A SUMMABILITY FACTOR THEOREM

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1. In all that follows $f(t)$ and $\phi(t)$ denote real functions, integrable L in every finite interval in $(1, \infty)$†.

We write, for $t \geqslant 1$,

$$\left.\begin{aligned} I_a f(t) &= f_a(t) = \frac{1}{\Gamma(a)} \int_1^t (t-u)^{a-1} f(u)\, du \quad (a > 0), \\ I_0 f(t) &= f_0(t) = f(t), \end{aligned}\right\} \tag{1.1}$$

$$\phi^{(\delta)}(t, x) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dt} \int_t^x (u-t)^{-\delta} \phi(u)\, du \quad (0 < \delta < 1,\ x > t), \tag{1.2}$$

$$\left.\begin{aligned} \phi^{(\delta)}(t) &= \lim_{x \to \infty} \phi^{(\delta)}(t, x) \qquad (0 < \delta < 1), \\ \phi^{(0)}(t) &= \phi(t), \\ \phi^{(\delta+s)}(t) &= (d/dt)^s \phi^{(\delta)}(t) \quad (0 \leqslant \delta < 1,\ s \text{ a positive integer}). \end{aligned}\right\} \tag{1.3}$$

At the point $t = 1$, d/dt denotes differentiation on the right.

It is clear that, for l a constant, if $\theta(u) = \phi(u) - l$, for all $u \geqslant 1$, and if, for $a > 0$ and $t \geqslant 1$, $\phi^{(a)}(t)$ exists, then

$$\theta^{(a)}(t) = \phi^{(a)}(t). \tag{1.4}$$

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† Throughout this paper every integral over a finite range is a Lebesgue integral and $\int_a^\infty$ denotes $\lim_{x \to \infty} \int_a^x$, if this limit exists, finite or infinite.

2. The following theorem, for λ an integer, is due in substance to Hardy* and, for λ not an integer, to Cossar†.

THEOREM A. *For* $\lambda \geqslant 0$, *if*, $\phi^{(\lambda)}(t)$ *is absolutely continuous*‡, $\int_1^\infty f(t)\, dt$ *is summable* (C, λ) [*or bounded* (C, λ)] *and*

(i) $\phi(t) \to l$ [*or is* $o(1)$] *as* $t \to \infty$,

(ii) $\int_1^\infty t^\lambda |\phi^{(\lambda+1)}(t)|\, dt < \infty$,

then $\int_1^\infty f(t)\phi(t)\, dt$ *is summable* (C, λ).

The object of this paper is to establish a result which is, in essence, the converse of Theorem A.

THEOREM 1. *For* $\lambda \geqslant 0$, *if* $\phi^{(\lambda)}(t)$ *is absolutely continuous and* $\int_1^\infty f(t)\phi(t)\, dt$ *is bounded* (C) [*or summable* (C)] *whenever* $\int_1^\infty f(t)\, dt$ *is summable* (C, λ) [*or bounded* (C, λ)], *then*

(i) *there is an absolutely continuous function* $\psi(t)$ *such that* $\psi(t) = \phi(t)$ p.p. *in* $(1, \infty)$ *and* $\psi(t) \to l$ [*or is* $o(1)$] *as* $t \to \infty$,

(ii) $\int_1^\infty t^\lambda |\phi^{(\lambda+1)}(t)|\, dt < \infty$.

The above theorems are analogues of well known theorems on series due to Bohr, Hardy, Fekete, Andersen and Bosanquet§.

We shall require the following lemmas.

3. LEMMA 1. *If* $\phi(t)$ *is absolutely continuous and* $\int_1^\infty |\phi'(t)|\, dt = \infty$, *then, for any non-negative integer* s, *there is a function* $f(t)$ *such that* $f^{(s)}(t)$ *is absolutely continuous*, $f(1) = f'(1) = \ldots = f^{(s)}(1) = 0$,

$$\int_1^\infty f(t)\, dt \text{ is convergent and } \int_1^\infty f(t)\phi(t)\, dt = \infty.$$

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* G. H. Hardy, *Messenger of Math.*, 40 (1911), 87–91 and 108–112.

