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THEOREMS ON SOME METHODS OF SUMMABILITY

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1. Introduction

I COMMENCE with a description of notation and terminology and give definitions of some of the matrices and methods of summability considered in this note.

Suppose throughout that s, s_n $(n = 0, 1, ...)$ are arbitrary complex numbers, that $\lambda \geqslant -1$, and that

$$\epsilon_n^\alpha = \binom{n+\alpha}{n}.$$

The method A_λ. Write $s_n \to s\ (A_\lambda)$ if either

(i) $\lambda > -1$ and $(1-x)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda s_n x^n \to s$ as $x \to 1-$

or (ii) $\lambda = -1$ and $-\dfrac{1}{\log(1-x)} \sum_{n=0}^{\infty} \dfrac{s_n}{n+1} x^{n+1} \to s$ as $x \to 1-$.

The Hausdorff matrix H. Let $\{\mu_n\}$ be a sequence of real numbers and let

$$t_n = \sum_{r=0}^{n} s_r \binom{n}{r} \sum_{\nu=0}^{n-r} (-1)^\nu \binom{n-r}{\nu} \mu_{r+\nu}.$$

Denote the matrix of the linear transformation from $\{s_n\}$ to $\{t_n\}$ by H, and write $H(s_n)$ for t_n. Then H is a Hausdorff matrix which is said to be 'generated by the sequence $\{\mu_n\}$'. I use the same symbol for both matrix and associated Hausdorff summability method, i.e. I write $s_n \to s\ (H)$ to mean that $H(s_n) \to s$. If $\mu_n \neq 0$, the Hausdorff matrix generated by $\{1/\mu_n\}$ is denoted by H^{-1}.

The product method $A_\lambda H$. Write $s_n \to s\ (A_\lambda H)$ if $H(s_n) \to s\ (A_\lambda)$.

The Hölder matrix (H, α). For any real α, this is the Hausdorff matrix generated by the sequence $\{(n+1)^{-\alpha}\}$.

The Cesàro matrix (C, α) $(\alpha > -1)$. This is the Hausdorff matrix generated by $\{1/\epsilon_n^\alpha\}$. In the specified range only, it is the matrix of the Cesàro method (C, α) [Hardy (**6**) 251]; and

$$(C, \alpha)(s_n) = \frac{1}{\epsilon_n^\alpha} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} s_r.$$

Quart. J. Math. Oxford (2), 9 (1958), 310-16.

The matrix (C^*, α). I define this to be the Hausdorff matrix generated by $\{1/\epsilon_n^\alpha\}$ when $\alpha > -1$, and by $\{\epsilon_n^{-\alpha}\}$ when $\alpha \leqslant -1$, so that

$$(C^*, \alpha) = \begin{cases} (C, \alpha) & (\alpha > -1), \\ (C, -\alpha)^{-1} & (\alpha \leqslant -1). \end{cases}$$

The matrix (C, α, β) $(\beta > -1, \alpha+\beta > -1)$. I use the notation

$$s_n^{\alpha,\beta} = \frac{1}{\epsilon_n^{\alpha+\beta}} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} \epsilon_r^\beta s_r,$$

and denote by (C, α, β) the matrix of the linear transformation from $\{s_n\}$ to $\{s_n^{\alpha,\beta}\}$. Note that, for $\alpha > -1$, $(C, \alpha, 0) = (C, \alpha)$.

Suppose that P and Q are summability methods (or Hausdorff matrices). P is said to be *regular* if $s_n \to s\ (P)$ whenever $s_n \to s$. If $s_n \to s\ (P)$ whenever $s_n \to s\ (Q)$, P is said to *include* Q, and we write $P \supseteq Q$. If $P \supseteq Q$ and $Q \supseteq P$, P and Q are said to be *equivalent*, and we write $P \simeq Q$.

The following results are known.

(I) *If H is a regular Hausdorff matrix, then* $A_\lambda H \supseteq A_\lambda$.

[$\lambda = 0$, Hausdorff (**7**) 191; see also Szász (**11**); $\lambda > -1$, Amir (**2**) 376; see also Borwein (**3**) 222; $\lambda = -1$, Borwein (**5**) 218.†]

(II) $A_\alpha \supseteq A_\beta \supseteq (C, \gamma)$ $(\beta > \alpha \geqslant -1;\ \gamma > -1)$.

[Borwein (**3**) and (**4**) 348.]

