ON MULTIPLICATION OF $(C, -\mu)$-SUMMABLE SERIES

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The main results proved in this paper are:

THEOREM 1. *If* $\lambda \geqslant 1$, $\lambda > \mu$, $\delta > 0$, *and the series* $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ *are summable* $(C, -\mu)$ *to* A, B *respectively and both are bounded* $(C, -\lambda)$, *and if* $c_n = \sum_{r=0}^n a_r b_{n-r}$, *then* $\sum_0^\infty c_n$ *is summable* $(C, -\lambda+\delta)$ *to* AB.

THEOREM 2. *If* $\mu \geqslant 0$, $\sum_0^\infty a_n$ *is absolutely summable* $(C, -\mu)$ *to* A *and* $\sum_0^\infty b_n$ *is summable* $(C, -\mu)$ *to* B, *and if* $c_n = \sum_{r=0}^n a_r b_{n-r}$, *then* $\sum_0^\infty c_n$ *is summable* $(C, -\mu)$ *to* AB.

Certain cases of these theorems are known. The cases $\mu = 1, 2, \ldots$, $\lambda = \mu+1$, $\delta = 1$ of Theorem 1 and $\mu = 1, 2, \ldots$ of Theorem 2 are due to Palmer [**6**]. The case $\mu = 0$, $\lambda = 1$ of Theorem 1 is due to Hardy ([**2**], 230-231), and the case $\mu = 0$ of Theorem 2 is Mertens' classical theorem.

1. *Notation and preliminary results.*

Let $s_n = \sum_{r=0}^n a_r$ $(n = 0, 1, \ldots)$, let $\{\mu_n\}$ be a sequence of real numbers and let

$$\sigma_n = \sum_{r=0}^n s_r \binom{n}{r} \sum_{\nu=0}^{n-r} (-1)^\nu \binom{n-r}{\nu} \mu_{r+\nu}.$$

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Denote the matrix of the linear transformation from $\{s_n\}$ to $\{\sigma_n\}$ by H. Then H is a Hausdorff matrix which is said to be generated by the sequence $\{\mu_n\}$. We write $H(s_n)$ for σ_n; and if $\mu_n \neq 0$, we denote by H^{-1} the Hausdorff matrix generated by $\{1/\mu_n\}$.

If $\sigma_n \to s$ (finite) we say that $\sum_0^\infty a_n$ is summable (H) to s and write $s_n \to s(H)$, and if, in addition, $\sum_1^\infty |\sigma_n - \sigma_{n-1}| < \infty$ we say that the series is absolutely summable (H) and write $s_n \to s|H|$ (see Knopp and Lorentz [4]). If $\sigma_n = O(1)$ we say that $\sum_0^\infty a_n$ is bounded (H) and write $s_n = O(1)$ (H). H is said to be regular if $s_n \to s(H)$ whenever $s_n \to s$.

Suppose now that K is a Hausdorff matrix generated by the real sequence $\{\nu_n\}$. Then it is known that HK is a Hausdorff matrix generated by $\{\mu_n \nu_n\}$, so that $HK = KH$. This result is proved in [2], Ch. XI, as are all other standard results about Hausdorff matrices quoted in this paper.

If $s_n \to s$ (H) whenever $s_n \to s$ (K) we write $H \supseteq K$, and if $s_n \to s|H|$ whenever $s_n \to s|K|$ we write $|H| \supseteq |K|$. If $H \supseteq K$ and $K \supseteq H$ we write $H \simeq K$, and if $|H| \supseteq |K|$ and $|K| \supseteq |H|$ we write $|H| \simeq |K|$.

It is known and easily demonstrated that, when $\mu_n \neq 0$, $K \supseteq H$ if and only if KH^{-1} is regular.

We use the notation:

$$\epsilon_n^\alpha = \binom{n+\alpha}{n}, \quad \Delta^\alpha s_n = \sum_{r=0}^{n} \epsilon_{n-r}^{-\alpha-1} s_r \quad (n = 0, 1, \ldots; \text{ any real } \alpha);$$

so that, for $n = 0, 1, \ldots$, $\Delta s_n = \Delta^1 s_n = s_n - s_{n-1}$ $(s_{-1} = 0)$.

