

# The Effect of Boundary Conditions and Mesh Size on the Accuracy of Finite Difference Solutions of Two-Point Boundary Problems

By DAVID BORWEIN and ANDREW R. MITCHELL, St. Andrews, Scotland<sup>1)</sup>

## 1. Introduction

In a recent communication, FOX and MITCHELL [1]<sup>2)</sup> showed that in certain problems, provided the finite difference interval is chosen with sufficient care, boundary value methods give good results when step-by-step solutions are unstable. The purpose of the present paper is to examine critically the accuracy of boundary value techniques based on finite difference methods as applied to the numerical solution of ordinary differential equations with two-point boundary conditions.

Fox [2], using the method of matrix inversion, examined this problem in some detail. In the present paper, matrix methods are not used. Instead, exact solutions of the finite difference replacement of the chosen differential equation are obtained and compared with corresponding exact solutions of the differential equation for a variety of mesh lengths and two-point boundary conditions.

In addition, the accuracy of approximate numerical solutions of the difference equation is examined. The latter solutions can be obtained either by an iterative process such as relaxation or by direct methods of solution of the associated set of simultaneous equations.

## 2. Conditions for Which the Difference Equation Has No Solution

Consider the differential equation

$$\frac{d^2y}{dx^2} + j \frac{dy}{dx} + ky = g(x) \quad (1)$$

<sup>1)</sup> Mathematics Dept., University of St. Andrews. A. R. MITCHELL is now at the California Institute of Technology, Pasadena, California.

<sup>2)</sup> Numbers in brackets refer to References, page 232.

over the range  $0 \leq x \leq L$ , with  $j$  and  $k$  constants and  $g(x)$  an arbitrary function of  $x$ . If the range is divided by  $N$  internal nodes, distance  $h$  apart, a simple finite difference replacement of (1) is

$$a y_{r+2} + b y_{r+1} + c y_r = f_r \quad [r = 0, 1, 2, \dots, (N - 1)], \quad (2)$$

where  $a = 1 + j h/2$ ,  $b = k h^2 - 2$ ,  $c = 1 - j h/2$ ,  $f_r = h^2 g\{(r + 1) h\}$  and  $y_r, y_{r+1}, y_{r+2}$  are the values of  $y$  at  $x = r h, (r + 1) h, (r + 2) h$  respectively. Throughout the paper, the values  $h = \pm 2/j$ , for which (2) becomes a first order equation, are excluded. A solution of (2) is

$$y_0 = A + B, \quad (3)$$

$$y_{n+1} = A \lambda^{n+1} + B \mu^{n+1} + \frac{1}{a} \cdot \frac{1}{\lambda - \mu} \sum_{r=0}^n (\lambda^r - \mu^r) f_{n-r} \quad (0 \leq n \leq N),$$

where  $\lambda, \mu$  are distinct roots of the auxiliary equation  $a x^2 + b x + c = 0$ , and  $A, B$  are constants to be determined by the boundary conditions.

It is convenient at this stage to derive certain relationships involving  $\lambda$  and  $\mu$ , which are used throughout the paper. We select

$$\lambda = \frac{-b + (b^2 - 4 a c)^{1/2}}{2 a}, \quad \mu = \frac{-b - (b^2 - 4 a c)^{1/2}}{2 a},$$

where  $\lambda + \mu = -b/a$  and  $\lambda \mu = c/a$ . Three cases are considered:

(1)  $\lambda, \mu$  complex: Put  $\lambda = (c/a)^{1/2} e^{i\theta}, \mu = (c/a)^{1/2} e^{-i\theta}$ , to obtain the result

$$\lambda^s - \mu^s = 2 i \left(\frac{c}{a}\right)^{s/2} \sin s \theta, \quad (4)$$

where  $s$  is a variable.