(III) $A_\lambda(C, \beta) \supseteq A_\lambda(C, \alpha) \supseteq (C, \gamma)$ $(\lambda = 0;\ \beta > \alpha > -1;\ \gamma > -1)$.

[Amir (**1**); see also Lord (**8**) 243.]

In § 2 of this note the result (I) is extended; also, conditions are obtained which are sufficient for $A_\lambda H$ to include K when H and K are Hausdorff methods.

In § 3 it is proved that $(C^*, \alpha) \simeq (H, \alpha)$ for all real α; and that, for $\beta > -1$, $\alpha+\beta > -1$, (C, α, β) is a Hausdorff matrix equivalent to (C^*, α).

In § 4 the results of § 2 and § 3 are applied to the method $A_\lambda(C^*, \alpha)$. It is proved, *inter alia*, that (III), with C^* in place of C, is true whenever $\lambda \geqslant -1$, $\beta > \alpha$, and that the methods $A_{-\alpha}$, $A_0(C^*, \alpha)$ are equivalent for all $\alpha < 1$.

2. The method $A_\lambda H$

Suppose in what follows that H, K are Hausdorff matrices generated respectively by the real sequences $\{\mu_n\}$, $\{\nu_n\}$, and denote by HK both the matrix product and the associated summability method. It is

† See the concluding remarks.

familiar that HK is a Hausdorff matrix generated by the sequence $\{\mu_n \nu_n\}$ and, consequently, that $HK = KH$. Further, it is known and easily proved that, when $\mu_n \neq 0$, $K \supseteq H$ if and only if KH^{-1} is regular.

The above results [for proofs see Hardy (**6**) ch. xi] are used without special mention in the rest of the paper.

THEOREM 1. *If* $\mu_n \neq 0$ *and* $A_\lambda \supseteq KH^{-1}$, *then* $A_\lambda H \supseteq K$.

Proof. Suppose that $s_n \to s\ (K)$ and let $t_n = H(s_n)$. Then

$$K(s_n) = KH^{-1}(t_n) \to s, \quad \text{so that} \quad t_n \to s\ (KH^{-1}).$$

It follows that $t_n \to s\ (A_\lambda)$, i.e. $s_n \to s\ (A_\lambda H)$; and the proof is complete.

THEOREM 2. *If either* (i) H *is regular or* (ii) $\mu_n \neq 0$ *and* $A_\lambda \supseteq H^{-1}$, *then* $A_\lambda H$ *is regular.*

Proof. Since, by (II), A_λ is regular, it follows from (I) that $A_\lambda H$ is regular when (i) is satisfied. When (ii) is satisfied the result follows from Theorem 1.

The next theorem is an extension of (I).

THEOREM 3. *If* H *is regular, then* $A_\lambda KH \supseteq A_\lambda K$.

Proof. Suppose that $s_n \to s\ (A_\lambda K)$, so that $t_n = K(s_n) \to s\ (A_\lambda)$. Hence, by (I), $H(t_n) = KH(s_n) \to s\ (A_\lambda)$. The theorem follows.

An immediate corollary of Theorem 3 is the theorem:

THEOREM 4. *If* $K \supseteq H$ *and* $\mu_n \neq 0$, *then* $A_\lambda K \supseteq A_\lambda H$.

3. The matrices (H, α), (C^*, α), (C, α, β)

It is known that the Hölder matrix (H, δ) is regular for $\delta \geqslant 0$ [(**6**) ch. xi], and it is evident that, for all real α, β,

$$(H,\alpha)(H,\beta) = (H, \alpha+\beta).$$

Hence, for $\delta > 0$ and all real α,

$$(H, \alpha+\delta) \supseteq (H, \alpha).$$

Further, it is known [(**6**) Theorem 211] that, for $\alpha > -1$,

$$(H, \alpha) \simeq (C, \alpha).$$

Since, for $\alpha > -1$, $(C^*, \alpha) = (C, \alpha)$; and, for $\alpha \leqslant -1$,

$$(C^*, \alpha) = (C, -\alpha)^{-1} \simeq (H, -\alpha)^{-1} = (H, \alpha),$$

we obtain

THEOREM 5. *For all real* α, $(C^*, \alpha) \simeq (H, \alpha)$.

Immediate consequences are the theorems:

THEOREM 6. *For* $\alpha > \beta$, $(C^*, \alpha) \supseteq (C^*, \beta)$.

