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J. Math. Anal. Appl. 293 (2004) 285–292

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

A one-sided Tauberian theorem for the Borel summability method ☆

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Received 4 September 2003

Submitted by K.A. Ames

Abstract

We establish a quantitative version of Vijayaraghavan's classical result and use it to give a short proof of the known theorem that a real sequence (s_n) which is summable by the Borel method, and which satisfies the one-sided Tauberian condition that $\sqrt{n}(s_n - s_{n-1})$ is bounded below must be convergent.

Keywords: Tauberian; Borel summability

1. Introduction and the main results

Suppose throughout that (s_n) is a sequence of real numbers, and that $s_n = \sum_{k=0}^{n} a_k$. Let $\alpha > 0$, let

$$p_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^\alpha},$$

and let

$$\sigma_\alpha(x) := \frac{1}{p_\alpha(x)} \sum_{k=0}^{\infty} \frac{s_k}{(k!)^\alpha} x^k \quad \text{for all } x \in \mathbb{R}.$$

☆ This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.01.006

Recall that the Borel summability method B is defined as follows:

$$s_n \to s(B) \quad \text{if } \sum_{k=0}^{\infty} \frac{s_k}{k!} x^k \text{ is convergent for all } x \in \mathbb{R}$$

and

$$\sigma_1(x) \to s \quad \text{as } x \to \infty.$$

For an inclusion result concerning the summability method based on $\sigma_\alpha(x)$ see [3, p. 29]. Our aim is to give a short proof of the following well-known Tauberian theorem for the Borel method [6, Theorem 241] and [4,9].

Theorem 1. *If $s_n \to s(B)$, and if $\sqrt{n}\, a_n \geqslant -c$ for some $c \geqslant 0$ and all $n \in \mathbb{N}$, then $s_n \to s$.*

Our proof depends largely on the next result which is an improvement of Vijayaraghavan's theorem [6, Theorem 238]; see also [8,9] in that it specifies bounds in its conclusion.

Theorem 2. *Let $\alpha > 0$, and suppose that*

$$\liminf_{n\to\infty} \sqrt{n}\, a_n \geqslant -c_1, \quad \text{where } 0 \leqslant c_1 < \infty, \tag{1}$$

and

$$\limsup_{n\to\infty} \left| \sigma_\alpha\left(n^\alpha \exp\left(\frac{\alpha}{2n}\right)\right)\right| = c_2 < \infty. \tag{2}$$

Then

$$\limsup_{n\to\infty} |s_n| \leqslant c_3\left(c_2 + c_1\left(2\delta + \frac{1}{\delta^2 \alpha \sqrt{2\pi\alpha}}\right)\right) \tag{3}$$

for all

$$\delta > \frac{2\sqrt{2}}{\sqrt{\alpha\pi}} \quad \text{with } c_3 = \left(1 - \frac{2\sqrt{2}}{\delta\sqrt{\alpha\pi}}\right)^{-1}.$$

2. An auxiliary result

We require the following lemma for our proofs.

Lemma. *Let $\alpha > 0, \delta > 0$, and let*

$$c_n(x) := \frac{1}{p_\alpha(x)} \cdot \frac{x^n}{(n!)^\alpha} \quad \text{for } n \in \mathbb{N}_0.$$

Moreover, suppose that $M, N \in \mathbb{N}$, $x := y^\alpha$ with

$$y = y(n) := n \exp\left(\frac{1}{2n}\right), \quad M = M(n), \quad N = N(n) \quad \text{for } n \in \mathbb{N},$$

and define

$$\Sigma_1 := \sum_{k=0}^{M} c_k(x), \qquad \Sigma_2 := \sum_{k=N}^{\infty} c_k(x), \quad \text{and} \quad \Sigma_3 := \sum_{k=N}^{\infty} \sum_{\nu=N}^{k} \frac{c_k(x)}{\sqrt{\nu}}.$$

Then

(i) $\displaystyle \limsup_{M\to\infty} \Sigma_1 \leqslant \frac{1}{\delta\sqrt{2\pi\alpha}}$ *whenever* $y \geqslant M + \delta\sqrt{M}$;

(ii) $\displaystyle \limsup_{n\to\infty} \Sigma_2 \leqslant \frac{1}{\delta\sqrt{2\pi\alpha}}$ *whenever* $N \geqslant y + \delta\sqrt{y}$;

(iii) $\displaystyle \limsup_{n\to\infty} \Sigma_3 \leqslant \frac{1}{\delta^2\alpha\sqrt{2\pi\alpha}}$ *whenever* $N \geqslant y + \delta\sqrt{y}$.