† J. Cossar, [1], *Journal London Math. Soc.*, 16 (1941), 56–68, proved the second version, of which the first is a consequence in virtue of (1.4).

‡ Where no interval of absolute continuity is specified, it is to be understood that the property pertains to every finite interval in $(1, \infty)$.

§ See L. S. Bosanquet, [1], *Journal London Math. Soc.*, 17 (1942), 166–173 and the references there given.

Case 1*. *Suppose* $\phi(t)$ *to be bounded in* $(1, \infty)$. For $b > a \geqslant 1$ we denote the upper bound, the lower bound and the variation of $\phi(t)$ in (a, b) by $M(\phi;\ a, b)$, $m(\phi;\ a, b)$ and $V(\phi;\ a, b)$, and write

$$w(\phi;\ a, b) = M(\phi;\ a, b) - m(\phi;\ a, b).$$

Using familiar results we now construct a strictly increasing unbounded sequence $\{x_r\}$, such that $x_1 = 1$ and

$$\sum_{r=2}^{\infty} w(\phi;\ x_{r-1}, x_r) = \infty. \tag{3.1}$$

We take $n_1 = 1$ and, allowing ν to assume successively the values 1, 2, 3, ..., choose numbers $x_{n_\nu}, x_{n_\nu+1}, \ldots, x_{n_{\nu+1}}$ such that

$$\nu = x_{n_\nu} < x_{n_\nu+1} < \ldots < x_{n_{\nu+1}} = \nu+1$$

and

$$\sum_{r=n_\nu+1}^{n_{\nu+1}} w(\phi;\ x_{r-1}, x_r) > V(\phi;\ \nu, \nu+1) - \frac{1}{\nu^2} = \int_\nu^{\nu+1} |\phi'(t)|\,dt - \frac{1}{\nu^2}. \tag{3.2}$$

We now obtain (3.1) from the second hypothesis and (3.2).

We write

$$M_r = M(\phi;\ x_r, x_{r+1}), \quad m_r = m(\phi;\ x_r, x_{r+1}), \quad w_r = w(\phi;\ x_r, x_{r+1}),$$

$$W_r = 1 + \sum_{\nu=1}^{r} w_\nu \quad \text{and} \quad M = M(|\phi|;\ 1, \infty).$$

Since $\phi(t)$ is continuous in $(1, \infty)$, it follows that, corresponding to any positive integer r, there are separated intervals i_r and j_r in the interior of (x_r, x_{r+1}), such that $|i_r| = |j_r| > 0$ and

$$\phi(t) \geqslant M_r - \tfrac{1}{4}w_r \text{ for } t \text{ in } i_r,$$

$$\phi(t) \leqslant m_r + \tfrac{1}{4}w_r \text{ for } t \text{ in } j_r.$$

We write $p_r = |i_r|$, $q_r = (p_r W_r)^{-1}$, and define

$$\begin{aligned} a(t) &= q_r \quad \text{for } t \text{ in } i_r \quad (r = 1, 2, \ldots), \\ &= -q_r \quad \text{for } t \text{ in } j_r \quad (r = 1, 2, \ldots), \\ &= 0 \quad \text{for other } t \text{ in } (1, \infty). \end{aligned}$$

It follows that, for $x_r \leqslant x < x_{r+1}$,

$$\left| \int_1^x a(t)\,dt \right| \leqslant p_r q_r = \frac{1}{W_r},$$

* *Cf.* W. L. C. Sargent, *Journal London Math. Soc.*, 23 (1948), 28–34.

