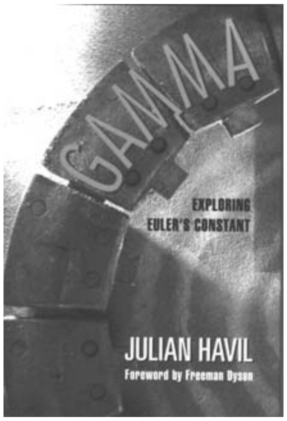
MEANDERING IN EULER'S NEIGHBOURHOOD

Book review by David Borwein, University of Western Ontario

GAMMA: EXPLORING EULER'S CONSTANT
by Julian Havil
Princeton 2003 xxiii + 266 pages



The Gamma of the title is Euler's constant

$$\gamma := \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right),$$

and the author describes his book as being an exploration of this constant and inescapably also an exploration of logarithms and the harmonic series. He expresses the hope that: "the material will appeal to a variety of people who have a little probability and statistics and a good calculus course behind them, and before that a rigorous course in algebra, if such a thing still exists: the motivated senior secondary student, who may well be seeing some of the ideas for the first time, the college student for whom the text may put flesh on what can sometimes be bare bones, the teacher for whom it might be a convenient synthesis of some nice ideas (and maybe the makings of a talk or two), and also for those who may have left mathematics behind and who wish to remind themselves why they used to find it so fascinating."

The book starts with a detailed description of Napier's invention of logarithms - historically interesting but not very illuminating mathematically. It continues with many mathematical items interlaced with historical snapshots. Much of the material is connected with Euler's voluminous contributions. Apart from logarithms and the harmonic series, topics visited are: harmonic series of primes, Madelung's constants, the Riemann Zeta function $\zeta(x)$, the Gamma function $\Gamma(x)$, continued fractions, Pell's equation, Euler-Maclaurin summation, Shannon's uncertainty measure, the Prime Number Theorem, the Riemann Hypothesis, and lots more. Some of the topics are dealt with in great detail, while others are only touched on.

Among the gems exhibited is Euler's formula

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

which he magically derived in 1735 (without bothering with rigor) by equating coefficients of $x3$ in the identity

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots.$$

Euler subsequently showed that

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},\ n = 1, 2, 3, \ldots,$$

where the B_{2n} are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} + \frac{x}{2} = \sum_{n=0}^{\infty} B_{2n}\frac{x^{2n}}{(2n)!}.$$

Another of Euler's formulas highlighted and proved in the book is the one linking the Riemann Zeta function with the increasing sequence of primes (p_n) with $p_1 = 2$, namely

$$\zeta(x) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-x}}, \quad x > 1,$$

which the author rightly describes as being the link through which analytic number theory came into being.

The Euler-Maclaurin summation formula was developed independently circa 1736 by the two mathematicians in the name. One of its general forms is stated in the book as:

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{f(1)+f(n)}{2} + \sum_{k=1}^{m} B_{2k}\frac{f^{(2k-1)}(n) - f^{(2k-1)}(1)}{(2k)!} + R_n(f,m),$$

where the remainder $R_n(f,m)$ is less in magnitude than

$$\frac{2}{(2\pi)^{2m}}\int_1^n |f^{(2m+1)}(x)|\,dx,$$

provided the derivative of order $2m+1$ of the function f exists and is continuous on $[1,n]$. In fact, a valid sharper bound for the remainder is

$$\frac{4}{(2\pi)^{2m+1}}\int_1^n |f^{(2m+1)}(x)|\,dx.$$

It is shown in the book how applying the summation formula to $f(x)=1/x$ yields

$$\sum_{k=1}^{n}\frac{1}{k}-\ln n-\frac{1}{2n}+\frac{1}{2}\sum_{k=1}^{m}\frac{B_{2k}}{kn^{2k}}$$

as an approximation to γ. Euler used this formula with $n=10$, $m=7$ to compute

$$\gamma = 0.5772156649015325\ \ldots,$$

of which the first 15 decimal places are correct, but the 16th is not - as can easily be checked in a flash by means of a symbolic computation product such as Maple. To this day it is not known whether γ is rational or not. Modern computational number theorists have shown that if gamma is rational it must have a denominator with many millions of digits.

