^{1-m-n}}{|\det S_\kappa|}\sum_{\gamma\in\Gamma}\epsilon_\gamma\left(\int_{\mathbb{R}}\frac{1}{u^{n+1}}\cos\left(s_{\kappa,\gamma;1}u-\frac{\pi}{2}(1+n)\right)g_{\rho_\kappa,\eta}(u)\,du\right)$$
$$\times\left(\prod_{j=2}^{m}\int_{\mathbb{R}} g_{\rho_\kappa,0}(u)\frac{\sin(s_{\kappa,\gamma;j}u)}{u}\,du\right)+O\big(\rho_\lambda^{-n}\big).$$

Consequently

$$\begin{aligned}
&\int_{Y_{\rho_\lambda}}\sum_{\kappa\in I}\alpha_\kappa f_{\kappa,\eta,0}(y)\,dy\\
&\quad=\int_{Y_{\rho_\lambda}}\sum_{\kappa\in I}\alpha_\kappa(s_1y)^{-n-1}g_{\infty,\eta}(s_1y)\left(\prod_{i=1}^{m+n}\sin(s_iy)\right)\left(\prod_{j=2}^{m}\big(s_{\kappa_j}y\big)^{-1}\right)dy\\
&\quad=\int_{Y_{\rho_\lambda}} g_{\infty,\eta}(s_1y)\prod_{i=1}^{m+n}\operatorname{sinc}(s_iy)\,dy\to\int_{\mathbb{R}^m} g_{\infty,\eta}(s_1y)\prod_{i=1}^{m+n}\operatorname{sinc}(s_iy)\,dy
\end{aligned}\tag{5}$$

as $\rho_\lambda\to\infty$.

But we also have that

$$\begin{aligned}
&\int_{Y_{\rho_\lambda}}\sum_{\kappa\in I}\alpha_\kappa f_{\kappa,\eta,0}(y)\,dy\\
&\quad=\sum_{\kappa\in I}\alpha_\kappa\frac{2^{1-m-n}}{|\det S_\kappa|}\sum_{\gamma\in\Gamma}\epsilon_\gamma\left(\int_{\mathbb{R}}\frac{1}{u^{n+1}}\cos\left(s_{\kappa,\gamma;1}u-\frac{\pi}{2}(1+n)\right)g_{\rho_\kappa,\eta}(u)\,du\right)\\
&\qquad\times\left(\prod_{j=2}^{m}\int_{\mathbb{R}} g_{\rho_\kappa,0}(u)\frac{\sin(s_{\kappa,\gamma;j}u)}{u}\,du\right)+O\big(\rho_\lambda^{-n}\big)\\
&\quad\to\frac{1}{2^n}\left(\frac{\pi}{2}\right)^{m-1}\sum_{\kappa\in I}\frac{\alpha_\kappa}{|\det S_k|}\sum_{\gamma\in\Gamma}\epsilon_\gamma\left(\prod_{j=2}^{m}\operatorname{sgn}(s_{\kappa,\gamma;j})\right)\\
&\qquad\times\int_{\mathbb{R}}\frac{1}{u^{n+1}}\cos\left(s_{\kappa,\gamma;1}u-\frac{\pi}{2}(1+n)\right)g_{\infty,\eta}(u)\,du
\end{aligned}\tag{6}$$

as $\rho_\lambda\to\infty$.

It follows from (5) and (6) that

$$\int_{\mathbb{R}^m} g_{\infty,\eta}(s_1y)\prod_{i=1}^{m+n}\operatorname{sinc}(s_iy)\,dy=\frac{1}{2^n}\left(\frac{\pi}{2}\right)^{m-1}\sum_{\kappa\in I}\frac{\alpha_\kappa}{|\det S_k|}\sum_{\gamma\in\Gamma}\epsilon_\gamma\left(\prod_{j=2}^{m}\operatorname{sgn}(s_{\kappa,\gamma;j})\right)$$
$$\times\int_{\mathbb{R}}\frac{1}{u^{n+1}}\cos\left(s_{\kappa,\gamma;1}u-\frac{\pi}{2}(1+n)\right)g_{\infty,\eta}(u)\,du.\tag{7}$$