Denote by (C, α, β) the matrix of the linear transformation from $\{s_n\}$ to $\{\sigma_n\}$ given by

$$\sigma_n = \frac{1}{\epsilon_n^{\alpha+\beta}} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} \epsilon_r^\beta s_r = \frac{1}{\epsilon_n^{\alpha+\beta}} \Delta^{-\alpha}(\epsilon_n^\beta s_n) \quad (\beta > -1, \alpha+\beta > -1).$$

Then $(C, \alpha, 0)$ $(\alpha > -1)$ is the Cesàro matrix (C, α) which, in the specified range only, is the matrix of the Cesàro summability method (C, α). It has been shown elsewhere (Borwein [1], Theorem 8) that (C, α, β) is the Hausdorff matrix generated by the sequence $\{\epsilon_n^\beta / \epsilon_n^{\alpha+\beta}\}$.

Let (C^*, α) be the Hausdorff matrix generated by $\{1/\epsilon_n^\alpha\}$ when $\alpha > -1$, and by $\{\epsilon_n^{-\alpha}\}$ when $\alpha \leqslant -1$; so that

$$(C^*, \alpha) = (C, \alpha) \quad (\alpha > -1),$$

$$(C^*, \alpha) = (C, -\alpha)^{-1} = (C, \alpha, -\alpha) \quad (\alpha \leqslant -1).$$

The following results are known (see [1], and the references there given):

(I) $(C^*, \alpha) \simeq (H, \alpha)$ (*all real* α),

(II) $(C, \alpha, \beta) \simeq (C^*, \alpha)$ $(\beta > -1, \alpha+\beta > -1)$,

(III) $(C^*, \alpha)(C^*, \beta) \simeq (C^*, \alpha+\beta)$ (*all real* α, β),

(IV) $(C^*, \alpha+\delta) \supseteq (C^*, \alpha)$ (*all real* α; $\delta > 0$).

Here and elsewhere (H, α) is the matrix of the Hölder method of order α; it is the Hausdorff matrix generated by the sequence $\{(n+1)^{-\alpha}\}$.

For the purposes of this paper it is convenient to make the following

DEFINITIONS. *The statements*

(i) $\sum_0^\infty a_n$ *is summable* (C, α) *to* A,

(ii) $\sum_0^\infty a_n$ *is bounded* (C, α),

(iii) $\sum_0^\infty a_n$ *is absolutely summable* (C, α) *to* A,

mean respectively

(i)* $\sum_0^\infty a_n$ *is summable* (C^*, α) *to* A,

(ii)* $\sum_0^\infty a_n$ *is bounded* (C^*, α),

(iii)* $\sum_0^\infty a_n$ *is absolutely summable* (C^*, α) *to* A;

where the starred statements are to be interpreted in accordance with the introductory remarks about Hausdorff matrices.

These definitions are the usual ones in the range $\alpha > -1$. For $\alpha \leqslant -1$, the definitions of (i) and (ii) differ from the standard ones given by Hausdorff. However Hausdorff [3] proved (i) and (ii) (defined in his sense) to be equivalent respectively to (i)* and (ii)* with (H, α) in place of (C^*, α). In virtue therefore of (I) and I_0 (below), Hausdorff's definitions of (i) and (ii) are equivalent to the above. (See Lyra [5] for the case when α is a negative integer.)

A definition of (iii) for fractional $\alpha < -1$ does not appear to have been given explicitly before; but for integral $\alpha \leqslant -1$ the above definition coincides with one given by Lyra [5].

We require the following lemma, of which part (i) is trivial, and part (ii) is due to Knopp and Lorentz ([4], 12) who effectively obtained it as a consequence of a more general result. An alternative (direct) proof of part (ii) is given here.

