WEIGHTED CONVOLUTION OPERATORS ON ℓ_p

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Abstract. The main results deal with conditions for the validity of the weighted convolution inequality $\sum_{n\in\mathbb{Z}} \left|b_n \sum_{k\in\mathbb{Z}} a_{n-k}x_k\right|^p \le C^p \sum_{k\in\mathbb{Z}} |x_k|^p$ when $p \ge 1$.

1. Introduction and main result.

We suppose throughout that

$$1 \le p \le \infty,\ \frac{1}{p} + \frac{1}{q} = 1;\ 1 \le r \le \infty,\ \frac{1}{r} + \frac{1}{s} = 1,$$

and observe the convention that $q = \infty$ when $p = 1$.

Given a two-sided complex sequence $x = (x_n)_{n\in\mathbb{Z}}$, we define

$$\|x\|_p := \left(\sum_{k\in\mathbb{Z}} |x_k|^p\right)^{1/p} \text{ for } 1 \le p < \infty, \text{ and } \|x\|_\infty := \sup_{n\in\mathbb{Z}} |x_n|;$$

and we say that $x \in \ell_p$ if $\|x\|_p < \infty$. Given a two-sided complex sequence $a = (x_n)$ and a two-sided complex sequence $b = (b_n)$ of weights, we define the weighted convolution linear transformation $y = (y_n) = \lambda x$ by

$$y_n := (\lambda x)_n := b_n \sum_{k\in\mathbb{Z}} a_{n-k}x_k,$$

and aim to obtain sufficient conditions for λ to be a bounded operator on ℓ_p. In other words, our objective is to establish conditions under which there is a positive constant C such that, for all $x \in \ell_p$,

$$\|y\|_p \le C\|x\|_p, \tag{1}$$

in which case the operator norm of λ, defined as $\|\lambda\|_p := \sup_{\|x\|_p\le 1} \|\lambda x\|_p \le C$. When $1 \le p < \infty$, (1) amounts to

$$\sum_{n\in\mathbb{Z}} \left|b_n \sum_{k\in\mathbb{Z}} a_{n-k}x_k\right|^p \le C^p \sum_{k\in\mathbb{Z}} |x_k|^p. \tag{2}$$

Our main result is the following

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Theorem. *If* $1 \le p \le \infty$, $1 \le r \le q$, $a \in \ell_r$, $b \in \ell_s$, *then* (1) *holds for all* $x \in \ell_p$ *with* $C = \|a\|_r \|b\|_s$.

Note that all the above concerns two-sided sequences. The situation is very different when one-sided sequences are considered. This amounts to having $a_n = b_n = x_n = 0$ for all $n < 0$. In this case (2) reduces to

$$\sum_{n=0}^{\infty} \left| b_n \sum_{k=0}^{n} a_{n-k} x_k \right|^p \le C^p \sum_{k=0}^{\infty} |x_k|^p,$$

and when $a_n \ge 0$, $A_n := a_0 + a_1 + \cdots + a_n > 0$ for $n \ge 0$, and $b_n := \dfrac{1}{A_n}$ for $n \ge 0$ we get the following known proposition about the Nörlund transform (see [1, Theorem 2] or [2, Theorem 1]).

Proposition. *If* $1 < p < \infty$ *and* $na_n = O(A_n)$ *as* $n \to \infty$, *then there is a positive constant* C *such that*

$$\sum_{n=0}^{\infty} \left| \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k \right|^p \le C^p \sum_{k=0}^{\infty} |x_k|^p.$$

2. Lemmas.

We prove two lemmas.

Lemma 1. *If* $1 < p < \infty$ *and* $\sum_{k \in \mathbb{Z}} c_k x_k$ *is convergent whenever* $\sum_{k \in \mathbb{Z}} |x_k|^p < \infty$, *then* $\sum_{k \in \mathbb{Z}} |c_k|^q < \infty$.

Proof. A version of this result with the stronger hypothesis that $\sum_{k \in \mathbb{Z}} c_k x_k$ is absolutely convergent whenever $x \in \ell_p$ appears as a problem in [3, p. 198, Problem 7] where ℓ_q is referred to as being the *Köthe-Toeplitz dual* of ℓ_p. It may well be that the result as stated is also known. We offer the following elementary non-functional analytic proof. The hypothesis is equivalent to the pair of statements:

$$\sum_{k=0}^{\infty} c_k x_k \text{ is convergent whenever } \sum_{k=0}^{\infty} |x_k|^p < \infty, \text{ and}$$
$$\sum_{k=1}^{\infty} c_{-k} x_{-k} \text{ is convergent whenever } \sum_{k=1}^{\infty} |x_{-k}|^p < \infty.$$

