ON BOREL-TYPE METHODS OF SUMMABILITY

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1. *Introduction.* Suppose throughout that l, a_n $(n=0, 1, \ldots)$ are arbitrary complex numbers, that α is a fixed positive number and that x is a variable in the interval $[0, \infty)$. Let

$$s_n = \sum_{r=0}^{n} a_r, \quad a(x) = \sum_{n=0}^{\infty} \frac{a_n x^{\alpha n}}{\Gamma(\alpha n+1)}, \quad s(x) = \sum_{n=0}^{\infty} \frac{s_n x^{\alpha n}}{\Gamma(\alpha n+1)}.$$

We shall be concerned with Borel-type methods of summability (B', α), $(\tilde{B}, \alpha)$ defined as follows:

$$s_n \to l\,(B', \alpha) \text{ if } \lim_{x\to\infty} \int_0^x e^{-t} a(t)\,dt = l,$$

$$s_n \to l\,(\tilde{B}, \alpha) \text{ if } \lim_{x\to\infty} \alpha e^{-x} s(x) = l.$$

Note that $(B', 1)$ and $(\tilde{B}, 1)$ are respectively the Borel integral and the Borel exponential methods.

The first of the above definitions appears in Hardy's book ([2], 222). The exponential-type method (B, α) there defined* as a companion to (B', α) differs from the method $(\tilde{B}, \alpha)$, which seems to have been first considered, in this context, by Włodarski [4].

It is known ([2], 82-3) that (B', α) is regular, *i.e.* $s_n \to l(B', \alpha)$ whenever $s_n \to l$; and it is a trivial consequence of known results ([2], Theorem 33; and Lemma 3(b) below) that $(\tilde{B}, \alpha)$ is also regular.

A known result, concerning the relative strengths of different (B', λ) methods, is:

If $\alpha > \beta > 0$, $s_n \to l\,(B', \alpha)$ and $\sum_0^\infty a_n x^{\beta n}/\Gamma(\beta n+1)$ is convergent for all $x > 0$, then $s_n \to l\,(B', \beta)$.

The case $2\beta = \alpha$ of this result is due to Hardy [3], and the general case to Good [1].

A companion result is:

If $\alpha > \beta > 0$, $s_n \to l\,(\tilde{B}, \alpha)$ and $\sum_0^\infty s_n x^{\beta n}/\Gamma(\beta n+1)$ is convergent for all $x > 0$, then $s_n \to l\,(\tilde{B}, \beta)$.

The case $\alpha = 2^{-k}$, $\beta = 2^{-k-m}$ $(k=0, 1, \ldots;\ m=1, 2, \ldots)$ of this result has been stated by Włodarski [4], and I have proved the general case in a paper to be published shortly (in *Proc. Cambridge Phil. Soc.*).

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* $s_n \to l(B, \alpha)$ if $\lim_{x\to\infty} e^{-x} \sum_0^\infty \sigma(n/\alpha)\, x^n/n! = l$, where $\sigma(t) = \sum_{r \leqslant t} a_r$.

[Mathematika 5 (1958), 128–133]

The above results have been included for interest only and are not used in the rest of the note, the object of which is to prove the following:

Theorem. *In order that*

$$s_n \to l\,(\tilde{B}, \alpha),$$

it is necessary and sufficient that

$$s_n \to l\,(B', \alpha) \text{ and } a_n \to 0\,(\tilde{B}, \alpha).$$

The case $\alpha = 1$ of this theorem is known ([2], 183).

2. *Preliminary results.* We suppose in what follows that $f(x)$ is a continuous function for $x \geqslant 0$, and use the notation

$$f_0(x) = f(x), \quad f_\delta(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} f(t)\,dt \quad (\delta > 0).$$

This notation will also be used with other letters in place of f.

We require three lemmas.

Lemma 1. *If*

(i) $\phi(x)$ *is continuous for* $x > 0$,

(ii) $\lim_{x\to\infty} \phi(x) = 0$,

(iii) $\int_0^\infty |\phi(x)|\,dx < \infty$,

(iv) $\lim_{x\to\infty} f(x) = l$,

then

$$\lim_{x\to\infty} \int_0^x f(t)\,\phi(x-t)\,dt = l \int_0^\infty \phi(t)\,dt.$$

This is a special case of a standard result ([2], Theorem 6).

Lemma 2. (a) *If* $\lim_{x\to\infty} e^{-x} f(x) = l$ *and* $\delta > 0$, *then* $\lim_{x\to\infty} e^{-x} f_\delta(x) = l$.

(b) *If* $\lim_{x\to\infty} \int_0^x e^{-t} f(t)\,dt = l$ *and* $\delta > 0$, *then* $\lim_{x\to\infty} \int_0^x e^{-t} f_\delta(t)\,dt = l$.