(2)  $\lambda, \mu$  real ( $\lambda \mu > 0$ ): Put  $\lambda/\mu = e^{2\theta}$ , to get

$$\lambda^s - \mu^s = \begin{cases} 2 \left(\frac{c}{a}\right)^{s/2} \sinh s \theta & (\lambda > 0, \mu > 0), \\ (-1)^s 2 \left(\frac{c}{a}\right)^{s/2} \sinh s \theta & (\lambda < 0, \mu < 0). \end{cases} \quad (5a)$$

(3)  $\lambda, \mu$  real ( $\lambda \mu < 0$ ): Put  $-\lambda/\mu = e^{2\theta}$ , to obtain

$$\lambda^s - \mu^s = \begin{cases} 2 \left(-\frac{c}{a}\right)^{s/2} \sinh s \theta & (s \text{ even}), \\ 2 \left(-\frac{c}{a}\right)^{s/2} \cosh s \theta & (s \text{ odd}). \end{cases} \quad (5b)$$

Two different types of two-point boundary conditions are considered: (I)  $y = 0, Y$  at  $x = 0, L$ . In this case, the solution given by (3) becomes

$$y_{n+1} = \left. \begin{aligned} & y - \frac{1}{a} \cdot \frac{1}{\lambda - \mu} \sum_{r=0}^N (\lambda^r - \mu^r) f_{N-r} \\ & \frac{\lambda^{N+1} - \mu^{N+1}}{\lambda^{N+1} - \mu^{N+1}} (\lambda^{n+1} - \mu^{n+1}) \\ & + \frac{1}{a} \cdot \frac{1}{\lambda - \mu} \sum_{r=0}^n (\lambda^r - \mu^r) f_{n-r}, \end{aligned} \right\} \quad (6)$$

where  $0 \leq n \leq N$ . No solution of (6) exists in general if

$$\lambda^{N+1} - \mu^{N+1} = 0. \quad (7)$$

Using (4) and (5) with  $s = N + 1$ , it is seen after some manipulation that (7) is satisfied only if  $\lambda$  and  $\mu$  are imaginary and

$$k h^2 = 2 \left[ 1 - \left( 1 - \frac{1}{4} j^2 h^2 \right)^{1/2} \cos \frac{K \Pi}{N + 1} \right] \quad (K = 1, 2, \dots, N). \quad (8)$$

Mesh lengths for which the difference equation has, in general, no solution are said to be critical.

(II)  $y = 0$  at  $x = 0$ , and  $dy/dx + \alpha y = \beta$  at  $x = L$ . The derivative  $dy/dx + \alpha y = \beta$  at  $x = L$  is replaced by the finite difference expression

$$2 y_N + (b - 2 a \alpha h) y_{N+1} = h^2 g\{(N + 1) h\} - 2 a \beta h.$$

The solution given by (3) then becomes

$$y_{n+1} = A (\lambda^{n+1} - \mu^{n+1}) + \frac{1}{a} \cdot \frac{1}{\lambda - \mu} \sum_{r=0}^n (\lambda^r - \mu^r) f_{n-r}, \quad (9)$$

where

$$A = \frac{f_N - 2 a \beta h - \frac{1}{\lambda - \mu} \left[ \left(\frac{b}{a} - 2 \alpha h\right) \sum_{r=0}^N (\lambda^r - \mu^r) f_{N-r} + \left(1 + \frac{c}{a}\right) \sum_{r=0}^{N-1} (\lambda^r - \mu^r) f_{N-1-r} \right]}{(b - 2 a \alpha h) (\lambda^{N+1} - \mu^{N+1}) + (a + c) (\lambda^N - \mu^N)}$$

and  $0 \leq n \leq N$ . Solution (9) does not in general exist if

$$(b - 2 a \alpha h) (\lambda^{N+1} - \mu^{N+1}) + (a + c) (\lambda^N - \mu^N) = 0. \quad (10)$$

If  $\lambda$  and  $\mu$  are imaginary, (10) is satisfied if the mesh length  $h$  satisfies the relationship

$$k h^2 = 2 \left[ 1 - \left( 1 - \frac{1}{4} j^2 h^2 \right)^{1/2} \cos \frac{K \Pi - \varphi}{N + 1} \right] \quad (K = 1, 2, \dots, N), \quad (11)$$

where

$$\tan \varphi = \frac{\{k(4 - kh^2) - j^2\}^{1/2}}{2\left(1 - \frac{1}{4}j^2h^2\right)\alpha + \frac{1}{2}(kh^2 - 1)}$$