Proof. First, note that $c_k(x)$ increases with k for $0 \leqslant k \leqslant y = x^{1/\alpha}$ and decreases for $k \geqslant y$, and that, for $0 \leqslant k \leqslant m \leqslant y$,

$$c_k(x) = c_m(x) \frac{(m(m-1)\dots(k+1))^\alpha}{x^{m-k}} \leqslant c_m(x)\left(\frac{m^\alpha}{x}\right)^{m-k} \leqslant c_m(x).$$

Hence, for $y \geqslant M + \delta\sqrt{M}$ with M large enough to ensure $M \leqslant n \leqslant y$, we have that

$$\Sigma_1 \leqslant c_M(x) \sum_{\nu=0}^{\infty} \left(\frac{M^\alpha}{x}\right)^\nu \leqslant c_n(x)\left(1 - \frac{M^\alpha}{y^\alpha}\right)^{-1},$$

where

$$\lim_{n\to\infty} c_n(x)\sqrt{n} = \sqrt{\frac{\alpha}{2\pi}}, \tag{4}$$

since

$$x = n^\alpha \exp\left(\frac{\alpha}{2n}\right),$$

by [2, Lemma 4.5.4], [5, p. 55] or [7]. Moreover

$$\frac{1}{\sqrt{n}}\left(1 - \frac{M^\alpha}{y^\alpha}\right)^{-1} \leqslant \frac{1}{\sqrt{M}}\left(1 - \frac{M^\alpha}{(M+\delta\sqrt{M})^\alpha}\right)^{-1}$$
$$= \frac{1}{\sqrt{M}}\left(1 - (1 + \delta M^{-1/2})^{-\alpha}\right)^{-1} \to \frac{1}{\alpha\delta} \quad \text{as } M \to \infty,$$

and this proves (i).

Next, we have that, for $y = x^{1/\alpha} \leqslant m + 1 \leqslant k + 1$,

$$c_k(x) = c_m(x) \frac{x^{k-m}}{((m+1)(m+2)\dots k)^\alpha} \leqslant c_m(x)\left(\frac{x}{(m+1)^\alpha}\right)^{k-m} \leqslant c_m(x).$$

Hence, for $N \geqslant y + \delta\sqrt{y}$, we have that

$$\Sigma_2 \leqslant c_N(x) \sum_{\nu=0}^{\infty} \left(\frac{x}{N^\alpha}\right)^\nu \leqslant c_n(x)\left(1 - \frac{y^\alpha}{N^\alpha}\right)^{-1},$$

where

$$\frac{1}{\sqrt{n}}\left(1-\frac{y^\alpha}{N^\alpha}\right)^{-1} \leqslant \frac{1}{\sqrt{n}}\left(1-\frac{y^\alpha}{(y+\delta\sqrt{y})^\alpha}\right)^{-1}$$
$$= \frac{1}{\sqrt{n}}\left(1-(1+\delta y^{-1/2})^{-\alpha}\right)^{-1} \to \frac{1}{\alpha\delta}$$
$$\text{as } y = n\exp\left(\frac{1}{2n}\right) \to \infty,$$

and this together with (4) implies (ii).

Finally, we see that, for $N \geqslant y + \delta\sqrt{y}$,

$$\Sigma_3 := \sum_{\nu=N}^{\infty} \frac{1}{\sqrt{\nu}} \sum_{k=\nu}^{\infty} c_k(x) \leqslant \sum_{\nu=N}^{\infty} \frac{c_\nu(x)}{\sqrt{\nu}}\left(1-\frac{x}{\nu^\alpha}\right)^{-1} \leqslant \frac{1}{\sqrt{N}}\left(1-\frac{x}{N^\alpha}\right)^{-1} \sum_{\nu=N}^{\infty} c_\nu(x).$$

Hence, by what we have shown before, we have that

$$\limsup_{n\to\infty} \Sigma_3 \leqslant \frac{1}{\alpha\delta} \cdot \frac{1}{\delta\sqrt{2\pi\alpha}},$$

which establishes (iii). □

3. Proofs of the theorems

Proof of Theorem 2. Let $\alpha > 0$ and $\delta > 2\sqrt{2}/\sqrt{\alpha\pi}$. Given $\varepsilon > 0$, choose $N_0 \in \mathbb{N}$ so large that

$$a_n \geqslant -(c_1+\varepsilon)\frac{1}{\sqrt{n}} \quad \text{for all } n \geqslant N_0$$

and

$$s_M > S_+(M) - \varepsilon \quad \text{and} \quad -s_N > S_-(N) - \varepsilon$$

for infinitely many integers M and N with $M \geqslant N_0$ and $N \geqslant N_0$, where

$$S_+(m) := \max_{N_0 \leqslant k \leqslant m} s_k \quad \text{and} \quad S_-(m) := \max_{N_0 \leqslant k \leqslant m} (-s_k) \text{ for } m \geqslant N_0.$$