Proof. Let $F(x) = \int_0^x e^{-t} f(t)\,dt$. Then

$$\Gamma(\delta)\, e^{-x} f_\delta(x) = \int_0^x e^{-t} f(t)\, (x-t)^{\delta-1} e^{-(x-t)}\,dt,$$

and so

$$\Gamma(\delta) \int_0^x e^{-t} f_\delta(t)\,dt = \int_0^x F(t)\, (x-t)^{\delta-1} e^{-(x-t)}\,dt.$$

Further $x^{\delta-1}e^{-x}$ is continuous for $x>0$ and tends to zero as $x\to\infty$; also

$$\int_0^\infty x^{\delta-1}e^{-x}\,dx=\Gamma(\delta).$$

In virtue of Lemma 1, results (a) and (b) follow.

Lemma 3. *If $\delta>4$, N is the integer such that $N\leqslant\delta/4<N+1$, and*

$$f(x)=\sum_{n=0}^\infty \frac{x^{\delta n}}{\Gamma(\delta n+1)},$$

then

(a) $\delta f(x)=\sum_{n=-N}^{N} e^{xw_n}+O(1),$

where $w_n=e^{i2n\pi/\delta}$;

(b) $\lim_{x\to\infty}\delta\, e^{-x}f(x)=1$;

(c) $e^{-x}\{f(x)-f_1(x)\}=O(e^{-\gamma x}),$

where $\gamma=2\sin^2\pi/\delta$.

Proof. Results (a) and (b) are known ([2], 198), (b) being an immediate consequence of (a). In view of (a),

$$f(x)-f_1(x)=\sum_{n=-N}^{N}(1-w_{-n})\,e^{xw_n}+O(1+x)=O(e^{x\cos(2\pi/\delta)}),$$

and result (c) follows.

Next, we prove that the following two statements are equivalent:

(A) $\sum_{n=0}^\infty \frac{a_n x^{\alpha n}}{\Gamma(\alpha n+1)}$ is convergent for all $x\geqslant 0$,

(B) $\sum_{n=0}^\infty \frac{s_n x^{\alpha n}}{\Gamma(\alpha n+1)}$ is convergent for all $x\geqslant 0$.

First assume (A). Then, given $\epsilon>0$, there is a positive integer N such that, for $n>N$,

$$|a_n|<\epsilon^n\,\Gamma(\alpha n+1)<\epsilon^{n+1}\,\Gamma(\alpha n+\alpha+1);$$

so that, for $n>N$,

$$|s_n|<\sum_{r=0}^{N}|a_r|+(n-N)\,\epsilon^n\,\Gamma(\alpha n+1).$$

Hence, for all n sufficiently large,

$$|s_n|<(2\epsilon)^n\,\Gamma(\alpha n+1),$$

and (B) follows.

Now assume (B). Then, since $\lim_{n\to\infty}\{\Gamma(\alpha n+\alpha+1)/\Gamma(\alpha n+1)\}^{1/n}=1$, $\sum_{n=0}^\infty s_n x^{\alpha n+\alpha}/\Gamma(\alpha n+\alpha+1)$ is convergent for all $x\geqslant 0$. Further,

$$\sum_{n=0}^\infty \frac{a_n x^{\alpha n}}{\Gamma(\alpha n+1)}=\sum_{n=0}^\infty \frac{s_n x^{\alpha n}}{\Gamma(\alpha n+1)}-\sum_{n=0}^\infty \frac{s_n x^{\alpha n+\alpha}}{\Gamma(\alpha n+\alpha+1)}, \tag{1}$$

and (A) follows.

We conclude this section with two useful identities. Suppose that (A) and (B) hold. It follows from (1) that

$$\begin{aligned} a(x)&=s(x)-\sum_{n=0}^\infty \frac{s_n}{\Gamma(\alpha n+1)\,\Gamma(\alpha)}\int_0^x t^{\alpha n}(x-t)^{\alpha-1}\,dt\\ &=s(x)-\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,dt\sum_{n=0}^\infty \frac{s_n t^{\alpha n}}{\Gamma(\alpha n+1)}\\ &=s(x)-s_\alpha(x), \end{aligned} \tag{2}$$

the inversion being legitimate since

$$\int_0^x (x-t)^{\alpha-1}\,dt\sum_{n=0}^\infty \frac{|s_n|\,t^{\alpha n}}{\Gamma(\alpha n+1)}<\infty.$$

Hence,

$$\begin{aligned} \int_0^x e^{-t}a(t)\,dt&=\int_0^x e^{-t}s(t)\,dt-\frac{1}{\Gamma(\alpha)}\int_0^x e^{-t}\,dt\int_0^t (t-u)^{\alpha-1}s(u)\,du\\ &=\int_0^x e^{-t}s(t)\,dt-\frac{1}{\Gamma(\alpha)}\int_0^x e^{-u}s(u)\,du\int_u^\infty (t-u)^{\alpha-1}e^{-(t-u)}\,dt\\ &\qquad+\frac{1}{\Gamma(\alpha)}\int_0^x e^{-u}s(u)\,du\int_x^\infty (t-u)^{\alpha-1}e^{-(t-u)}\,dt\\ &=\frac{1}{\Gamma(\alpha)}\int_0^x e^{-u}s(u)\,du\int_{x-u}^\infty t^{\alpha-1}e^{-t}\,dt. \end{aligned} \tag{3}$$