THEOREM 7. *For all real* α, β, $(C^*, \alpha)(C^*, \beta) \simeq (C^*, \alpha+\beta)$.

The cases $\alpha > -1$ of Theorem 5 and $\alpha > \beta > -1$ of Theorem 6 are, of course, standard. The cases $\alpha = -1, -2, \ldots$ of Theorem 5, $-1 \geqslant \alpha > \beta$ of Theorem 6, and $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$ of Theorem 7 have been given respectively by Lyra (**9**), Mears (**10**), and Zygmund (**12**).

We now prove

THEOREM 8. *If* $\beta > -1$, $\alpha+\beta > -1$, *then*

$$(C, \alpha, \beta) = (C, \alpha+\beta)(C, \beta)^{-1},$$

so that (C, α, β) *is the Hausdorff matrix generated by the sequence* $\{\epsilon_n^\beta / \epsilon_n^{\alpha+\beta}\}$.

Proof. It is well known and easily verified that, for $\beta > -1$, $\alpha+\beta > -1$,

$$\frac{1}{\epsilon_n^{\alpha+\beta}} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} \epsilon_r^\beta s_r^{\beta,0} = s_n^{\alpha+\beta,0},$$

whence $(C, \alpha, \beta)(C, \beta) = (C, \alpha+\beta)$. The theorem follows.

In consequence of Theorems 5 and 8 we have

THEOREM 9. *For* $\beta > -1$, $\alpha+\beta > -1$, $(C, \alpha, \beta) \simeq (C^*, \alpha)$.

The case $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$ of this result was proved in essence by Zygmund (**12**).

So far we have considered the Cesàro method (C, α) only in the range $\alpha > -1$. The standard definition† of this method in the range $\alpha \leqslant -1$ is:

$$s_n \to s\ (C, \alpha) \quad \text{if} \quad \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} s_r = s\epsilon_n^\alpha + o(n^\alpha) \quad \text{and} \quad s_n \to s\ (A_0).$$

This definition is due to Hausdorff (**7**) who proved the methods (C, α) and (H, α) to be equivalent (for all real α).

Hence, in virtue of Theorem 5, we have

THEOREM 10. *The methods* (C, α) *and* (C^*, α) *are equivalent for all real* α.

The case $\alpha = -1, -2, \ldots$ of this result has been proved by Lyra [(**9**), Satz 1].

4. The method $A_\lambda(C^*, \alpha)$

In what follows I use the notations

$$s_n^\alpha = s_n^{*\alpha} = \frac{1}{\epsilon_n^\alpha} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} s_r \quad (\alpha > -1),$$

$$s_n^{*\alpha} = \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} \epsilon_r^{-\alpha} s_r \quad (\alpha \leqslant -1).$$

I also use the first notation with a in place of s.

† For a full discussion of this and other definitions see Lyra (**9**).

Note that $s_n^\alpha = (C,\alpha)(s_n)$ $(\alpha > -1)$, and, in view of Theorem 8 with $\beta = -\alpha$, $s_n^{*\alpha} = (C^*,\alpha)(s_n)$ (all real α). [Cf. Mears (**10**), (11).]

The next theorem generalizes (III).

THEOREM 11. *If* $\alpha > \beta$ *and* γ *is real, then*

$$A_\lambda(C^*,\alpha) \supseteq A_\lambda(C^*,\beta) \supseteq (C^*,\gamma).$$

Proof. The first inclusion is a consequence of Theorems 4 and 6. Further, in virtue of (II) and Theorems 6 and 7, we have

$$A_\lambda \supseteq (C^*,\gamma-\beta) \simeq (C^*,\gamma)(C^*,\beta)^{-1},$$

so that, by Theorem 1, $A_\lambda(C^*,\beta) \supseteq (C^*,\gamma)$.

A corollary of Theorem 11 is that $A_\lambda(C^*,\alpha)$ is regular for all real α. The next theorem shows that for $\beta > -1$ the strength of the method $A_\beta(C^*,\alpha+\beta)$ is independent of β.