By one-dimensional partial integration, it is now easy to establish that

$$\begin{aligned} C_{\kappa,\gamma}(\eta) &:= \int_{\mathbb{R}} \frac{1}{u^{n+1}} \cos\left(s_{\kappa,\gamma;1}u - \frac{\pi}{2}(1+n)\right) g_{\infty,\eta}(u)\, du \\ &= \frac{1}{n!}\sum_{r=1}^{n}(r-1)!\frac{\phi_{\kappa,\gamma}^{(n-r)}(\eta)}{\eta^r} + \frac{2(s_{\kappa,\gamma;1})^n}{n!}\int_{\eta}^{\infty}\frac{\sin(s_{\kappa,\gamma;1}u)}{u}\, du. \end{aligned}$$

where

$$\begin{aligned} \phi_{\kappa,\gamma}(\eta) &:= \cos\left(s_{\kappa,\gamma;1}\eta - \frac{\pi}{2}(1+n)\right) + (-1)^{n+1}\cos\left(s_{\kappa,\gamma;1}\eta + \frac{\pi}{2}(1+n)\right) \\ &= \sin\left(s_{\kappa,\gamma;1}\eta - \frac{\pi}{2}n)\right) + (-1)^n \sin\left(s_{\kappa,\gamma;1}\eta + \frac{\pi}{2}n\right), \end{aligned}$$

whence

$$\begin{aligned} \phi_{\kappa,\gamma}^{(n-r)}(\eta) &= (s_{\kappa,\gamma;1})^{n-r}\left\{\sin\left(s_{\kappa,\gamma;1}\eta - \frac{\pi}{2}r\right) + (-1)^r \sin\left(s_{\kappa,\gamma;1}\eta + \frac{\pi}{2}r\right)\right\} \\ &= \begin{cases} 2(s_{\kappa,\gamma;1})^{n-r}(-1)^{r/2}\sin(s_{\kappa,\gamma;1}\eta) & \text{if } r \text{ is even} \\ 2(s_{\kappa,\gamma;1})^{n-r}(-1)^{(1+r)/2}\cos(s_{\kappa,\gamma;1}\eta) & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

It follows from (7), by dominated convergence of the left-hand integral, that

$$\begin{aligned} &\lim_{\eta\to 0+}\frac{1}{2^n}\left(\frac{\pi}{2}\right)^{m-1}\sum_{\kappa\in I}\frac{\alpha_\kappa}{|\det S_k|}\sum_{\gamma\in\Gamma}\epsilon_\gamma\left(\prod_{j=2}^{m}\operatorname{sgn}(s_{\kappa,\gamma;j})\right)C_{\kappa,\gamma}(\eta) \\ &= \int_{\mathbb{R}^m}\prod_{i=1}^{m+n}\operatorname{sinc}(s_i y)\, dy = \sigma(S), \end{aligned}$$

and hence that

$$\sigma(S) - \frac{1}{2^{n-1}n!}\left(\frac{\pi}{2}\right)^m \sum_{\kappa\in I}\frac{\alpha_\kappa}{|\det S_\kappa|}\sum_{\gamma\in\Gamma}\epsilon_\gamma(s_{\kappa,\gamma;1})^n\prod_{j=1}^{m}\operatorname{sgn}(s_{\kappa,\gamma;j}) = \lim_{\eta\to 0+}F(\eta). \tag{8}$$

where

$$\begin{aligned} F(\eta) &:= \frac{1}{2^n n!}\left(\frac{\pi}{2}\right)^{m-1}\sum_{\kappa\in I}\frac{\alpha_\kappa}{|\det S_\kappa|}\sum_{\gamma\in\Gamma}\epsilon_r\left(\prod_{j=2}^{m}\operatorname{sgn}(s_{\kappa,\gamma;j})\right)\sum_{r=1}^{n}(r-1)!\frac{\phi_{\kappa,\gamma}^{(n-r)}(\eta)}{\eta^r} \\ &=: \sum_{\gamma\in\Gamma,\kappa\in I}\omega_{\kappa,\gamma}\sum_{r=1}^{n}(r-1)!\frac{\phi_{\kappa,\gamma}^{(n-r)}(\eta)}{\eta^r} = \frac{1}{\eta^n}\sum_{\gamma\in\Gamma,\kappa\in I}\omega_{\kappa,\gamma}\sum_{r=0}^{n-1}(n-r-1)!\phi_{\kappa,\gamma}^{(r)}(\eta)\eta^r. \end{aligned}$$

It follows from (8) that the meromorphic function $F(\eta)$ can have no pole at the origin and so must in fact be entire provided $F(0) := \lim_{\eta\to 0} F(\eta)$.