and hence, since by (3.1) $W_r \to \infty$ as $r \to \infty$,

$$\int_1^\infty a(t)\,dt = 0. \tag{3.3}$$

On the other hand

$$\int_{x_r}^{x_{r+1}} a(t)\,\phi(t)\,dt \geqslant p_r q_r (M_r - \tfrac{1}{4}w_r) - p_r q_r (m_r + \tfrac{1}{4}w_r) = \tfrac{1}{2}\frac{w_r}{W_r},$$

while, for $x_r \leqslant x < x_{r+1}$,

$$\left| \int_x^{x_{r+1}} a(t)\,\phi(t)\,dt \right| \leqslant 2Mp_r q_r = 2\frac{M}{W_r}.$$

Hence, since $\sum_{r=1}^{\infty} \frac{w_r}{W_r} = \infty$ and $\frac{1}{W_r} \to 0$ as $r \to \infty$,

$$\int_1^\infty a(t)\,\phi(t)\,dt = \infty. \tag{3.4}$$

Now let $a_1 = 1$ and arrange the boundary points of the sequences of intervals $\{i_r\}$ and $\{j_r\}$ into a strictly increasing unbounded sequence $a_2, a_3, \ldots$.

It follows from the definitions of $a(t)$ and $\{a_r\}$ that the relations

$$\mu_r = a(t) \text{ for } a_r < t < a_{r+1} \quad (r = 1, 2, \ldots)$$

determine constants $\mu_1, \mu_2, \mu_3, \ldots$, of which those with odd suffixes vanish and those with even suffixes do not.

We now write, for $r = 2, 3, \ldots$,

$$A_r = 1 + M(|\phi|;\ a_r, a_{r+1}), \quad B_r = |\mu_r| + |\mu_{r-1}|,$$

$$c_r = \min\left\{\frac{1}{r^2 A_r B_r},\ (a_{r+1} - a_r)\right\}, \quad b_r = a_r + c_r,$$

and define

$$\begin{aligned} f(t) &= \mu_{r-1}\left\{1 - \left(\frac{t-a_r}{c_r}\right)^{s+1}\right\}^{s+1} + \mu_r\left\{1 - \left(\frac{b_r - t}{c_r}\right)^{s+1}\right\}^{s+1} \quad \text{for } a_r \leqslant t \leqslant b_r \\ &\qquad (r = 2, 3, \ldots), \\ &= a(t) \quad \text{for other } t \text{ in } (1, \infty). \end{aligned}$$

Clearly $f^{(s)}(t)$ is absolutely continuous, $f(1) = f'(1) = \ldots = f^{(s)}(1) = 0$ and

$$\int_1^\infty |f(t) - a(t)|\,dt = \sum_{r=2}^{\infty} \int_{a_r}^{b_r} |f(t) - a(t)|\,dt \leqslant \sum_{r=2}^{\infty} B_r c_r \leqslant \sum_{r=2}^{\infty} \frac{1}{r^2} < \infty, \tag{3.5}$$

$$\int_1^\infty |f(t) - a(t)|\,|\phi(t)|\,dt \leqslant \sum_{r=2}^{\infty} A_r B_r c_r \leqslant \sum_{r=2}^{\infty} \frac{1}{r^2} < \infty. \tag{3.6}$$

The result for this case then follows from (3.3), (3.4), (3.5) and (3.6).

Case 2. *Suppose* $\phi(t)$ *to be unbounded in* $(1, \infty)$. Since $\phi(t)$ is continuous and unbounded in $(1, \infty)$, it is plain that there is a sequence of separated intervals $i_1, i_2, i_3, \ldots$, in $(2, \infty)$, such that their boundary points form a strictly increasing unbounded sequence and

$$|\phi(t)| > r \quad \text{for } t \text{ in } i_r \quad (r = 1, 2, \ldots).$$

Clearly $\phi(t)$ is of one sign in each i_r.

