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Tauberian theory, a century of developments, by Jacob Korevaar, Springer-Verlag, Berlin, Heidelberg, 2004, xv+483 pp., \$109.00, ISBN 3-540-21058-X

Just over a century ago, in 1897, Tauber [11] proved the following "corrected converse" of Abel's theorem:

Theorem T. *If* (i) $\sum_{n=0}^{\infty} a_n x^n \to \ell$ *as* $x \to 1-$, *and*

(T$_0$) $\quad na_n = o(1)$,

then (ii) $\sum_{n=0}^{\infty} a_n = \ell$.

Abel's theorem, of course, is the familiar result that (ii)⇒(i) without condition (T$_0$). Subsequently Hardy and Littlewood proved numerous other such converse theorems, and they coined the term *Tauberian* to describe them.

In summability language Theorem T can be expressed as:

If $\sum_{n=0}^{\infty} a_n = \ell\,(A)$, *where* A *denotes the Abel summability method, and if the Tauberian condition* (T$_0$) *holds, then* $\sum_{n=0}^{\infty} a_n = \ell$.

The simplest example of an Abel summable series that is not convergent is given by $a_n := (-1)^n$, for which $\sum_{n=0}^{\infty} a_n = \frac{1}{2}\,(A)$.

Tauber's innocent looking theorem was the start of a veritable Tauberian jungle of results which Korevaar, in the book under review, has made a very worthwhile effort to organize and present in a coherent manner. The book's 483 pages are densely packed and there are around 800 references. Rather than trying for a comprehensive description of its contents, this review will cut a reasonably narrow path through part of the jungle, in the hope that it will give the non-expert reader a view of what the subject is about. There are a few proofs, but they can be skipped by the reader primarily interested in the statement of the results.

In 1914 Hardy and Littlewood [3] proved the following generalization of Theorem T in which the strong "two-sided" Tauberian condition (T$_0$) is replaced by the much weaker "one-sided" condition (T$_1$):

Theorem H-L. *If* $\sum_{n=0}^{\infty} a_n x^n \to \ell$ *as* $x \to 1-$, *and*

(T$_1$) $\quad na_n \le C$, *a positive constant,*

then $\sum_{n=0}^{\infty} a_n = \ell$.

Note that by changing sign throughout, the Tauberian condition (T$_1$) could be expressed as $na_n \ge -C$. An interesting, and non-trivial, illustration of the potency

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of Theorem H-L is a proof that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z},$$

which is absolutely convergent and defines the Riemann zeta function $\zeta(z)$ when $\Re z > 1$, is not Abel summable on the line $z = 1 + it$. This amounts to observing that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+it}}$$

cannot be Abel summable, for if it were, Theorem H-L (or even a weaker two-sided version of it) would imply that the series is actually convergent, which it cannot be since Hardy [2, §7.9] has shown that, for fixed $t \neq 0$,

$$\sum_{n=1}^{N} \frac{1}{n^{1+it}} = \frac{i}{tN^{it}} + \ell + o(1) \text{ as } N \to \infty,$$

where ℓ is finite and independent of N. In fact ℓ turns out to be $\zeta(1 + it)$.

Karamata [6] simplified Hardy and Littlewood's proof of Theorem H-L in 1930, and in 1952 Wielandt [15] elegantly modified Karamata's proof as follows:

Suppose, without loss in generality, that $\sum_{n=0}^{\infty} a_n x^n \to 0$ as $x \to 1-$. Let $\mathfrak{F}$ be the linear space of real functions f for which

$$\sum_{n=0}^{\infty} a_n f(x^n) \to 0 \text{ as } x \to 1- .$$

Then every real polynomial p with $p(0) = 0$ is in $\mathfrak{F}$. Let $g := \chi_{[1/2,1]}$, the characteristic function of $[1/2, 1]$. Given $\varepsilon > 0$, there are real polynomials p_1, p_2 with $p_1(0) = p_2(0) = 0$ and $p_1(1) = p_2(1)$ such that $p_1(x) \leq g(x) \leq p_2(x)$ for $0 \leq x \leq 1$, and