To complete the proof of Theorem 4, it remains only to show that $F(0) = 0$. Evidently we can write

$$F(\eta) = \frac{1}{\eta^n} \sum_{j=n}^{\infty} a_j \eta^j,$$

where the power series is convergent for all $\eta \in \mathbb{C}$. Now

$$F(0) = a_n = \frac{1}{n!} \lim_{\eta \to 0} \left(\frac{d}{d\eta} \right)^n \eta^n F(\eta),$$

and, by Leibnitz's rule [3, p. 378],

$$\begin{aligned} \left(\frac{d}{d\eta} \right)^n \eta^n F(\eta) &= \sum_{\gamma \in \Gamma, \kappa \in I} \omega_{\kappa,\gamma} \sum_{r=0}^{n-1} (n-r-1)! \sum_{j=0}^{r} \binom{n}{j} \phi_{\kappa,\gamma}^{(r+n-j)}(\eta) \eta^{r-j} \\ &= \sum_{\gamma \in \Gamma, \kappa \in I} \omega_{\kappa,\gamma} \sum_{r=0}^{n-1} (n-r-1)! \binom{n}{r} \phi_{\kappa,\gamma}^{(n)}(\eta) + O(\eta) \\ &\to 0 \text{ as } \eta \to 0, \end{aligned}$$

since

$$\phi_{\kappa,\gamma}^{(n)}(\eta) = 2(s_{\kappa,\gamma;1})^n \sin(s_{\kappa,\gamma;1}\eta).$$

We have thus shown that the limit in (8) is zero, and this completes the proof. □

The quantities in Theorem 4 can all be expressed as determinants. For example by application of Cramer's rule

$$s_{\kappa,\gamma;1} = \frac{\det\left(s_\gamma \; s_{\kappa_2} \; \dots \; s_{\kappa_m}\right)}{\det \, S_k},$$

and the sgn term is similarly expressible.

We note that in Mathematica or Maple, it is possible via Theorem 4, to compute integrals/volumes with $m = 5$ and $n = 6$, for example, quite rapidly.

Example. (a) Let V denote the volume of $\{x \in \mathbb{R}^6 : |p_i x| \leq 1, i = 1 \dots 11\}$, where p_i is the i-th column of the matrix

$$P = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 9 & 10 & -1 & -3 & 7 \\ 0 & 10 & 0 & 0 & 0 & 0 & -2 & -1 & -8 & 2 & -6 \\ 0 & 0 & 10 & 0 & 0 & 0 & -9 & 7 & -5 & 5 & 1 \\ 0 & 0 & 0 & 10 & 0 & 0 & 5 & -2 & -9 & -8 & -9 \\ 0 & 0 & 0 & 0 & 10 & 0 & -10 & -2 & -3 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & 10 & -8 & 9 & 2 & 7 & -10 \end{pmatrix}.$$

By definition, $V = \nu(P)$. Then, using Theorem 2 and Corollary 1,

$$\nu(P) = 10^{-6}\nu\left(\frac{P}{10}\right) = 10^{-6}\frac{2^6}{\pi^5}\sigma\left(\frac{S}{10}\right) = 10^{-1}\frac{2^6}{\pi^5}\sigma(S),$$

where

$$S = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 9 & -2 & -9 & 5 & -10 & -8 \\ 0 & 10 & 0 & 0 & 0 & 10 & -1 & 7 & -2 & -2 & 9 \\ 0 & 0 & 10 & 0 & 0 & -1 & -8 & -5 & -9 & -3 & 2 \\ 0 & 0 & 0 & 10 & 0 & -3 & 2 & 5 & -8 & 6 & 7 \\ 0 & 0 & 0 & 0 & 10 & 7 & -6 & 1 & -9 & -4 & -10 \end{pmatrix}.$$

Thus

$$\nu(P) = \frac{32}{5\pi^5}\int_{\mathbb{R}^5}\prod_{i=1}^{11}\operatorname{sinc}(s_i y)\,dy,$$

where s_i is the i-th column of S.