If  $\lambda$  and  $\mu$  are real, then (10) takes the form

$$\left[\frac{b - (b^2 - 4ac)^{1/2}}{b + (b^2 - 4ac)^{1/2}}\right]^{N+1} = \frac{2ac\alpha h + b(1-c) - (b^2 - 4ac)^{1/2}}{2ac\alpha h + b(1-c) + (b^2 - 4ac)^{1/2}}, \quad (12)$$

where  $b^2 > 4ac$ . In order to illustrate the existence of mesh lengths satisfying (12), consider the case  $j = 0$ . If  $kh^2 > 4$ , (12) can be solved to give

$$\delta = \frac{2 \sinh \nu}{\tanh 2(N+1)\nu}, \quad (13)$$

where

$$\delta = \frac{2\alpha}{k^{1/2}} \quad \text{and} \quad \nu = \cosh^{-1} \frac{k^{1/2}h}{2},$$

and where for a prescribed value of  $N$ , there is a value of  $\delta > 1/(N+1)$  corresponding to any value of  $k^{1/2}h > 2$ . If  $k < 0$ , (12) can be solved to give

$$\delta = -\frac{2 \cosh \nu}{\tanh 2(N+1)\nu}, \quad (14)$$

where

$$\delta = \frac{2\alpha}{(-k)^{1/2}} \quad \text{and} \quad \nu = \sinh^{-1} \frac{(-k)^{1/2}h}{2},$$

and where for any value of  $N$ , there is a negative value of  $\delta$  corresponding to any value of  $(-k)^{1/2}h > 0$ .

### 3. Correspondence Between Exact Solutions of the Differential and Difference Equations

In the previous section, critical values of the mesh length are given for which the difference equation has in general no solution. These critical lengths depend on the coefficients of the difference equation and the boundary conditions of the problem. Now the range of the problem is given by

$$L = (N+1)h \quad (15)$$

and so corresponding to each critical mesh length there is a critical range of problem given by (15), for which the difference equation, applied at  $N$  internal nodes, has in general no solution. In the present section, the differential equation is examined for critical ranges, and the latter compared with the values obtained in the previous section for the difference equation.

The differential equation (1) has solution

$$y = e^{-jx/2} [A e^{(j^2/4 - k)^{1/2}x} + B e^{-(j^2/4 - k)^{1/2}x}] + p(x), \quad (16)$$

if  $j^2/4 > k$ , and

$$y = e^{-jx/2} \left[ A \sin\left(k - \frac{1}{4}j^2\right)^{1/2}x + B \cos\left(k - \frac{1}{4}j^2\right)^{1/2}x \right] + p(x), \quad (17)$$

if  $j^2/4 < k$ , where in each case  $p(x)$  is a particular integral of (1) and  $A, B$  are constants to be determined by the boundary conditions.

(I) If  $y = 0, Y$  at  $x = 0, L$ , the differential equation has no solution in general, if  $j^2/4 < k$ , when

$$k^{1/2}L = \frac{K\pi}{\left(1 - \frac{1}{4}j^2/k\right)^{1/2}} \quad (K = 1, 2, \dots). \quad (18)$$

Now returning to the difference equation, if (15) is used to eliminate  $h$ , (8) can be solved to give

$$\frac{4k}{j^2} \left(1 - \frac{j^2L^2}{4(N+1)^2}\right)^{1/2} = \cos \frac{K\pi}{N+1} \pm \left\{ \cos^2 \frac{K\pi}{N+1} + \frac{8k}{j^2} \left(\frac{2k}{j^2} - 1\right) \right\}^{1/2} \quad (19)$$

$$= \cos \frac{K\pi}{N+1} \pm \left\{ \left(\frac{4k}{j^2} - 1\right)^2 - \sin^2 \frac{K\pi}{N+1} \right\}^{1/2}. \quad (20)$$

If  $k > j^2/2$ , a unique solution exists for  $L$  for each pair of values of  $K$  and  $N$ . This is obtained from (19) with the positive sign. For fixed  $K$ , as  $N$  tends to infinity, each value of  $L$  tends to the corresponding critical length of the differential equation given by (18). If  $k < j^2/2$ , it follows that  $0 < (4k/j^2 - 1) < 1$  and so from (20), for prescribed  $K$  and  $N$ , the number of values of  $L$  is 0, 1, or 2 depending on  $K$  and  $N$ . Provided  $N$  is sufficiently large, however, there are two values of  $L$  for each  $K$ . The value given by (20) with the positive sign again tends to the corresponding critical length of the differential equation, and the other value tends to infinity, as  $N$  tends to infinity.