$$\int_0^1 \frac{p_2(t) - p_1(t)}{t(1-t)}\, dt < \frac{\varepsilon}{C}.$$

Then, by (T_1),

$$\begin{aligned}\sum_{n=0}^{\infty} a_n g(x^n) - \sum_{n=0}^{\infty} a_n p_1(x^n) &\leq C \sum_{n=1}^{\infty} \frac{p_2(x^n) - p_1(x^n)}{n} \\ &= C \sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n} q(x^n) \leq C\,(1-x) \sum_{n=0}^{\infty} x^n q(x^n),\end{aligned}$$

where

$$q(x) := \frac{p_2(x) - p_1(x)}{x(1-x)} =: \sum_{k=0}^{m} b_k x^k.$$

Further, as $x \to 1-$,

$$(1-x)\sum_{n=0}^{\infty} x^n q(x^n) = \sum_{k=0}^{m} b_k \frac{1-x}{1-x^{k+1}} \to \sum_{k=0}^{m} \frac{b_k}{k+1} = \int_0^1 q(t)\, dt < \frac{\varepsilon}{C}.$$

Hence

$$\limsup_{x\to 1-} \sum_{n=0}^{\infty} a_n g(x^n) < \varepsilon,$$

and likewise

$$\liminf_{x\to 1-} \sum_{n=0}^{\infty} a_n g(x^n) > -\varepsilon.$$

It follows that $g \in \mathfrak{F}$, and therefore, for $N = \lfloor -\log 2/\log x \rfloor$,

$$\sum_{n=0}^{\infty} a_n g(x^n) = \sum_{n=0}^{N} a_n \to 0 \text{ as } x \to 1- .$$

Another proof of Theorem H-L is by means of Wiener's powerful Tauberian theorem involving Fourier transforms [16] which he published in 1932:

Theorem W. *If* $K \in L(-\infty,\infty)$, $\phi \in L^{\infty}(-\infty,\infty)$,

$$\int_{-\infty}^{\infty} e^{-itx} K(t)\,dt \neq 0 \quad \forall x \in (-\infty,\infty), \text{ and}$$

$$\int_{-\infty}^{\infty} K(x-t)\phi(t)\,dt = o(1) \text{ as } x \to \infty,$$

then, $\forall H \in L(-\infty,\infty)$,

$$\int_{-\infty}^{\infty} H(x-t)\phi(t)\,dt = o(1) \text{ as } x \to \infty. \tag{1}$$

To prove Theorem H-L with $\ell = 0$ by means of Theorem W, let

$$s(x) := \sum_{n\le x} a_n, \qquad \text{and} \qquad F(x) := \sum_{n=0}^{\infty} a_n x^n.$$

Then, by hypothesis, $F(x) = o(1)$ as $x \to 1-$, and it follows (fairly easily) from this and (T_1) that $s(x) = O(1)$, and hence that, for $t > 0$,

$$F(e^{-t}) = \sum_{n=0}^{\infty} a_n e^{-nt} = \int_0^{\infty} e^{-tx}\,ds(x) = t\int_0^{\infty} e^{-tx} s(x)\,dx.$$

Now take $\phi(x) := s(e^x)$ and $K(x) := \exp(-x - e^{-x})$. Then

$$\int_{-\infty}^{\infty} K(x-t)\phi(t)\,dt = F\big(\exp(-e^{-x})\big) = o(1) \text{ as } x \to \infty,$$

and, $\forall x \in (-\infty,\infty)$,

$$\int_{-\infty}^{\infty} e^{-itx} K(t)\,dt = \int_0^{\infty} u^{ix} e^{-u}\,du = \Gamma(1+ix) \neq 0.$$