Performing the calculation from Theorem 4 for $\sigma(S)$ on a work station, we determine that $\nu(P)$ equals

1783555333298996761896629034429151640987432075715436721335976340904268
30954976107932353821685447822038731283351005300285791701641123713818
20826358461393954862567727

divided by

1271798037608528688337022500854279611126999889183337607935819035761877
62830324849292543701198920841061437021681539693375054209434572479321
674998957268149032759500800000000

which is approximately $1.4023888074656644090660969336515301763 \times 10^{-5}\ldots$.

This entailed computing the various intermediate quantities defined above Theorem 4.

(b) For illustration, we go through the following smaller computation in detail.

We wish to evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{sinc}(x)\operatorname{sinc}(y)\operatorname{sinc}(x+2y)\operatorname{sinc}\left(-x+\frac{y}{2}\right)dx\,dy = \sigma\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & \frac{1}{2} \end{pmatrix}.$$

This involves the following sub-matrices

$$S_{1,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_{1,3} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad S_{1,4} = \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix},$$

with inverses

$$S_{1,2}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_{1,3}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad S_{1,4}^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix},$$

The corresponding determinant ratios are

$$\alpha_{1,2} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2}{\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} -1 & 0 \\ \frac{1}{2} & 1 \end{vmatrix}} = -1, \quad \alpha_{1,3} = \frac{\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}^2}{\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}\begin{vmatrix} -1 & 1 \\ \frac{1}{2} & 2 \end{vmatrix}} = \frac{8}{5},$$

$$\alpha_{1,4} = \frac{\begin{vmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{vmatrix}^2}{\begin{vmatrix} 0 & -1 \\ 1 & \frac{1}{2} \end{vmatrix}\begin{vmatrix} 1 & -1 \\ 2 & \frac{1}{2} \end{vmatrix}} = -\frac{1}{6}.$$

The remaining intermediate quantities are best given in tabular form:

γ	ϵ_γ	s_γ
$+1, +1, +1$	$+1$	$1, \frac{7}{2}$
$+1, +1, -1$	-1	$3, \frac{5}{2}$
$+1, -1, +1$	-1	$-1, -\frac{1}{2}$
$+1, -1, -1$	$+1$	$1, -\frac{3}{2}$
$-1, +1, +1$	-1	$1, \frac{3}{2}$
$-1, +1, -1$	$+1$	$3, \frac{1}{2}$
$-1, -1, +1$	$+1$	$-1, -\frac{5}{2}$
$-1, -1, -1$	-1	$1, -\frac{7}{2}$

and

γ	ϵ_γ	$s_{\kappa,\gamma}(\kappa = 1,2)$	$s_{\kappa,\gamma}(\kappa = 1,3)$	$s_{\kappa,\gamma}(\kappa = 1,4)$
$+1, +1, +1$	$+1$	$1, \frac{7}{2}$	$-\frac{3}{4}, \frac{7}{4}$	$8, 7$
$+1, +1, -1$	-1	$3, \frac{5}{2}$	$\frac{7}{4}, \frac{5}{4}$	$8, 5$
$+1, -1, +1$	-1	$-1, -\frac{1}{2}$	$-\frac{3}{4}, -\frac{1}{4}$	$-2, 1$
$+1, -1, -1$	$+1$	$1, -\frac{3}{2}$	$\frac{7}{4}, -\frac{3}{4}$	$-2, -3$
$-1, +1, +1$	-1	$1, \frac{3}{2}$	$\frac{1}{4}, \frac{3}{4}$	$4, 3$
$-1, +1, -1$	$+1$	$3, \frac{1}{2}$	$\frac{11}{4}, \frac{1}{4}$	$4, 1$
$-1, -1, +1$	$+1$	$-1, -\frac{5}{2}$	$\frac{1}{4}, -\frac{5}{4}$	$-6, -5$
$-1, -1, -1$	-1	$1, -\frac{7}{2}$	$\frac{11}{4}, -\frac{7}{4}$	$-6, -7$