The following table illustrates the manner in which a critical range of the difference equation approaches the corresponding critical range of the differential equation as  $N$  tends to infinity. For convenience, the values  $j = 0, k = 9$  are chosen.

Values of Critical Range

$K \backslash N$	3	7	15	31	$\frac{1}{3}K\pi$
1	1.0204	1.0408	1.0458	1.0464	1.0472
2	1.8856	2.0408	2.0810	2.0915	2.0944
3	2.4636	2.9632	3.0963	3.1302	3.1416

(II) If  $y = 0$  at  $x = 0$  and  $dy/dx + \alpha y = \beta$  at  $x = L$ , the differential equation has no solution in general when

$$k^{1/2} L = \frac{K \Pi - \varphi}{\left(1 - \frac{1}{4} j^2/k\right)^{1/2}} \quad (K = 1, 2, \dots), \quad (21)$$

where

$$\tan \varphi = \frac{\left(k - \frac{1}{4} j^2\right)^{1/2}}{\alpha - \frac{1}{2} j},$$

if  $j^2/4 < k$ , and no solution in general when

$$e^{(j^2-4k)^{1/2}L} = \frac{2\alpha - j - (j^2 - 4k)^{1/2}}{2\alpha - j + (j^2 - 4k)^{1/2}} \quad (22)$$

if  $j^2/4 > k$ . By comparison with (I), it seems probable that a critical range of the difference equation computed from (11) approaches the corresponding critical range of the differential equation given by (21) or tends to infinity as  $N$  tends to infinity. Consider now critical ranges of the difference equation obtained from (12). Suppose first  $j^2/4 < k$ , then there are values of  $\alpha$  for which (12) yields no critical ranges for any  $N$ . For any other value of  $\alpha$ , it seems likely that there is a corresponding critical range  $L_N$  for all sufficiently large  $N$ . It then follows that

$$L_N = \frac{2(N+1)}{k} \left(k - \frac{1}{4} j^2\right)^{1/2},$$

and hence  $L_N$  tends to infinity as  $N$  tends to infinity. If  $j^2/4 > k$ , however, the position is more complicated. Again there are values of  $\alpha$  for which (12) fails to yield a critical range. For any other value of  $\alpha$  it seems that for each sufficiently large  $N$ , there are two possible critical ranges, one of which tends to the corresponding critical range of the differential equation given by (22), and the other tends to infinity as  $N$  tends to infinity.

#### 4. Behaviour of Solutions Near Critical Points

Three examples are now given of corresponding solutions worked out with ranges which are nearly critical for either the difference or the differential equation. In all examples, the coefficients of the difference equation are exact, and the value  $N = 3$  is chosen in order to reduce the computation. The differential equation in all examples is

$$\frac{d^2y}{dx^2} + ky = kx \quad (k > 0), \quad (23)$$

which, although simple, is adequate to illustrate some of the results of the previous section.

(1) In the first example the boundary conditions are  $y = 0$ ,  $Y$  at  $x = 0, L$ . The solution of the difference equation is given by (6) with  $a = 1$ , where

$$\lambda, \mu = 1 - \frac{1}{2} k h^2 \pm i k^{1/2} h \left(1 - \frac{1}{4} k h^2\right)^{1/2},$$

and

$$f_s = (s+1) k h^3 \quad (s = 0, 1, \dots, N).$$

Choose  $k^{1/2} h = 0.765$ , a value close to the critical value  $2 \sin \pi/8$  for the case of three internal nodes, and the solution becomes

$$\begin{aligned} y_1 &= -444.77 L + 445.02 Y, \\ y_2 &= -629.10 L + 629.60 Y, \\ y_3 &= -444.97 L + 445.72 Y, \end{aligned}$$

where  $y_1, y_2, y_3$  are the values of  $y$  at the three internal nodes. The solution of the differential equation is given by

$$y = \frac{Y - L}{\sin k^{1/2} L} \sin k^{1/2} x + x,$$

with  $k^{1/2} L = 3.06$ , and so at corresponding points,

$$\begin{aligned} y_1 &= -8.2477 L + 8.4977 Y, \\ y_2 &= -11.7598 L + 12.2598 Y, \\ y_3 &= -8.4409 L + 9.1909 Y. \end{aligned}$$