LEMMA 1. *Let* $\mu_n = \int_0 t^n d\chi(t)$, *where* $\chi(t)$ *is a real function of bounded variation in* [0, 1]. *Let* H *be the Hausdorff matrix generated by the sequence* $\{\mu_n\}$, *and let* $\sigma_n = H(s_n)$.

(i) *If* $s_n = O(1)$, *then* $\sigma_n = O(1)$.

(ii) *If* $\sum_0^\infty |\Delta s_n| < \infty$, *then* $\sum_0^\infty |\Delta \sigma_n| < \infty$.

Proof. It is familiar and readily verified that

$$\sigma_n = \sum_{r=0}^{n} \binom{n}{r} s_r \int_0^1 t^r (1-t)^{n-r} d\chi(t).$$

Consequently, if $|s_n| < M < \infty$, then $|\sigma_n| \leqslant M \int_0^1 |d\chi(t)| < \infty$; and this establishes (i).

Now

$$(C, 1)^{-1}(\sigma_n) = \sigma_n + n\Delta\sigma_n = (C, 1)^{-1} H(s_n) = H(C, 1)^{-1}(s_n)$$
$$= H(s_n) + H(n\Delta s_n).$$

Hence
$$\Delta\sigma_n = \frac{1}{n} \sum_{r=1}^{n} \binom{n}{r} r\Delta s_r \int_0^1 t^r (1-t)^{n-r} d\chi(t) \quad (n \geqslant 1),$$

and so
$$\sum_{n=1}^{N} |\Delta\sigma_n| \leqslant \int_0^1 |d\chi(t)| \sum_{r=1}^{N} |\Delta s_r| \sum_{n=r}^{N} \binom{n}{r} \frac{r}{n} t^r (1-t)^{n-r}$$
$$\leqslant \int_0^1 |d\chi(t)| \sum_{r=1}^{N} |\Delta s_r| t^r \sum_{n=r}^{\infty} \binom{n-1}{r-1} (1-t)^{n-r}$$
$$= \int_0^1 |d\chi(t)| \sum_{r=1}^{N} |\Delta s_r|.$$

Result (ii) follows.

It is known that a necessary and sufficient condition for a Hausdorff matrix H generated by a real sequence $\{\mu_n\}$ to be regular is that

$$\mu_n = \int_0^1 t^n d\chi(t) \quad (n \geqslant 0),$$

where $\chi(t)$ is a real function of bounded variation in [0, 1] such that $\chi(0+) = \chi(0)$ and $\chi(1) - \chi(0) = 1$.

We therefore have the following corollary of Lemma 1.

LEMMA 2. *If* H, K *are real Hausdorff matrices and the former is generated by a sequence with no vanishing terms, and if* $K \supseteq H$, *then* $|K| \supseteq |H|$, *and* $s_n = O(1)$ (K) *whenever* $s_n = O(1)$ (H).

In consequence of Lemma 2 and results (I) to (IV), we now have the following results:

|I| $\quad |C^*, \alpha| \simeq |H, \alpha| \quad$ (*all real* α),

|II| $\quad |C, \alpha, \beta| \simeq |C^*, \alpha| \quad (\beta > -1,\ \alpha+\beta > -1)$,

|III| $\quad |(C^* \alpha)(C^*, \beta)| \simeq |C^*, \alpha+\beta| \quad$ (*all real* α, β),

|IV| $\quad |C^*, \alpha+\delta| \supseteq |C^*, \alpha| \quad$ (*all real* α; $\delta > 0$);

and also the corresponding results involving boundedness which we label I_0, II_0, III_0, IV_0.

The cases $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$ of |III|, and $\alpha > -1$ of |I| and |IV| are known (see Knopp and Lorentz [4]). Lyra [5] has proved |I| for $\alpha = -1, -2, \ldots$ and |IV| for $\alpha = -1, -2, \ldots, \delta = 1$. Various cases of results I_0 to IV_0 are also known.

We require two additional lemmas.

LEMMA 3. *If* $\beta > \alpha$, $\delta > 0$, *and* $\sum_0^\infty a_n$ *is bounded* (C, α) *and summable* (C, β) *to* A, *then the series is summable* $(C, \alpha+\delta)$ *to* A.