Note that the sequences $(S_+(m))$ and $(S_-(m))$ are nondecreasing, and that $\max(S_+(m), S_-(m)) \geqslant |s_k|$ for $N_0 \leqslant k \leqslant m$. We consider two cases which exhaust all possibilities (cf. [6, pp. 308–311]).

Case 1. $S_+(m) \geqslant S_-(m)$ for infinitely many integers m.

Then there are infinitely many integers $M \geqslant N_0$ such that

$$s_M > S_+(M) - \varepsilon \quad \text{and} \quad S_+(M) \geqslant S_-(M). \tag{5}$$

We choose such M, and then integers n and N satisfying

$$\begin{cases} M + \delta\sqrt{M} \leqslant y := n\exp\left(\frac{1}{2n}\right) < M + \delta\sqrt{M} + 2, \\ y + \delta\sqrt{y} \leqslant N < y + \delta\sqrt{y} + 2, \end{cases} \tag{6}$$

and we put $x := y^\alpha$. Then $\sqrt{N} \leqslant \sqrt{M} + \delta + 2/\sqrt{M}$, because

$$N < \left(\sqrt{y} + \frac{\delta}{2} + \frac{1}{\sqrt{y}}\right)^2 \quad \text{and} \quad y < \left(\sqrt{M} + \frac{\delta}{2} + \frac{1}{\sqrt{M}}\right)^2.$$

We split $\sigma_\alpha(x)$ as follows:

$$\sigma_\alpha(x) := \sum_{\nu=1}^{4} \tau_\nu(x),$$

where

$$\tau_1(x) := \sum_{k=0}^{N_0} s_k c_k(x), \qquad \tau_2(x) := \sum_{k=N_0+1}^{M} s_k c_k(x),$$
$$\tau_3(x) := \sum_{k=M+1}^{\infty} s_M c_k(x), \qquad \tau_4(x) := \sum_{k=M+1}^{\infty} (s_k - s_M) c_k(x).$$

We see immediately that

$$\tau_1(x) \to 0 \quad \text{as } M \to \infty.$$

In what follows we use the notation of the lemma. By (5), we have that $-s_k \leqslant S_-(k) \leqslant S_-(M) \leqslant S_+(M) < s_M + \varepsilon$ for $0 \leqslant k \leqslant M$, and hence that

$$\tau_2(x) \geqslant -(s_M+\varepsilon)\Sigma_1.$$

Next, we observe that

$$\tau_3(x) = s_M(1-\Sigma_1).$$

Finally, we see that

$$\tau_4(x) = \sum_{k=M+1}^{\infty} \sum_{\nu=M+1}^{k} a_\nu c_k(x) \geqslant -(c_1+\varepsilon) \sum_{k=M+1}^{\infty} \sum_{\nu=M+1}^{k} \frac{c_k(x)}{\sqrt{\nu}}$$
$$= -(c_1+\varepsilon)\left(\tau_{4,1}(x) + \tau_{4,2}(x)\right),$$

where

$$\tau_{4,1}(x) := \sum_{k=M+1}^{\infty} \sum_{\nu=M+1}^{\min(k,N)} \frac{c_k(x)}{\sqrt{\nu}} \leqslant \sum_{k=M+1}^{\infty} c_k(x) \int_M^N \frac{dt}{\sqrt{t}}$$
$$= 2(\sqrt{N} - \sqrt{M}) \sum_{k=M+1}^{\infty} c_k(x) \leqslant 2\left(\delta + \frac{2}{\sqrt{M}}\right)$$

and

$$\tau_{4,2}(x) := \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} \frac{c_k(x)}{\sqrt{\nu}} \leqslant \Sigma_3.$$

Collecting the above results, we see that

$$\sigma_\alpha(x) \geqslant \tau_1(x) + s_M(1 - 2\Sigma_1) - \varepsilon\Sigma_1 - (c_1+\varepsilon)\left(2\delta + \frac{4}{\sqrt{M}} + \Sigma_3\right). \tag{7}$$