We now define

$$a(t) = (r^2 |i_r|)^{-1} \operatorname{sgn}\{\phi(t)\} \quad \text{for } t \text{ in } i_r \quad (r = 1, 2, \ldots),$$
$$= 0 \quad \text{for other } t \text{ in } (1, \infty).$$

It follows that

$$\int_1^\infty |a(t)|\, dt = \sum_{r=2}^\infty \frac{1}{r^2} < \infty \tag{3.7}$$

and

$$\int_1^\infty a(t)\phi(t)\, dt \geqslant \sum_{r=1}^\infty \frac{1}{r} = \infty. \tag{3.8}$$

The proof from here continues as in Case 1.

LEMMA 2*. *If* $\int_1^\infty f(t)\phi(t)\, dt$ *is bounded* (C) *whenever* $\int_1^\infty f(t)\, dt$ *is convergent, then* $\phi(t)$ *is essentially bounded in* $(1, \infty)$.

Assuming the lemma false, we can obtain a contradiction by using an adapted version of the argument in Case 2 of Lemma 1, in which the sequence of intervals is replaced by a sequence of non-null sets of finite measure.

4. LEMMA 3. *For* $\lambda \geqslant 0$, *if* $\int_1^\infty f(t)\, dt$ *is summable* (C, λ), *then*

$$tf(t) \to 0 \quad (C, \lambda+1) \quad \text{as} \quad t \to \infty.$$

This follows from the identity

$$\frac{1}{t^{\lambda+1}} \int_1^t (t-u)^\lambda u f(u)\, du = \frac{1}{t^\lambda} \int_1^t (t-u)^\lambda f(u)\, du - \frac{1}{t^{\lambda+1}} \int_1^t (t-u)^{\lambda+1} f(u)\, du.$$

LEMMA 4. *For* $\lambda \geqslant 0$, $p+\lambda > -1$, $p+q > -1$, *if* $(t-u)^{\lambda-1} f(u)$ *is integrable* L *in* $(1, t)$, *for all* $t > 1$, *and* $f(t) = o(t^p)$ (C, λ) *as* $t \to \infty$, *then* $t^q f(t) = o(t^{p+q})$ (C, λ) *as* $t \to \infty$.

* *Cf.* L. S. Bosanquet and H. Kestelman, *Proc. London Math. Soc.*, (2), 45 (1939), 90.

This is due to Bosanquet*.

LEMMA 5. *For* $0 < \delta \leqslant 1$, $1 < x \leqslant t$,

$$\frac{1}{\Gamma(\delta)} \left| \int_1^x (t-u)^{\delta-1} f(u)\, du \right| \leqslant \max_{1 \leqslant u \leqslant x} |f_\delta(u)|,$$

where max *denotes the essential upper bound.*

This is due substantially to M. Riesz†.

LEMMA 6. *For* $\lambda > 0$, *if* $\int_1^\infty t^{-\lambda} f_\lambda(t)\, dt$ *is convergent, then* $\int_1^\infty f(t)\, dt$ *is summable* (C, λ) *to zero.*

For

$$f_{\lambda+1}(x) = \int_1^x t^\lambda . t^{-\lambda} f_\lambda(t)\, dt$$
$$= x^\lambda\{l + o(1)\} - \lambda \int_1^x t^{\lambda-1}\{l + o(1)\}\, dt$$
$$= o(x^\lambda).$$

5. LEMMA 7. *If* $\phi(t)$ *is essentially bounded in* $(1, \infty)$ *and, for* $0 < \delta < 1$, $\phi^{(\delta)}(t)$ *is absolutely continuous, then there is an absolutely continuous function* $\psi(t)$, *such that* $\psi(t) = \phi(t)$ p.p. *in* $(1, \infty)$.