Though γ is not as familiar a constant as e, π, or i, it creeps surreptitiously into many places in mathematics, as is shown in the book. For example, in the Weierstrass infinite product expansion of the gamma function,

$$\frac{1}{\Gamma(x)}=xe^{\gamma x}\prod_{r=1}^{\infty}\left(1+\frac{x}{r}\right)e^{-x/r},$$

from which it follows that $\Gamma'(1)=-\gamma$. Other appearances of γ noted in the book are:

$$\mathrm{Li}(x):=\int_2^x\frac{1}{\ln u}\,du=\gamma+\ln(\ln x)+\sum_{r=1}^{\infty}\frac{\ln^r x}{rr!},$$

$$\mathrm{Ci}(x):=\int_x^{\infty}\frac{\cos u}{u}\,du=-\gamma-\ln x+\sum_{r=1}^{\infty}\frac{(-x2)^r}{2r(2r)!},$$

$$\int_0^1\ln\left(\ln\frac{1}{u}\right)du=-\gamma,\quad \int_0^{\infty}e^{-u}\ln^2 u\,du=\frac{\pi^2}{6}+\gamma^2,$$

$$\sum_{r=2}^{\infty}\frac{\zeta(r)-1}{r}=1-\gamma,$$

and there are many more.

Chapters 15 and 16, the final two chapters in the book, are devoted to a discussion of the Prime Number Theorem that

$$\pi(x):=\sum_{p_n<x}1\sim\frac{x}{\ln x}\qquad\text{as } x\to\infty$$

and the Riemann Hypothesis that all the non-trivial zeros of $\zeta(z)$ lie on the line $z = 1/2$ in the complex plane. No proof of the theorem is offered in the book, and, of course, the famous hypothesis remains as one of the pre-eminent open problems in mathematics today. The Prime Number Theorem is one of the most fundamental and beautiful results in number theory. It was originally proved in 1896 by Hadamard and de la Vallée Poissin, independently, using properties of the complex zeros of $\zeta(z)$. Their proofs were lengthy and complicated. In 1980 Donald J. Newman developed a very simple and elegant Tauberian argument needed for an analytic proof of the theorem. Based on Newman's approach, a completely self-contained, yet concise analytic proof of the Prime Number Theorem was published by Don Zagier in *The American Mathematical Monthly* (October 1997, pages 705-708). It is a masterpiece of excellent mathematical exposition and is accessible to anyone with a minimum background in complex analysis. For this article Zagier was awarded the Chauvenet prize by the Mathematical Association of America.

The author mentions that Alte Selberg gave the first so-called "elementary" proof (i.e.,avoiding complex variable theory) in 1949, but fails to mention that Paul Erdös independently produced such a proof at about the same time.

The author's writing style is pleasing and clear. The material is reasonably free of typos – though there are some and also a few easily correctable logical flaws. A less obvious error is the author's assertion that the 3-dimensional Madelung constant associated with the crystallographic structure of NaCl is given by the sum of the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{r_3(n)}{\sqrt{n}}$$

where $r_3(n)$ is the number of ways of writing n as the sum of three squares. It is known, however, that the n-th term of this series does not tend to zero, so that the series is divergent, whereas its 2-dimensional counterpart is convergent and does measure the 2-dimensional Madelung constant. Figures 3.1 and 3.2 certainly confirm this dichotomy.

There are many other diagrams in the book – some helpful in understanding the mathematics, and some (to me) confusing. There are appendices containing some basic real and complex function theory to help the reader who is either vague about or has never known the parts needed to follow the mathematical ideas in the text. An aspect of the book I enjoyed *inter alia* was that it prodded me into revisiting the Euler-Maclaurin summation formula and to understanding for the first time the real nature of the remainder term therein.

To sum up, the book contains a wealth of interesting material, both mathematical and historical. It is not a book that one would normally read from cover to cover, but anyone with curiosity and some basic mathematical knowledge could browse through it and pick parts which would be fascinating and instructive. The book would make a nice gift for anyone mathematically inclined.v