Further, $\phi(x) = O(1)$, and it follows from (T_1) that, given $\delta > 0$, $\exists x_0$ such that

$$\phi(y) - \phi(x) \le 2\delta \text{ for } x_0 \le x \le y \le x + \delta.$$

Taking $H := \delta^{-1}\chi_{[0,\delta]}$ and then $H := \delta^{-1}\chi_{[-\delta,0]}$ in (1), we obtain respectively

$$\limsup_{x\to\infty} \phi(x) \le 2\delta \qquad \text{and} \qquad \liminf_{x\to\infty} \phi(x) \ge -2\delta,$$

from which it follows that $\phi(x) \to 0$ and hence that $s(x) \to 0$ as $x \to \infty$.

Wiener's theorem yields Tauberian theorems for many standard summability methods.

Karamata proved various Tauberian theorems, the most famous being the following one [7] about Laplace transforms, which he proved in 1931:

Theorem K. *Let A be a non-decreasing, unbounded function on $[0,\infty)$ with $A(0) \geq 0$, and let L be a slowly varying function (i.e., $\forall t > 0$, $L(xt)/L(x) \to 1$ as $x \to \infty$). Then, for $\sigma \geq 0$,*

$$B(x) := \int_0^\infty e^{-t/x} dA(t) \sim x^\sigma L(x) \text{ as } x \to \infty$$

(i.e., B is regularly varying with index σ) if and only if

$$A(x) \sim \frac{x^\sigma L(x)}{\Gamma(1+\sigma)} \text{ as } x \to \infty.$$

From this theorem Karamata derived:

Theorem $\mathbf{K}_1$. *Let A be a non-decreasing, unbounded and regularly varying function on $[0,\infty)$ with $A(0) \geq 0$, and let the function s be continuous and bounded below on $[0,\infty)$. If*

$$\int_0^\infty e^{-yt} s(t)\, dA(t) \sim \ell \int_0^\infty e^{-yt}\, dA(t) \text{ as } y \to 0+\,, \tag{2}$$

then

$$\frac{1}{A(x)} \int_0^x s(t)\, dA(t) \to \ell \text{ as } x \to \infty. \tag{3}$$

This is also a Tauberian theorem since (3) $\Rightarrow$ (2) without the one-sided boundedness condition on s. It follows from a theorem established by Korenblum [8] in 1955 that the condition in Theorem K_1 that A be regularly varying can be replaced by the weaker condition

$$\frac{A(y)}{A(x)} \to 1 \text{ when } \frac{y}{x} \to 1,\ y > x \to \infty \tag{4}$$

(i.e., $\log A(x)$ is slowly oscillating). From this extension of Theorem K_1, the reviewer [1] was able to prove:

Theorem DB. *Let A be a non-decreasing, unbounded function on $[0,\infty)$ with $A(0) \geq 0$, and let the function s be continuous $[0,\infty)$. If (2) and (4) are satisfied, and in addition*

$$\liminf\{s(y) - s(x)\} \geq 0 \text{ when } \frac{y}{x} \to 1,\ y > x \to \infty, \tag{5}$$

then $s(x) \to \ell$ as $x \to \infty$.

The proof uses a variant of a method developed by Vijayaraghavan [12], [13] in 1926 (see [2, Theorem 238]) to first deduce that $s(x)$ is bounded. Theorem DB can be specialized by taking

$$A(x) := n \text{ for } n \leq x < n+1,\ n = 0, 1, \ldots, \text{ and}$$

$$s(n) := s_n := \sum_{k=0}^{n} a_k,$$

to obtain as a corollary the following result which Schmidt [9] established in 1925:

Corollary. *If*

$$\sum_{n=0}^{\infty} s_n e^{-ny} \sim \ell \sum_{n=0}^{\infty} e^{-ny} = \frac{\ell}{1-e^{-y}} \text{ as } y \to 0+ \tag{6}$$

and

$$\liminf(s_m - s_n) \geq 0 \text{ when } \frac{m}{n} \to 1,\ m > n \to \infty \tag{7}$$

(*i.e.*, s_n *is slowly decreasing*), *then* $s_n \to \ell$.