Suppose that $0 < \epsilon < x-1$ and $1 \leqslant t < x$. Since $\phi(t)$ is essentially bounded in $(1, \infty)$, it follows from (1.3) that

$$\phi^{(\delta)}(t) = \phi^{(\delta)}(t, x) + \frac{\delta}{\Gamma(1-\delta)} \int_x^\infty (u-t)^{-\delta-1} \phi(u)\, du, \tag{5.1}$$

where the final integral is clearly an absolutely continuous function of t in $(1, x-\frac{1}{2}\epsilon)$. Hence

$$\phi^{(\delta)}(t, x) \text{ is absolutely continuous for } t \text{ in } (1, x-\tfrac{1}{2}\epsilon). \tag{5.2}$$

Denoting the essential upper bound of $|\phi(t)|$ in $(1, \infty)$ by M, we deduce from (5.1) that

$$|\phi^{(\delta)}(t, x)| \leqslant |\phi^{(\delta)}(t)| + \frac{M}{\Gamma(1-\delta)} (x-t)^{-\delta}$$

and thus

$$\phi^{(\delta)}(t, x) \text{ is integrable } L \text{ in } (1, x). \tag{5.3}$$

* L. S. Bosanquet, *Journal London Math. Soc.*, 23 (1948), 35-38. The replacement of O by o presents no difficulty.

† See L. S. Bosanquet, *Journal London Math. Soc.*, 16 (1941), 146-148, for full references.

Now by a result used by Cossar*,

$$-\phi(t) \equiv \frac{1}{\Gamma(\delta)} \int_t^x (u-t)^{\delta-1} \phi^{(\delta)}(u, x)\, du$$

$$= \frac{1}{\Gamma(\delta)} \int_t^{x-\frac{1}{2}\epsilon} (u-t)^{\delta-1} \phi^{(\delta)}(u, x)\, du + \frac{1}{\Gamma(\delta)} \int_{x-\frac{1}{2}\epsilon}^x (u-t)^{\delta-1} \phi^{(\delta)}(u, x)\, du. \quad (5.4)$$

In virtue of (5.2) and (5.3) the latter two integrals in (5.4) are absolutely continuous functions of t in $(1, x-\epsilon)$ and consequently, on writing

$$\psi(t) = -\frac{1}{\Gamma(\delta)} \int_t^x (u-t)^{\delta-1} \phi^{(\delta)}(u, x)\, du,$$

it is clear that $\psi(t)$ is independent of x and has the properties required in the lemma.

LEMMA 8†. *If $\phi(t)$ is bounded in $(1, \infty)$ and absolutely continuous, then, for $0 < \delta < 1$,*

(i) $\phi^{(\delta)}(t)$ *exists* p.p. *in* $(1, \infty)$,

(ii) $\dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_t^\infty (u-t)^{-\delta} \phi'(u)\, du = \phi^{(\delta)}(t)$ p.p. *in* $(1, \infty)$,

(iii) $\dfrac{1}{\Gamma(\delta)} \displaystyle\int_1^\infty t^{\delta-1} |\phi^{(\delta)}(t)|\, dt \leqslant \int_1^\infty |\phi'(t)|\, dt.$

It is well known that, for absolutely continuous $\phi(t)$, $\phi^{(\delta)}(t, x)$ exists for almost all t in $(1, x)$ and thus (i) follows from (5.1).

Let n denote a positive integer. It is familiar that, for $1 \leqslant t < n$,

$$\int_t^n (u-t)^{-\delta} \phi'(u)\, du \equiv -\frac{d}{dt} \int_t^n (u-t)^{-\delta} du \int_u^n \phi'(v)\, dv$$

$$= \frac{d}{dt} \int_t^n (u-t)^{-\delta} \phi(u)\, du + (n-t)^{-\delta} \phi(n). \quad (5.5)$$

Let z_n be the set of t in $(1, n)$ for which either (5.5) is not an equality or $\phi^{(\delta)}(t)$ is not defined. Then $z = \sum_{n=2}^{\infty} z_n$ is null. For t in $(1, \infty) - z$, $\int_t^n (u-t)^{-\delta} \phi'(u)\, du$ exists for $n > t$ and thus, since $\phi(t)$ is bounded in $(1, \infty)$, $\int_t^\infty (u-t)^{-\delta} \phi'(u)\, du$ is convergent. Consequently, letting $n \to \infty$ in (5.5), we obtain (ii).