Since ε is an arbitrary positive number, and

$$\liminf_{M\to\infty} s_M + \varepsilon \geqslant \lim_{m\to\infty} S_+(m) = \lim_{m\to\infty} \max\bigl(S_+(m), S_-(m)\bigr) \geqslant \limsup_{m\to\infty} |s_m|,$$

it follows from (7) that

$$\liminf_{M\to\infty} s_M \Bigl(1 - 2\limsup_{M\to\infty} \Sigma_1\Bigr) \leqslant \limsup_{M\to\infty} \sigma_\alpha(x) + c_1\Bigl(2\delta + \limsup_{M\to\infty} \Sigma_3\Bigr)$$

and hence, by the lemma, that

$$\limsup_{m\to\infty} |s_m| \left(1 - \frac{\sqrt{2}}{\delta\sqrt{\alpha\pi}}\right) \leqslant c_2 + c_1\left(2\delta + \frac{1}{\delta^2\alpha\sqrt{2\pi\alpha}}\right),$$

which yields assertion (3) in Case 1.

Case 2. $S_+(m) < S_-(m)$ for all $m \geqslant N_1 \geqslant N_0$.

We choose integers M, n, N to satisfy (6) as in Case 1. In addition, we choose $N \geqslant N_1$ such that $-s_N > S_-(N) - \varepsilon$, which is evidently possible for large N. We now split $\sigma_\alpha(x)$ as follows:

$$\sigma_\alpha(x) := \sum_{\nu=1}^{6} \tau_\nu(x),$$

where

$$\tau_1(x) := \sum_{k=0}^{N_1} s_k c_k(x), \qquad \tau_2(x) := \sum_{k=N_1+1}^{M} s_k c_k(x),$$

$$\tau_3(x) := \sum_{k=M+1}^{\infty} s_N c_k(x), \qquad \tau_4(x) := \sum_{k=M+1}^{N-1} (s_k - s_N) c_k(x),$$

$$\tau_5(x) := \sum_{k=N}^{\infty} (-2s_N) c_k(x), \qquad \tau_6(x) := \sum_{k=N}^{\infty} (s_k + s_N) c_k(x).$$

We see immediately that

$$\tau_1(x) \to 0 \quad \text{as } N \to \infty.$$

In what follows we again use the notation of the lemma. In this case we have that $s_k \leqslant S_+(k) \leqslant S_+(N) < S_-(N)$ for $0 \leqslant k \leqslant M$ with $N > M > N_1$, and hence, since $-s_N + \varepsilon > S_-(N) \geqslant 0$, that

$$\tau_2(x) \leqslant (-s_N + \varepsilon)\Sigma_1.$$

Next, we observe that

$$\tau_3(x) = s_N(1 - \Sigma_1).$$

Further, we see that

$$\begin{aligned}\tau_4(x) &= \sum_{k=M+1}^{N-1} \sum_{\nu=k+1}^{N} (-a_\nu) c_k(x) \leqslant (c_1+\varepsilon) \sum_{k=M+1}^{N-1} \sum_{\nu=M+1}^{k} \frac{c_k(x)}{\sqrt{\nu}} \\ &\leqslant (c_1+\varepsilon) \sum_{k=M+1}^{\infty} c_k(x) \int_M^N \frac{dt}{\sqrt{t}} = 2(c_1+\varepsilon)(\sqrt{N} - \sqrt{M}) \sum_{k=M+1}^{\infty} c_k(x) \\ &\leqslant 2(c_1+\varepsilon)\left(\delta + \frac{2}{\sqrt{M}}\right)\end{aligned}$$

and that

$$\tau_5(x) = -2s_N\Sigma_2.$$

Finally, we observe that, for $k \geqslant N \geqslant N_1 \geqslant N_0$, either $s_k \leqslant S_+(k) < S_-(k) = \max_{N_0 \leqslant \nu \leqslant k}(-s_\nu) = -s_m$ for some $m \in (N, k]$, in which case we have that

$$s_k + s_N \leqslant s_N - s_m = \sum_{\nu=N+1}^{m} (-a_\nu) \leqslant (c_1+\varepsilon) \sum_{\nu=N+1}^{k} \frac{1}{\sqrt{\nu}},$$

or $s_k \leqslant S_-(N) < -s_N + \varepsilon$. It follows that

$$\tau_6(x) \leqslant (c_1+\varepsilon) \sum_{k=N}^{\infty} \sum_{\nu=N}^{k} \frac{c_k(x)}{\sqrt{\nu}} + \varepsilon\Sigma_2 = (c_1+\varepsilon)\Sigma_3 + \varepsilon\Sigma_2.$$