Suppose $\sum_{k=0}^{\infty} |c_k|^q = \infty$. Let $D_n := \sum_{k=0}^{n} |c_k|^q$. Assume without loss in generality that $D_0 > 0$, and take

$$x_k := \begin{cases} \dfrac{|c_k|^{q-1}}{D_k} \dfrac{|c_k|}{c_k} & \text{when } c_k \ne 0 \\ 0 & \text{otherwise,} \end{cases}$$

Then, by the Abel-Dini theorem,

$$\sum_{k=0}^{\infty} c_k x_k = \sum_{k=0}^{\infty} \frac{|c_k|^q}{D_k} = \infty \text{ while } \sum_{k=0}^{\infty} |x_k|^p = \sum_{k=0}^{\infty} \frac{|c_k|^q}{D_k^p} < \infty,$$

contrary to hypothesis. Thus we must have $\sum_{k=0}^{\infty} |c_k|^q < \infty$, and likewise $\sum_{k=1}^{\infty} |c_{-k}|^q < \infty$. □

Lemma 2. *If $1 \le p < \infty\,, 1 < r \le q$, and some finite $t \ge 1$ is such that*

$$\sum_{n\in\mathbb{Z}} \left| b_n \sum_{k\in\mathbb{Z}} a_{n-k} x_k \right|^p < \infty$$

whenever $a \in \ell_r$, $x \in \ell_p$, $b \in \ell_t$, then $t \le s$.

Proof. Suppose, to the contrary, that $t > s$, and let $3\varepsilon := \dfrac{1}{s} - \dfrac{1}{t}$. Let

$$a_n := \begin{cases} (n+1)^{-\frac{1}{r}-\varepsilon} & \text{for } n \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$x_n := \begin{cases} (n+1)^{-\frac{1}{p}-\varepsilon} & \text{for } n \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$b_n := \begin{cases} (n+1)^{-\frac{1}{t}-\varepsilon} & \text{for } n \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $a \in \ell_r$, $x \in \ell_p$, $b \in \ell_t$, but

$$\sum_{n\in\mathbb{Z}} \left| b_n \sum_{k\in\mathbb{Z}} a_{n-k} x_k \right|^p = \sum_{n=0}^{\infty} \left((n+1)^{-\frac{1}{t}-\varepsilon} \sum_{k=0}^{n} (n+1-k)^{-\frac{1}{r}-\varepsilon} (k+1)^{-\frac{1}{p}-\varepsilon} \right)^p$$
$$\ge \sum_{n=0}^{\infty} \left((n+1)^{-\frac{1}{t}-\varepsilon} (n+1)(n+1)^{-\frac{1}{r}-\varepsilon} (n+1)^{-\frac{1}{p}-\varepsilon} \right)^p = \sum_{n=0}^{\infty} (n+1)^{-1} = \infty.$$

□

3 Proof of the Theorem.

Case 1. $1 < p < \infty$. For inequality (2) to be meaningful and non-trivial, observe that, for any n for which $b_n \ne 0$, $\sum_{k\in\mathbb{Z}} a_{n-k} x_k$ has to be convergent whenever $\sum_{k\in\mathbb{Z}} |x_k|^p < \infty$. It thus follows from Lemma 1 that we must have $\sum_{k\in\mathbb{Z}} |a_{n-k}|^q =$

$\sum_{k\in\mathbb{Z}} |a_k|^q < \infty$. This explains why we make the restriction $1 \le r \le q$ in the hypothesis, and Lemma 2 shows why it is not sufficient to require $b \in \ell_t$ for any $t > s$.

An application of Hölder's inequality yields

$$\left|\sum_{k\in\mathbb{Z}} a_{n-k}x_k\right|^p \le \|a\|_r^{r(p-1)} \sum_{k\in\mathbb{Z}} |a_{n-k}|^{(q-r)(p-1)}|x_k|^p,$$

and hence that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \left|b_n \sum_{k\in\mathbb{Z}} a_{n-k}x_k\right|^p &\le \|a\|_r^{r(p-1)}\|x\|_p^p \sum_{n\in\mathbb{Z}} |b_n|^p|a_{n-k}|^{(q-r)(p-1)}\\ &\le \|a\|_r^{r(p-1)}\|x\|_p^p \cdot \|a\|_r^{(q-r)(p-1)}\|b\|_s^p\\ &= \|a\|_r^p\|b\|_s^p\|x\|_p^p,\end{aligned}$$

since $\|x\|_p^p = \sum_{k\in\mathbb{Z}} |x_k|^p < \infty$ and $\|b\|_s^s = \sum_{n\in\mathbb{Z}} |b_n|^s < \infty$, and this establishes (1) with $C = \|a\|_r\|b\|_s$. Note that Hölder's inequality with $\tilde{r} = \dfrac{r}{(q-r)(p-1)}$, $\tilde{s} = \dfrac{s}{p}$ is used in the penultimate step above.