* Equation (8.4) in Cossar's paper, [1], is valid for the $\phi(t)$ we consider.

† *Cf.* Cossar, [1], Lemma 5.

To complete the lemma we observe that, in virtue of (ii),

$$\int_1^\infty t^{\delta-1} |\phi^{(\delta)}(t)|\, dt \leqslant \frac{1}{\Gamma(1-\delta)} \int_1^\infty t^{\delta-1} dt \int_t^\infty (u-t)^{-\delta} |\phi'(u)|\, du$$

$$\leqslant \frac{1}{\Gamma(1-\delta)} \int_1^\infty |\phi'(u)|\, du \int_0^u t^{\delta-1} (u-t)^{-\delta}\, dt = \Gamma(\delta) \int_1^\infty |\phi'(u)|\, du.$$

LEMMA 9*. *For $0 < \delta < 1$, if $\phi(t)$ is absolutely continuous,*

$$\int_1^\infty |\phi'(t)|\, dt < \infty,$$

and $\phi(t) = o(1)$ as $t \to \infty$, then, for $t \geqslant 1$,

$$\frac{1}{\Gamma(\delta)} \int_t^\infty (u-t)^{\delta-1} \phi^{(\delta)}(u)\, du = -\phi(t).$$

It follows from Lemma 8 (ii) that

$$\frac{1}{\Gamma(\delta)} \int_t^\infty (u-t)^{\delta-1} \phi^{(\delta)}(u)\, du = \frac{1}{\Gamma(\delta)\Gamma(1-\delta)} \int_t^\infty (u-t)^{\delta-1} du \int_u^\infty (v-u)^{-\delta} \phi'(v)\, dv$$

$$= \frac{1}{\Gamma(\delta)\Gamma(1-\delta)} \int_t^\infty \phi'(v)\, dv \int_t^v (u-t)^{\delta-1} (v-u)^{-\delta}\, du = \int_t^\infty \phi'(v)\, dv.$$

The inversion is justified by the absolute convergence of the final integral and the result now follows since $\phi(t) = o(1)$ as $t \to \infty$.

LEMMA 10. *For $\lambda > 1$, s the integer such that $s < \lambda \leqslant s+1$, $\delta = \lambda - s$, if $\phi^{(\lambda-1)}(t)$ is absolutely continuous, $\int_1^\infty t^{\delta-1} \phi^{(\delta)}(t)\, dt$ is convergent and $\int_1^\infty t^{\lambda-1} |\phi^{(\lambda)}(t)|\, dt < \infty$, then for $r = 0, 1, \ldots, s-1$,*

(i) $\phi^{(\delta+r)}(t) = o(1)$ *as* $t \to \infty$, (ii) $\displaystyle\int_1^\infty t^{\delta+r-1} |\phi^{(\delta+r)}(t)|\, dt < \infty$,

(iii) $\dfrac{1}{\Gamma(\delta)} \displaystyle\int_t^\infty (u-t)^{\delta-1} \phi^{(\delta)}(u)\, du = \dfrac{(-1)^{r+1}}{\Gamma(\delta+r+1)} \int_t^\infty (u-t)^{\delta+r} \phi^{(\delta+r+1)}(u)\, du \ (t \geqslant 1).$

Conclusions (i) and (ii) are well known†.

* *Cf.* Cossar, [1]. Lemma 8.

† Cossar, [1], Lemmas 4, 9 and 10.