Collecting the above results, we see that

$$\begin{aligned}\sigma_\alpha(x) \leqslant\ & \tau_1(x) + s_N(1 - 2\Sigma_1 - 2\Sigma_2) + 2(c_1+\varepsilon)\left(\delta + \frac{2}{\sqrt{M}}\right) \\ & + (c_1+\varepsilon)\Sigma_3 + \varepsilon.\end{aligned} \tag{8}$$

Since ε is an arbitrary positive number, and

$$\liminf_{N\to\infty}(-s_N) + \varepsilon \geqslant \lim_{m\to\infty} S_-(m) = \lim_{m\to\infty} \max\bigl(S_+(m), S_-(m)\bigr) \geqslant \limsup_{m\to\infty} |s_m|,$$

it follows from (8) that

$$\begin{aligned}&\liminf_{N\to\infty}(-s_N)\Bigl(1 - 2\limsup_{N\to\infty}\Sigma_1 - 2\limsup_{N\to\infty}\Sigma_2\Bigr) \\ &\quad \leqslant \limsup_{N\to\infty}\bigl(-\sigma_\alpha(x)\bigr) + c_1\Bigl(2\delta + \limsup_{N\to\infty}\Sigma_3\Bigr),\end{aligned}$$

and hence, by the lemma, that

$$\limsup_{m\to\infty} |s_m| \left(1 - \frac{2\sqrt{2}}{\delta\sqrt{\alpha\pi}}\right) \leqslant c_2 + c_1\left(2\delta + \frac{1}{\delta^2\alpha\sqrt{2\pi\alpha}}\right),$$

which yields assertion (3) in Case 2. □

We now discuss consequences of Theorem 2. The corresponding two-sided result is [2, Lemma 4.5.5] and [7, Lemma 5], and the arguments from now on are much the same as those in the references.

Proposition 1 (Cf. the o-Tauberian theorem [2, Corollary 4.3.8]). *Suppose that* $s_n \to s(B)$, *and that* $\liminf_{n\to\infty} \sqrt{n}\, a_n \geqslant 0$. *Then* $s_n \to s$.

Proof. We may assume without loss of generality that $s = 0$, so that $\lim_{x\to\infty} \sigma_1(x) = 0$. Then Theorem 2 can be applied with $c_1 = c_2 = 0$, $\alpha = 1$, and any $\delta > 2\sqrt{2}/\sqrt{\pi}$, to yield $\limsup_{n\to\infty} |s_n| = 0$, i.e., $s_n \to 0$. □

Observe that we did not need the full proof of (4) in [2] or [7] which involved asymptotic approximations valid for all $\alpha > 0$. For the case $\alpha = 1$, only Stirling's formula is used.

Proposition 2 (Boundedness). *Suppose that* $\sigma_\alpha(x)$ *is bounded as* $x \to \infty$ *for some* $\alpha > 0$, *and that condition* (1) *of Theorem* 2 *holds. Then the sequence* (s_n) *is bounded.*

Proof. The result follows from Theorem 2 with any $\delta > 2\sqrt{2}/\sqrt{\alpha\pi}$. □

Proof of Theorem 1. We may again assume without loss of generality that $s = 0$, i.e., that $s_n \to 0(B)$. Then, by Proposition 2, (s_n) is bounded, and it follows from [2, Theorem 4.5.2 and Proof of Theorem 4.5.1 on p. 200] (see also [7] and [1]) that

$$\sigma_\alpha\left(n^\alpha \exp\left(\frac{\alpha}{2n}\right)\right) \to 0 \quad \text{as } n \to \infty$$

for all $\alpha > 0$. Hence, by Theorem 2 with $c_2 = 0$,

$$\limsup_{n\to\infty} |s_n| \leqslant \left(1 - \frac{2\sqrt{2}}{\delta\sqrt{\alpha\pi}}\right)^{-1} c_1 \left(2\delta + \frac{1}{\delta^2\alpha\sqrt{2\pi\alpha}}\right)$$

for all $\alpha > 0$ and $\delta > 2\sqrt{2}/\sqrt{\alpha\pi}$. Letting $\delta \to 0$, $\alpha \to \infty$, subject to $\delta\sqrt{\alpha} \to \infty$, we obtain the required conclusion that $s_n \to 0$. □

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