Case 2. $p = 1$, $q = \infty$ or $p = \infty$, $q = 1$. When $p = 1$ the result follows by changing the order of summation in (2) and then applying Hölder's inequality, and when $p = \infty$ the desired conclusion is even more immediate. □

We have shown that if $1 \le p < \infty$, $1 < r \le q$, $a \in \ell_r$, then (2) holds for all $x \in \ell_p$ provided $b \in \ell_s$, but may fail to hold if $b \in \ell_t$ with a finite $t > s$. In the following section we show by means of an example that, if $1 < p < \infty$, then (2) may hold for all $x \in \ell_p$ when $b \notin \ell_t$ for any finite $t > 1$.

4. Example.

Suppose $1 < p < \infty$. Let $A_n := a_0 + a_1 + \cdots + a_n$ for $n \ge 0$, where

$$a_n := \begin{cases} \dfrac{1}{n+1} & \text{for } n \ge 0\\ 0 & \text{otherwise,}\end{cases}$$

let

$$b_n := \begin{cases} \dfrac{1}{A_n} & \text{for } n \ge 0\\ 0 & \text{otherwise,}\end{cases}$$

and let

$$y_n := \left|b_n \sum_{k\in\mathbb{Z}} a_{n-k}x_k\right| = \left|b_n \sum_{k=0}^{\infty} a_k x_{n-k}\right| \le y_{1,n} + y_{2,n},$$

where

$$y_{1,n} := \left| \frac{1}{A_n} \sum_{k=0}^{n} a_k x_{n-k} \right| \text{ and } y_{2,n} := \left| \frac{1}{A_n} \sum_{k=n+1}^{\infty} a_k x_{n-k} \right|.$$

Note that $\sum_{k\in\mathbb{Z}} |a_k| = \infty$ and $\|a\|_r^r = \sum_{k\in\mathbb{Z}} |a_k|^r < \infty$ for all $r > 1$. Suppose that the sequence $x = (x_n) \in \ell_p$. Since

$$A_n \sim \log n \text{ and } \frac{na_n}{A_n} \sim \frac{1}{\log n} = O(1) \text{ as } n \to \infty,$$

it follows from the Proposition that

$$\sum_{n=0}^{\infty} y_{1,n}^p \le C_1 \sum_{k=0}^{\infty} |x_k|^p \le C_1 \|x\|_p^p.$$

Further, by Hölder's inequality,

$$\begin{aligned}
\sum_{n=0}^{\infty} y_{2,n}^p &\le \|x\|_p^p \sum_{n=0}^{\infty} \frac{1}{A_n^p} \left(\sum_{k=n+1}^{\infty} a_k^q \right)^{p-1} \le \|x\|_p^p \sum_{n=0}^{\infty} \frac{1}{A_n^p} \left(\int_{n+1}^{\infty} \frac{dt}{t^q} \right)^{p-1} \\
&= (q-1)^{1-p} \|x\|_p^p \sum_{n=0}^{\infty} \frac{(n+1)^{(q-1)(1-p)}}{A_n^p} = (q-1)^{1-p} \|x\|_p^p \sum_{n=0}^{\infty} \frac{a_n}{A_n^p} \\
&= C_2 \|x\|_p^p,
\end{aligned}$$

where $C_2 = (q-1)^{1-p} \sum_{n=0}^{\infty} \frac{a_n}{A_n^p} < \infty$. Hence

$$\sum_{n\in\mathbb{Z}} y_n^p = \sum_{n=0}^{\infty} y_n^p \le 2^p \sum_{n=0}^{\infty} (y_{n,1}^p + y_{n,2}^p) \le 2^p (C_1 + C_2) \|x\|_p^p.$$

Thus (2) is satisfied but $b \notin \ell_t$ for any finite $t > 1$, since $\|b\|_t^t = \sum_{n=0}^{\infty} \frac{1}{A_n^t} = \infty$.

A similar but slightly more complicated argument can be used to show that we could get the same result by taking, for any real α,

$$a_n := \begin{cases} \dfrac{\log^{\alpha}(n+1)}{n+1} & \text{for } n \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

in the example.

References

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