This result is known for the cases $\alpha \geqslant -1$ (see Hardy [2], Theorems 45 and 70, and the references given on p. 127) and $\alpha = -2, -3, \ldots$, $\beta = \alpha+2$, $\delta = 1$ (Lyra [5], 559).

Proof. Let $s_n = \sum_{r=0}^{n} a_r$, then

$$(C^*, \alpha)(s_n) = O(1) \ (C, 0).$$

Further, $s_n \to A(C^*, \beta)$, and so, by (III),

$$(C^*, \alpha)(s_n) \to A \ (C, \beta-\alpha).$$

Hence, by one of the known cases of the lemma,

$$(C^*, \alpha)(s_n) \to A \ (C, \delta),$$

so that, by (III), $s_n \to A \ (C^*, \alpha+\delta)$.

LEMMA 4. *Suppose that* $0 \leqslant \alpha < 1$, $0 < \delta \leqslant 1$, *and let*

$$P_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \epsilon_{n-r}^{-\alpha} x_{n-r} y_r, \quad Q_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \epsilon_{n-r}^{1-\alpha-\delta} \epsilon_r^{\delta-1} x_{n-r} y_r,$$

$$R_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \epsilon_{n-r}^{1-\alpha} x_{n-r} \Delta y_r.$$

(i) *If* $x_n \to \xi$, $y_n \to \eta$, *then* $P_n \to \xi\eta$.

(ii) *If* $x_n = O(1)$, $y_n \to 0$, *then* $Q_n \to 0$.

(iii) *If* $x_n \to \xi$, $y_n \to \eta$ *and* $\sum_0^\infty |\Delta x_n| < \infty$, *then* $R_n \to \xi\eta$.

(iv) *If* $x_n \to \xi$, $y_n \to \eta$ *and* $\sum_0^\infty |\Delta y_n| < \infty$, *then* $R_n \to \xi\eta$.

Parts (i) and (ii) are simple consequences of variants of Toeplitz's theorem.

Proof of (iii). Note that

$$R_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} y_r \Delta(\epsilon_{n-r}^{1-\alpha} x_{n-r}).$$

Now

$$\frac{\Delta(\epsilon_{n-r}^{1-\alpha} x_{n-r})}{\epsilon_n^{1-\alpha}} \to 0 \text{ as } n \to \infty,$$

and

$$\frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \Delta(\epsilon_{n-r}^{1-\alpha} x_{n-r}) = x_n \to \xi.$$

Further

$$\frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} |\Delta(\epsilon_{n-r}^{1-\alpha} x_{n-r}| \leqslant \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} (\epsilon_r^{1-\alpha} |\Delta x_r| + \epsilon_r^{-\alpha} |x_{r-1}|) \quad (x_{-1} = 0)$$

$$\leqslant \sum_{r=0}^{\infty} |\Delta x_r| + \sup_{r \geqslant 0} |x_r| < \infty.$$

Consequently, by Toeplitz's theorem, $R_n \to \xi\eta$.

Proof of (iv). We now observe that

$$\frac{\epsilon_r^{1-\alpha} \Delta y_{n-r}}{\epsilon_n^{1-\alpha}} \to 0 \text{ as } n \to \infty,$$

and, since $y_n \to \eta$, that

$$\frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \epsilon_{n-r}^{1-\alpha} \Delta y_r = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \epsilon_{n-r}^{-\alpha} y_r \to \eta.$$

Also

$$\frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} |\epsilon_{n-r}^{1-\alpha} \Delta y_r| \leqslant \sum_{r=0}^{\infty} |\Delta y_r| < \infty.$$

Hence, by Toeplitz's theorem, $R_n \to \xi\eta$.