In consequence now of (i) with $r=0$ and (ii) with $r=1$,

$$\frac{1}{\Gamma(\delta)}\int_t^\infty (u-t)^{\delta-1}\phi^{(\delta)}(u)\,du = -\frac{1}{\Gamma(\delta)}\int_t^\infty (u-t)^{\delta-1}du\int_u^\infty \phi^{(\delta+1)}(v)\,dv$$

$$= -\frac{1}{\Gamma(\delta)}\int_t^\infty \phi^{(\delta+1)}(v)\,dv\int_t^v (u-t)^{\delta-1}du$$

$$= -\frac{1}{\Gamma(\delta+1)}\int_t^\infty (v-t)^{\delta}\phi^{(\delta+1)}(v)\,dv.$$

This proves (iii) for $r=0$ and repetition, if $r>0$, yields the complete result.

6. For the purpose of the following lemma we write, for $r \geqslant 0$, $\lambda>0$, $x\geqslant 1$,

$$\left.\begin{aligned} h(x,r) &= \frac{1}{\Gamma(r+1)}\int_1^x (x-t)^r f(t)\,dt\int_x^\infty (u-t)^{\lambda-1}\phi(u)\,du,\\ h(x) &= h(x,0), \end{aligned}\right\} \tag{6.1}$$

$$H(x,r) = \frac{1}{\Gamma(r+1)}\int_1^x (x-t)^r|f(t)|\,dt\int_x^\infty (u-t)^{\lambda-1}|\phi(u)|\,du. \tag{6.2}$$

Lemma 11. *For $\lambda>0$, s the integer such that $s<\lambda\leqslant s+1$, if $\int_1^\infty f(t)\,dt$ is bounded (C,λ) and $\int_1^\infty t^{\lambda-1}|\phi(t)|\,dt<\infty$, then*

$$h(x)=o(1)\ \ (C,s+1)\ \ \textit{as}\ \ x\to\infty.$$

We prove first that, for $y\geqslant 1$, $r\geqslant 0$,

$$\int_1^y H(x,r)\,dx<\infty \quad\text{and}\quad H(y,r+1)<\infty. \tag{6.3}$$

This follows since $H(x,r)$ is dominated by

$$\frac{x^{1-\lambda}}{\Gamma(r+1)}\int_1^x (x-t)^{r+\lambda-1}|f(t)|\,dt\int_x^\infty u^{\lambda-1}|\phi(u)|\,du \quad\text{if}\quad 0<\lambda<1,$$

and by

$$\frac{1}{\Gamma(r+1)}\int_1^x (x-t)^r|f(t)|\,dt\int_x^\infty u^{\lambda-1}|\phi(u)|\,du \quad\text{if}\quad \lambda\geqslant 1.$$

Now, in virtue of (6.3), we have, for $y\geqslant 1$, $r\geqslant 0$,

$$\Gamma(r+1)\int_1^y h(x,r)\,dx = \int_1^y dx\int_x^\infty \phi(u)\,du\int_1^x (u-t)^{\lambda-1}(x-t)^r f(t)\,dt$$

$$= \int_1^y dx\int_x^y \phi(u)\,du\int_1^x (u-t)^{\lambda-1}(x-t)^r f(t)\,dt + \int_1^y dx\int_y^\infty \phi(u)\,du\int_1^x (u-t)^{\lambda-1}(x-t)^r f(t)\,dt$$

$$= \int_1^y \phi(u)\,du\int_1^u dx\int_1^x (u-t)^{\lambda-1}(x-t)^r f(t)\,dt + \int_y^\infty \phi(u)\,du\int_1^y dx\int_1^x (u-t)^{\lambda-1}(x-t)^r f(t)\,dt$$

$$= \frac{1}{r+1}\int_1^y \phi(u)\,du\int_1^u (u-t)^{\lambda+r}f(t)\,dt + \frac{1}{r+1}\int_y^\infty \phi(u)\,du\int_1^y (u-t)^{\lambda-1}(y-t)^{r+1}f(t)\,dt$$

$$= \frac{\Gamma(\lambda+r+1)}{r+1}\int_1^y \phi(u)f_{\lambda+r+1}(u)\,du + \Gamma(r+1)h(y,r+1). \tag{6.4}$$