THEOREM 12. *If* $\beta > -1$ *and* α *is any real number, then*

$$A_0(C^*,\alpha) \simeq A_\beta(C^*,\alpha+\beta).$$

Proof. It follows from the formal identity

$$\sum_{n=0}^{\infty} s_n x^n = (1-x)^\beta \sum_{n=0}^{\infty} \epsilon_n^\beta s_n^\beta x^n \quad (0 < x < 1;\ \beta > -1),$$

that, if one of the series is convergent throughout the interval (0, 1), then so is the other. Consequently, $s_n \to s\ (A_0)$ if and only if $s_n^\beta \to s\ (A_\beta)$; and so $s_n^{*\alpha} \to s\ (A_0)$ if and only if $(C^*,\beta)(s_n^{*\alpha}) \to s\ (A_\beta)$. Hence,

$$A_0(C^*,\alpha) \simeq A_\beta(C^*,\alpha)(C^*,\beta)$$

and, since $(C^*,\alpha)(C^*,\beta) \simeq (C^*,\alpha+\beta)$, application of Theorem 4 yields the required result.

Put $\beta = -\alpha$ in the above theorem to get the corollary

$$A_0(C^*,\alpha) \simeq A_{-\alpha} \quad (\alpha < 1),$$

i.e. for $\alpha < 1$,

$$(1-x)\sum_{n=0}^{\infty} s_n^{*\alpha} x^n \to s \quad \text{as} \quad x \to 1-$$

if and only if

$$(1-x)^{1-\alpha}\sum_{n=0}^{\infty} \epsilon_n^{-\alpha} s_n x^n \to s \quad \text{as} \quad x \to 1-.$$

The final two theorems are concerned with the method $A_{-1}(C,\alpha)$.

Suppose that $$s_n = \sum_{r=0}^{n} a_r \quad (n = 0, 1,\ldots).$$

For $\alpha > 0$, I define the logarithmic method of summability (L,α) as follows:

$$s_n \to s\ (L,\alpha) \quad \text{or} \quad \sum_{n=0}^{\infty} a_n = s\ (L,\alpha)$$

if $$-\frac{1}{\log(1-x)}\sum_{n=0}^{\infty} a_n^\alpha x^{n+1} \to \alpha s \quad \text{as} \quad x \to 1-.$$

Note that the methods $(L, 1)$ and A_{-1} are identical.

We require two lemmas, of which the first is known [(**5**) Theorem 2] and the second is a simple consequence of a known result [(**5**) Lemma 1].

LEMMA 1. $A_{-1} \supseteq A_0(C,1)$.

LEMMA 2. *If* $p > 0$, $q > 0$ *and* $s_n \to s\ (A_{-1})$, *then*

$$\frac{n+p}{n+q} s_n \to s\ (A_{-1}).$$

THEOREM 13. *For* $\alpha > 0$, $(L,\alpha) \simeq A_{-1}(C,\alpha-1)$.

Proof. Since $s_n \to s\ (L,\alpha)$ if and only if $(n+1)a_n^\alpha \to \alpha s\ (A_{-1})$, the required result is a consequence of Lemma 2 and the easily verified identity

$$(n+1)a_n^\alpha = \frac{n+1}{n+\alpha}\alpha s_n^{\alpha-1} \quad (\alpha > 0).$$

THEOREM 14. *For* $\alpha > \beta > 0$, $(L,\alpha) \supseteq (L,\beta) \supseteq A_0(C,\beta)$.

Proof. In virtue of Theorems 11 and 13,

$$(L,\alpha) \supseteq (L,\beta) \simeq A_{-1}(C,\beta-1).$$

Further, by Lemma 1 and Theorems 4 and 7,

$$A_{-1}(C,\beta-1) \supseteq A_0(C,1)(C,\beta-1) \simeq A_0(C,\beta).$$

This completes the proof.

I am indebted to the referee for suggestions which enabled me to simplify the presentation of the material in § 3, and also for supplying most of the references there given.

Remarks on results (I) (added 10 July 1958). Professor C. T. Rajagopal has kindly sent me a reprint of his paper 'Product of two summability methods', *J. Indian Math. Soc.* 18 (1954) 89–105. This paper, which I had not seen previously, predates my paper (**5**) but not Amir's paper (**2**). In it Professor Rajagopal deduces, from his Theorem 1, the case $\lambda > -1$ of (I), and also (without giving details) a result equivalent to the following:

if H is a regular Hausdorff matrix and $t_n = H(s_n)$, *then* $t_{n+1} \to s\ (A_{-1})$ *whenever* $s_{n+1} \to s\ (A_{-1})$.

This is proved directly in (**5**) and is the key result used there in establishing the case $\lambda = -1$ of (I).

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