2. *Proofs of the main theorems.*

Let $\mu = m + \alpha$, where m is a non-negative integer and $0 \leqslant \alpha < 1$; and let

$$s_n = \sum_{r=0}^{n} a_r, \quad t_n = \sum_{r=0}^{n} b_r, \quad c_n = \sum_{r=0}^{n} a_r b_{n-r}, \quad u_n = \sum_{r=0}^{n} c_n = \sum_{r=0}^{n} s_r b_{n-r}.$$

Then

$$(C, 1)(u_n) = \frac{1}{n+1} \sum_{r=0}^{n} s_r t_{n-r},$$

and, by (II) and (III), *a necessary and sufficient condition for* $\sum_1^\infty c_n$ *to be summable* $(C, -\mu)$ *to* AB, *is that*

$$v_n = (C, -\mu-1, m+2)(C, 1)(u_n)$$

should tend to AB.

Now

$$v_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \epsilon_{n-r}^{-\mu-2} \epsilon_r^{m+2} \frac{1}{r+1} \sum_{\nu=0}^{r} s_\nu t_{r-\nu},$$

and

$$\frac{1}{r+1} \epsilon_r^{m+2} = \frac{1}{m+2} \epsilon_{m+1}^{r+1} = \frac{1}{m+2} \sum_{p=0}^{m+1} \epsilon_\nu^{m+1-p} \epsilon_{r-\nu}^{p} \quad (\nu = 0, 1, \ldots, r).$$

Also, it is well known and easily verified that, for all real θ, ϕ,

$$\Delta^{\theta+\phi}\left(\sum_{r=0}^{n} x_r y_{n-r}\right) = \sum_{r=0}^{n} \Delta^\theta x_r \; \Delta^\phi y_{n-r}.$$

Hence (*cf.* Palmer [6], 262),

$$(m+2) v_n = \sum_{p=0}^{m+1} \frac{1}{\epsilon_n^{1-\alpha}} \Delta^{\mu+1}\left(\sum_{r=0}^{n} \epsilon_r^{m+1+p} s_r \epsilon_{n-r}^{p} t_{n-r}\right) = X_n + Y_n + Z_n,$$

where

$$X_n = \sum_{p=1}^{m} \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \Delta^{m+1-p}(\epsilon_r^{m+1-p} s_r) \; \Delta^{p+\alpha}(\epsilon_{n-r}^{p} t_{n-r})$$

when $m \geqslant 1$ and $X_n = 0$ when $m = 0$, and for any real β,

$$Y_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \Delta^{\mu+\beta}(\epsilon_{n-r}^{\mu+1-\alpha} s_{n-r}) \; \Delta^{1-\beta} t_r,$$

$$Z_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \Delta^{\mu+\beta}(\epsilon_{n-r}^{\mu+1-\alpha} t_{n-r}) \; \Delta^{1-\beta} s_r.$$

Proof of Theorem 1. Suppose that μ, δ of the hypotheses are such that

$$0 < \delta < 1, \quad \mu = \lambda - \delta.$$

In view of (IV) and Lemma 3, these extra conditions can be imposed without loss in generality.

We now have to show that $\sum_0^\infty c_n$ is summable $(C, -\mu)$ to AB, and, since $\mu > 0$, this is equivalent to showing that $v_n \to AB$.

Since $s_n \to A$ $(C^*, -\mu)$, $t_n \to B$ $(C^*, -\mu)$, it follows from (II) and (IV) that, if $m \geqslant 1$, $p = 1, 2, \ldots, m$,

$$\Delta^{m+1-p}(\epsilon_n^{m+1-p} s_n) \to A, \quad \frac{1}{\epsilon_n^{-\alpha}} \Delta^{p+\alpha}(\epsilon_n{}^p t_n) \to B.$$

Consequently, by Lemma 4(i),

$$X_n \to mAB \quad (m \geqslant 0).$$

Put $\beta = \delta$ in the expression for Y_n to get

$$Y_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \Delta^{\lambda}(\epsilon_{n-r}^{\mu+1-\alpha} s_{n-r})\ \Delta^{1-\delta}(t_r - B) + \frac{B}{\epsilon_n^{1-\alpha}} \Delta^{\mu}(\epsilon_n^{\mu+1-\alpha} s_n).$$

Note that, by (II), the second term tends to AB. Further, by (II) and (IV), since $1-\delta \leqslant \lambda - \delta = \mu$,

$$\Delta^{1-\delta}(t_n - B) = o(\epsilon_n^{\delta-1}),$$

and, by II_0, since $s_n = O(1)\ (C^*, -\lambda)$,

$$\Delta^{\lambda}(\epsilon_n^{\mu+1-\alpha} s_n) = O(\epsilon_n^{1-\alpha-\delta});$$

and consequently, by Lemma 4(ii), the first term tends to zero.