(2) The boundary conditions of the previous example are retained, but this time a range is chosen which is close to a critical range of the differential equation. Choose  $k^{1/2} L = 3.14$ , which is close to the critical value  $\pi$ , and the solution of the differential equation yields

$$\begin{aligned} y_1 &= -443.16 L + 443.41 Y, \\ y_2 &= -626.85 L + 627.35 Y, \\ y_3 &= -443.38 L + 444.13 Y. \end{aligned}$$

The solution of the difference equation with  $k^{1/2} h = 0.785$  gives at corresponding points

$$\begin{aligned} y_1 &= 8.7352 L - 8.4852 Y, \\ y_2 &= 12.2416 L - 11.7416 Y, \\ y_3 &= 8.5126 L - 7.7626 Y. \end{aligned}$$

In the above two examples, it should be noted that although the solutions disagree violently for general  $Y$ , there is exact correspondence if  $Y = L$ . The reason for this becomes evident when the solutions of the difference and differential equations are rearranged for general  $N$  to give

$$y_{n+1} = \frac{Y - L}{\lambda^{N+1} - \mu^{N+1}} (\lambda^{n+1} - \mu^{n+1}) + (n + 1) h$$

and

$$y_{n+1} = \frac{Y - L}{\sin k^{1/2} L} \sin(n + 1) k^{1/2} h + (n + 1) h$$

respectively. Thus in examples (1) and (2), although the range of the problem may be nearly critical for either the difference or the differential equation, a fortunate boundary value  $Y$  may still enable reasonable agreement to be obtained between the exact solutions.

(3) In the final example, the boundary conditions are  $y = 0$  at  $x = 0$  and  $dy/dx + \alpha y = \beta$  at  $x = L$ . The solution of the difference equation is given by (9) with  $a = 1$ ,  $b = k h^2 - 2$ ,

$$\lambda, \mu = 1 - \frac{1}{2} k h^2 \pm k^{1/2} h \left( \frac{1}{4} k h^2 - 1 \right)^{1/2},$$

and

$$f_s = (s + 1) k h^3 \quad (s = 0, 1, \dots, N).$$

Choose  $k^{1/2} h = 2.5$  and  $\delta = 1.5$ , values which almost satisfy (13) and so this example is worked out near critical conditions of the difference equation. The solution of the difference equation for  $N = 3$  is

$$\begin{aligned} Y_1 &= -2175 + 256 \beta, \\ Y_2 &= 9250 - 1088 \beta, \\ Y_3 &= -37125 + 4368 \beta, \\ Y_4 &= 148550 - 17476 \beta, \end{aligned}$$

where  $Y_r = 4 y_r/L$  ( $r = 1, \dots, 4$ ). The solution of the differential equation (23) is

$$y = \frac{\beta - (1 + \alpha L)}{\alpha \sin k^{1/2} L + k^{1/2} \cos k^{1/2} L} \sin k^{1/2} x + x,$$

and so the corresponding values are

$$\begin{aligned} Y_1 &= 2.63023 - 0.19179 \beta, \\ Y_2 &= -0.61205 + 0.3073 \beta, \\ Y_3 &= 5.55507 - 0.3006 \beta, \\ Y_4 &= 2.51808 + 0.17434 \beta. \end{aligned}$$

In general, the lack of agreement between the two solutions is even more marked than in the previous examples. However, the solutions correspond exactly if  $\beta = 8.5$ , and so even although the values of  $k^{1/2} h$  and  $\delta$  are near critical values for the difference equation, a fortuitous value of  $\beta$  may still enable reasonable agreement to be obtained between the exact solutions of the difference and differential equations.

### 5. Errors in Approximate Solutions of Finite Difference Equations

So far, numerical solutions of the difference equation (2) have been obtained directly from the exact solution (3). In practice, however, an exact solution of the difference equation is rarely available, and approximate numerical solutions are obtained either by an iterative process such as relaxation or by direct methods of solution of the associated set of simultaneous equations. The accuracy of such numerical solutions of (2) is now investigated.