The composite calculation of

$$c_\kappa := \frac{\alpha_\kappa}{|\det S_\kappa|} \sum_{\gamma \in \Gamma} (s_{\kappa,\gamma;1})^n \epsilon_\gamma \prod_{j=1}^{m} \operatorname{sgn}(s_{\kappa,\gamma;j})$$

can now be completed, for the κ's below:

$$\kappa = 1, 2: \quad c_\kappa = \frac{-1}{1}(+1^2 - 3^2 - (-1)^2 - 1^2 - 1^2 + 3^2 + (-1)^2 + 1^2) = 0.$$

$$\kappa = 1, 3: \quad c_\kappa = \frac{8/5}{2}\left(\frac{-(-3)^2 - 7^2 - (-3)^2 - 7^2 - 1^2 + 11^2 - 1^2 + 11^2}{16}\right) = \frac{31}{5}.$$

$$\kappa = 1, 4: \quad c_\kappa = \frac{-1/6}{1/2}(+8^2 - 8^2 - (-2)^2 + (-2)^2 - 4^2 + 4^2 + (-6)^2 - (-6)^2) = 0.$$

Hence

$$\sigma(S) = \frac{1}{2 \cdot 2}\frac{\pi^2}{4}\sum_{\kappa \in I} c_\kappa = \frac{31\pi^2}{80}.$$

Thus, by Theorem 2,

$$\frac{2^n}{\pi^m}\sigma(S) = \frac{31}{20} = \nu\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & \frac{1}{2} \end{pmatrix},$$

as is confirmed by the picture in Fig. 1 of the right hand quantity, in which the area of the inscribed hexagon is indeed $\frac{31}{20}$.

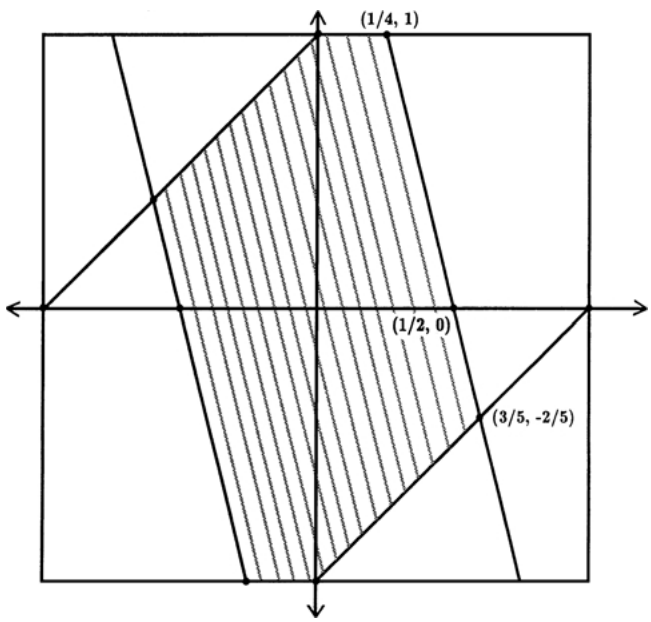

Figure 1. The corresponding polyhedron ($n = m = 2$).

Remark. Implicitly above we have used the evaluation

$$\int_0^\infty \frac{\sin y}{y} dy = \frac{\pi}{2}.$$

There are several well-known proofs [1]. It also follows on taking the limit, via Binet's mean value theorem [3, p. 328], of the absolutely convergent integral

$$\int_0^\infty \frac{\sin y}{y^{1+\varepsilon}} dy = \frac{\pi}{2} \frac{\sec\left(\frac{\pi}{2}\varepsilon\right)}{\Gamma(1+\varepsilon)}.$$

It seems worth recording the following Mellin transform based proof.

Proof: Maple happily evaluates the second integral to a form which simplifies to that we have given. A conventional proof follows by using the Γ-function to write

$$\int_0^\infty \frac{\sin y}{y^{1+\varepsilon}}\, dy = \frac{1}{\Gamma(\varepsilon+1)} \int_0^\infty dx \int_0^\infty \sin(x) \exp(-xt) t^\varepsilon\, dt.$$

One now interchanges the variables and evaluates the inner integral to $t^\varepsilon/(t^2+1)$. The outer integral now evaluates precisely to the claimed form. □

Finally, we note that many others have researched expressions and algorithms for computing volumes of polyhedra determined by their linear inequalities. In particular, Lasserre [2] derives a nice expression, implemented by the computer program VINCI, currently available online at `www.math.uni-augsburg.de/enge/vinci/manual/manual.html.`

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