3. *Proof of the theorem. Necessity.* The hypothesis is:

$$\lim_{x\to\infty}\alpha e^{-x}s(x)=l.$$

In virtue of (2) and Lemma 2(a), it follows that

$$\lim_{x\to\infty}e^{-x}a(x)=0.$$

Consider now identity (3). Note that $\int_x^\infty t^{\alpha-1}e^{-t}\,dt$ is a continuous function for $x>0$ which tends to zero as $x\to\infty$, and that

$$\int_0^\infty dx\int_x^\infty t^{\alpha-1}e^{-t}\,dt=\alpha\Gamma(\alpha).$$

Hence, by Lemma 1,

$$\lim_{x\to\infty}\int_0^x e^{-t}a(t)\,dt = l.$$

This completes the proof of the necessity part of the theorem.

Sufficiency. The hypotheses are:

$$\lim_{x\to\infty}\int_0^x e^{-t}a(t)\,dt = l, \tag{4}$$

$$\lim_{x\to\infty} e^{-x}a(x) = 0. \tag{5}$$

Suppose that

$$k\alpha = \delta > 4,$$

where k is an integer; and let

$$b(x) = \sum_{r=0}^{k-1} a_{\alpha r}(x), \quad B(x) = \int_0^x e^{-t}b(t)\,dt, \quad f(x) = \sum_{n=0}^{\infty}\frac{x^{\delta n}}{\Gamma(\delta n+1)},$$

$$\phi(x) = \frac{d}{dx}\{e^{-x}f_{\delta-1}(x)\}.$$

Then, in virtue of (4) and Lemma 2(b),

$$\lim_{x\to\infty} B(x) = kl; \tag{6}$$

and, by (2),

$$b(x) = s(x) - s_\delta(x).$$

Also,

$$f_{\delta-1}(x) = \sum_{n=1}^{\infty}\frac{1}{\Gamma(\delta n-\delta+1)\,\Gamma(\delta-1)}\int_0^x (x-t)^{\delta-2}t^{\delta n-\delta}\,dt$$

$$= \sum_{n=1}^{\infty}\frac{x^{\delta n-1}}{\Gamma(\delta n)}.$$

Further, it is easily verified that $\lim_{m\to\infty} s_{\delta m}(x) = 0$, and so

$$s_\delta(x) = \sum_{n=1}^{\infty}\{s_{\delta n}(x) - s_{\delta n+\delta}(x)\} = \sum_{n=1}^{\infty} b_{\delta n}(x)$$

$$= \sum_{n=1}^{\infty}\frac{1}{\Gamma(\delta n)}\int_0^x t^{\delta n-1}b(x-t)\,dt$$

$$= \int_0^x b(x-t)f_{\delta-1}(t)\,dt,$$

the inversion being legitimate since $\int_0^x |b(x-t)|f_{\delta-1}(t)\,dt < \infty$. A partial integration yields

$$e^{-x}s_\delta(x) = \int_0^x B(x-t)\,\phi(t)\,dt. \tag{7}$$

We prove next that $\phi(x)$ satisfies the conditions of Lemma 1. By Lemma 3(b),

$$\lim_{x\to\infty}\delta e^{-x}f(x) = 1.$$

Hence, by Lemma 2(a),

$$\lim_{x\to\infty}\int_0^x \phi(t)\,dt = \lim_{x\to\infty} e^{-x}f_{\delta-1}(x) = 1/\delta$$

and

$$\lim_{x\to\infty}\phi(x) = \lim_{x\to\infty} e^{-x}\{f_{\delta-2}(x) - f_{\delta-1}(x)\} = 0.$$

Further $\phi(x)$ is continuous for $x \geqslant 0$, and, in view of Lemma 3(c),

$$\int_0^\infty |\phi(x)|\,dx \leqslant \frac{1}{\Gamma(\delta-2)}\int_0^\infty e^{-x}dx\int_0^x (x-t)^{\delta-3}|f(t)-f_1(t)|\,dt$$

$$= \int_0^\infty e^{-t}|f(t)-f_1(t)|\,dt < \infty.$$

Consequently, it follows from (6) and (7), by Lemma 1, that

$$\lim_{x\to\infty}\alpha e^{-x}s_\delta(x) = l.$$

Note that so far we have only used hypothesis (4). To complete the proof we have only to observe that, in virtue of (2) and Lemma 2(a), a consequence of hypothesis (5) is that

$$\lim_{x\to\infty} e^{-x}\{s(x) - s_\delta(x)\} = 0;$$

whence

$$\lim_{x\to\infty}\alpha e^{-x}s(x) = l.$$

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