The error equation corresponding to (2) is

$$a \varepsilon_{r+2} + b \varepsilon_{r+1} + c \varepsilon_r = R_r \quad [r = 0, 1, 2, \dots, (N - 1)], \tag{24}$$

where  $\varepsilon_r, \varepsilon_{r+1}, \varepsilon_{r+2}$  are the errors in  $y_r, y_{r+1}, y_{r+2}$  respectively and  $R_r$  is the residual at the  $(r + 1)$ -th node. Residuals exist for any approximate numerical solution of (2), whether it is obtained by an iterative process such as relaxation or by a direct method. A solution of (24) is

$$\varepsilon_0 = A + B, \tag{25}$$

$$\varepsilon_{n+1} = A \lambda^{n+1} + B \mu^{n+1} + \frac{1}{a} \cdot \frac{1}{\lambda - \mu} \sum_{r=0}^n (\lambda^r - \mu^r) R_{n-r} \quad (0 \leq n \leq N),$$

where  $A, B$  are constants to be determined by the boundary conditions.

Consider first the case where the value of the function  $y$  is given at both ends of the range. If the boundary values require no rounding off, then

$$\varepsilon_0 = \varepsilon_{N+1} = 0,$$

and so (25) yields

$$a (\lambda - \mu) \varepsilon_{n+1} = \sum_{r=0}^n (\lambda^r - \mu^r) R_{n-r} - \frac{\sum_{r=0}^N (\lambda^r - \mu^r) R_{N-r}}{\lambda^{N+1} - \mu^{N+1}} (\lambda^{n+1} - \mu^{n+1}) \quad (0 \leq n \leq N - 1). \tag{26}$$

If  $\lambda, \mu$  are imaginary, it follows from (4) that the coefficients of the residuals may be large in modulus when either  $\theta$  approximately satisfies (7) or  $h$  is near

the value  $2/j$ . In the following two examples, the errors given by (25) are evaluated for  $j = 0$  and  $N = 3$ .

(1) In the first example,  $h^{1/2} h = 0.765$ , and the errors at the internal nodes are respectively

$$\begin{aligned} \epsilon_1 &= -[445.73 R_0 + 629.61 R_1 + 445.02 R_2], \\ \epsilon_2 &= -[629.61 R_0 + 890.75 R_1 + 629.61 R_2], \\ \epsilon_3 &= -[44.502 R_0 + 629.61 R_1 + 445.73 R_2]. \end{aligned}$$

The errors are now examined in relation to the corresponding values of  $y$  obtained from the exact solutions of the difference and differential equations in the previous section, and two distinct cases arise. If  $Y$  is approximately equal to  $L$ , the exact solutions are in good agreement, but approximate numerical solutions of the difference equation are likely to involve substantial errors. On the other hand, if  $Y$  is not nearly equal to  $L$ , the exact solutions are in poor agreement, but the ratio  $\epsilon/y$  in an approximate numerical solution is likely to be considerably reduced.

(2) In the second example,  $h^{1/2} h = 0.785$ , and the errors are respectively

$$\begin{aligned} \epsilon_1 &= -[7.7623 R_0 + 11.7418 R_1 + 8.4852 R_2], \\ \epsilon_2 &= -[11.7418 R_0 + 16.2475 R_1 + 11.7418 R_2], \\ \epsilon_3 &= -[8.4852 R_0 + 11.7418 R_1 + 7.7623 R_2]. \end{aligned}$$

These errors are comparatively small, and so the approximate numerical and exact solutions of the difference equation are likely to be in good agreement for any boundary value  $Y$ .

If  $\lambda, \mu$  are real, it follows from (5) that the errors given by (26) are unlikely to be large, provided  $h$  is not near the value  $2/j$ .