Hence $Y_n \to AB$, and similarly $Z_n \to AB$.

We have thus shown that $X_n + Y_n + Z_n = (m+2)v_n \to (m+2)AB$, and the proof is complete.

Proof of Theorem 2. As above, it follows from the hypotheses of the theorem that

$$X_n \to mAB.$$

Let

$$w_n = \frac{\Delta^{\mu}(\epsilon_n^{\mu+1-\alpha} s_n)}{\epsilon_n^{1-\alpha}}.$$

Now $s_n \to A\ |C^*, -\mu|$, $t_n \to B\ (C^*, -\mu)$ and so, by |II| and (IV),

$$w_n \to A, \quad \sum_0^{\infty} |\Delta w_n| < \infty, \quad t_n \to B.$$

Hence, by Lemma 4(iii),

$$Y_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \epsilon_{n-r}^{1-\alpha} w_{n-r} \Delta t_r \to AB.$$

Further, by (II) and |IV|,

$$\frac{1}{\epsilon_n^{1-\alpha}} \Delta^{\mu}(\epsilon_n^{\mu+1-\alpha} t_n) \to B, \quad s_n \to A, \quad \sum_0^{\infty} |\Delta s_n| < \infty;$$

so that, by Lemma 4(iv),

$$Z_n = \frac{1}{\epsilon_n^{1-\alpha}} \sum_{r=0}^{n} \Delta^{\mu}(\epsilon_{n-r}^{\mu+1-\alpha} t_{n-r})\ \Delta s_r \to AB.$$

Hence $v_n \to AB$, and the theorem is established.

3. *Additional results.*

We prove finally that the conclusions in Theorems 1 and 2 cannot be sharpened (*cf.* Palmer [6]), even when the hypotheses on $\sum_0^{\infty} a_n$ are replaced by the more restrictive hypothesis

(a) $\sum_0^{\infty} a_n$ is absolutely summable $(C, -\kappa)$ for every real κ.

Let $a_0 = 1$, $a_n = 0$ $(n \geqslant 1)$. Then $\sum_0^{\infty} a_n$ satisfies (a) and, for any series $\sum_0^{\infty} b_n$,

$$c_n = \sum_{r=0}^{n} a_r b_{n-r} = b_n.$$

Hence it is sufficient for our purpose to show that, given any real λ, μ, there are series $\sum_0^{\infty} b_n'$, $\sum_0^{\infty} b_n''$ such that

(b′) for every $\delta > 0$, $\sum_0^{\infty} b_n'$ is summable $(C, -\lambda+\delta)$ and is bounded but not summable $(C, -\lambda)$,

(b″) $\sum_0^{\infty} b_n''$ is summable $(C, -\mu)$ but is neither absolutely summable $(C, -\mu)$ nor bounded $(C, -\mu-\gamma)$ for any $\gamma > 0$.

Now it is familiar that the series $\sum_0^{\infty} (-1)^n$ satisfies (b′) with $\lambda = 0$, and that $\sum_0^{\infty} (-1)^n/\log(n+2)$ satisfies (b″) with $\mu = 0$. Consequently, in view of results (III), |III| and III_0, we can take

$$b_n' = \Delta(C^*, \lambda)\Big(\sum_{r=0}^{n} (-1)^r\Big), \quad b_n'' = \Delta(C^*, \mu)\Big(\sum_{r=0}^{n} (-1)^r/\log(r+2)\Big).$$

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