Note that (6) is equivalent to

$$(1-x)\sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty} a_n x^n \to \ell \text{ as } x \to 1-$$

and that $na_n > -C \Rightarrow (7)$, so that the corollary generalizes Theorem H-L.

Another classical Tauberian result concerns the Borel method B defined by:

$$\sum_{n=0}^{\infty} a_n = \ell\,(B), \text{ or } s_n \to \ell\,(B),$$

$$\text{if } e^{-x}\sum_{n=0}^{\infty} \frac{s_n x^n}{n!} \to \ell \text{ as } x \to \infty, \qquad \text{where } s_n := \sum_{k=0}^{n} a_k.$$

Theorem B. *If* $\sum_{n=0}^{\infty} a_n = \ell\,(B)$, *and*

(T$_2$) $\sqrt{n}a_n \leq C$,

then $\sum_{n=0}^{\infty} a_n = \ell$.

The version of this result with (T$_2$) replaced by the stronger two-sided condition $\sqrt{n}a_n = O(1)$ was proved by Hardy and Littlewood [4] in 1916. In 1925 Schmidt [10] showed that (T$_2$) can be relaxed to

$$\liminf(s_m - s_n) \geq 0 \text{ when } 0 < \sqrt{m} - \sqrt{n} \to 0,\ n \to \infty.$$

Various Tauberian theorems have been used in assorted proofs of the prime number theorem. A particularly interesting one is the following one proved in 1931 by Ikehara [5], a student and colleague of Wiener's:

Theorem I-W. *Suppose that the function F has the following properties:*

(i) *For* $\Re z > 1$, $F(z) = \int_0^{\infty} e^{-zt}A(t)\,dt$, *where A is a non-decreasing function with* $A(0) \geq 0$.

(ii) *For* $\Re z > 1$, $z \neq 1$, $F(z) = G(z) + \dfrac{1}{z-1}$, *where $G(z)$ is continuous on the half-plane* $\Re z \geq 1$.

Then $e^{-t}A(t) \to 1$ *as* $t \to \infty$.

The prime number theorem can be proved with the aid of Theorem I-W as follows: Let

$$A(t) := \psi(e^t), \text{ where } \psi(x) := \sum_{p^n \leq x} \log p.$$

The p's in the sum defining the Chebyshev function ψ are the odd primes, and it is known that the prime number theorem, viz.,

$$\pi(x) := \sum_{p \le x} \sim \frac{x}{\log x} \text{ as } x \to \infty,$$

is equivalent to $\psi(x) \sim x$ as $x \to \infty$.

For $\Re z > 1$, we have [14, ch. V, §17] that

$$F(z) = \int_0^\infty e^{-zt} A(t)\, dt = \int_1^\infty u^{-z-1} \psi(u)\, du = -\frac{\zeta'(z)}{z\zeta(z)} = G(z) + \frac{1}{z-1},$$

the function G satisfying the requirements of Theorem I-W since the Riemann zeta function $\zeta(z)$ has no zeros in the half plane $\Re z \ge 1$ and is holomorphic in the whole plane, except for a simple pole at $z = 1$ with residue 1. Hence, by Theorem I-W, $e^{-t}\psi(e^t) \to 1$ as $t \to \infty$ and so $\psi(x) \sim x$ as $x \to \infty$.

Most of the above topics are dealt with in the book (though not always in the same manner). Among the many others there are the spectacular "high indices" theorems, Gelfand's algebraic treatment of Wiener theory and the author's own distributional approach. There is also his new unified theory for the Borel and "circle" methods of summability. And much more, including a chapter on Tauberian remainder theory. Though not an easy read, the book is a *must have* for anyone seriously interested in Tauberian theory.

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