Consider next the case where the values of the function and its derivative are given, one at either end of the range. The appropriate error equation, from (25), is

$$a(\lambda - \mu) \epsilon_{n+1} = \sum_{r=0}^n (\lambda^r - \mu^r) R_{n-r} + C(\lambda^{n+1} - \mu^{n+1}), \tag{27}$$

where

$$C = \frac{2a(\lambda - \mu) R_N - \left[ (b - 2a\alpha h) \sum_{r=0}^N (\lambda^r - \mu^r) R_{N-r} + (a + c) \sum_{r=0}^{N-1} (\lambda^r - \mu^r) R_{N-1-r} \right]}{(b - 2a\alpha h)(\lambda^{N+1} - \mu^{N+1}) + (a + c)(\lambda^N - \mu^N)},$$

and  $R_r$  ( $0 \leq r \leq N$ ) is the residual at the  $(r + 1)$ -th node. The magnitudes of the coefficients in (27) are obtained using (4) if  $\lambda, \mu$  are complex and (5) if  $\lambda, \mu$  are real. The coefficients are in general considerably greater in modulus when

$\lambda, \mu$  are real, and may differ widely in magnitude, particularly when the range of the problem is near a critical range of the difference equation, or  $h$  is near one or other of the values  $\pm 2/j$ . As an illustration of this unequal 'weighting' of the residuals, consider an example where  $N = 3, j = 0, h^{1/2} h = 2.5, \delta = 1.5$ , and the boundary conditions are  $y = 0$  at  $x = 0$  and  $y' + \alpha y = \beta$  at  $x = L$ . The values of  $\lambda, \mu$  are  $-1/4$  and  $4$  respectively and the errors, using (27), are given by

$$\begin{aligned} \epsilon_1 &= 4 R_0 - 16 R_1 + 64 R_2 - 256 R_3, \\ \epsilon_2 &= -16 R_0 + 68 R_1 - 272 R_2 + 1088 R_3, \\ \epsilon_3 &= 64 R_0 - 272 R_1 + 1092 R_2 - 4368 R_3, \\ \epsilon_4 &= -256 R_0 + 1088 R_1 - 4368 R_2 + 17476 R_3, \end{aligned}$$

where the coefficients are exact. It is clear that the value of  $R_3$  has a much greater influence on the errors than the value of  $R_0$ , and consequently in any approximate numerical method of solution of the difference equation, the emphasis must be placed on reducing  $R_3$  as much as possible. Unequal 'weighting' of the residuals in the expressions for the errors, as illustrated in this example, may complicate matters considerably as far as direct numerical methods of solution of the difference equation are concerned, since in such methods it is difficult to attempt to eliminate residuals in a prescribed order. In iterative methods such as relaxation, however, a prescribed order of procedure for eliminating residuals is an advantage. In this particular example, in which  $j = 0$ , the residuals which must be eliminated occur at the nodes near the end where the derivative is specified. This is likely to be so for general  $j$  provided the mesh length is not near the value  $-2/j$ .

### 6. Concluding Remarks

In the present paper, it is shown that there are several major difficulties which may confront the computer who is attempting to solve an ordinary differential equation with two-point boundary conditions by difference methods. Although the differential equation examined here is of a particular type, nevertheless, the difficulties described are certain to exist for many other general types of equation.

In the expressions obtained for the errors in the numerical solution in terms of the residuals, some of the coefficients are large in modulus while others are relatively small, when the range of the problem is near a critical range of the difference equation. This unequal 'weighting' makes it essential that the residuals with large coefficients should be reduced as much as possible, a task which is more easily accomplished by an iterative rather than a direct process.

Finally, in the solution of initial value problems by difference methods, it is known that the chance of serious errors arising in numerical solutions is increased considerably if the differential equation is replaced by a higher order difference equation (TODD [3], RUTISHAUSER [4], MITCHELL and CRAGGS [5]). It seems likely that the use of higher order difference replacements will also produce additional complications in the numerical solution of two-point boundary problems.

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### Zusammenfassung

Es werden inhomogene Randwertaufgaben bei einer gewöhnlichen linearen Differentialgleichung 2. Ordnung mit konstanten Koeffizienten untersucht. Es gibt dann bekanntlich Ausnahmefälle, wo die Länge des Grundintervalls kritisch ist, das heisst die Randwertaufgabe im allgemeinen keine Lösung besitzt. Auch wenn die Differentialgleichung durch eine Differenzgleichung approximiert wird, hat letztere kritische Intervalle, und es zeigt sich, dass in der Nähe eines kritischen Falles die Lösungen der Differential- und der Differenzgleichung erheblich voneinander abweichen können. Es wird ferner bewiesen, dass bei Verfeinerung der Maschenlänge die kritischen Intervalle der Differenzgleichung gegen diejenigen der Differentialgleichung streben oder über alle Grenzen wachsen.

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