It follows from (6.4) that, for $x\geqslant 1$,

$$I_{s+1}h(x) = h(x,s+1)+\sum_{r=0}^{s}\frac{\Gamma(\lambda+r+1)}{(r+1)!}I_{s+1-r}\{\phi(x)f_{\lambda+r+1}(x)\}. \tag{6.5}$$

It is clear from the hypotheses that, for $r\geqslant 0$,

$$\int_1^\infty t^{\lambda-1}\phi(t)\,t^{-\lambda-r}f_{\lambda+r+1}(t)\,dt$$

is convergent. Hence, for $r=0,1,\ldots,s$, by Lemma 3,

$$x^{-r}\phi(x)f_{\lambda+r+1}(x)=o(1)\ \ (C,s+1-r)\quad\text{as}\quad x\to\infty,$$

and thus, by Lemma 4,

$$\phi(x)f_{\lambda+r+1}(x)=o(x^r)\ \ (C,s+1-r)\quad\text{as}\quad x\to\infty.$$

Consequently the summation term in (6.5) is $o(x^{s+1})$ as $x\to\infty$, and thus, to complete the lemma, we must prove that $h(x,s+1)=o(x^{s+1})$ as $x\to\infty$.

In view of the second inequality in (6.3), we have, for $x\geqslant 1$,

$$\Gamma(s+2)x^{-s-1}h(x,s+1) = \int_x^\infty u^{\lambda-1}\phi(u)\,du\int_1^x f(t)\left(1-\frac{t}{u}\right)^{\lambda-1}\left(1-\frac{t}{x}\right)^{s+1}dt$$

and therefore, because of the condition on $\phi(t)$, it is sufficient to prove the inner integral bounded independently of u and x, for $u>x>2$ say.

We have, for $u > x > 2$, on integrating $s+1$ times by parts,

$$\int_1^x f(t)\left(1-\frac{t}{u}\right)^{\lambda-1}\left(1-\frac{t}{x}\right)^{s+1} dt$$

$$= (-1)^{s+1}\int_1^x f_{s+1}(t)\left(\frac{d}{dt}\right)^{s+1}\left\{\left(1-\frac{t}{u}\right)^{\lambda-1}\left(1-\frac{t}{x}\right)^{s+1}\right\} dt = \sum_{r=0}^{s+1} c_r X_r, \quad (6.6)$$

where $c_0, c_1, \ldots, c_{s+1}$ are constants and

$$X_r = u^{-r}x^{r-s-1}\int_1^x f_{s+1}(t)\left(1-\frac{t}{u}\right)^{\lambda-r-1}\left(1-\frac{t}{x}\right)^{r} dt$$

$$= u^{-r}x^{r-\lambda}\int_1^x (x-t)^{\lambda-s-1} f_{s+1}(t)\left\{\left(1-\frac{t}{x}\right)^{r+s+1-\lambda}\left(1-\frac{t}{u}\right)^{\lambda-1-r}\right\} dt. \quad (6.7)$$

For $u > x > 2$, $x \geqslant t \geqslant 0$, the term in the curled brackets is a decreasing function of t, since, for $x > t \geqslant 0$, the derivative with respect to t of its logarithm, $(r+s+1-\lambda)(x-u)/(t-x)(t-u)+s/(t-u)$, is negative (except in the trivial case $\lambda = 1$, $s = 0$, $r = 0$, when it is indentically zero).

It follows then from (6.7), on applying first the Second Mean Value Theorem and then Lemma 5 that, for $r = 0, 1, \ldots, s+1$, $u > x > 2$,

$$|X_r| = x^{-\lambda}\left(\frac{x}{u}\right)^r \left|\int_1^{\xi}(x-t)^{\lambda-s-1}f_{s+1}(t)\,dt\right| \quad (1 < \xi < x)$$

$$\leqslant x^{-\lambda}\max_{1\leqslant t\leqslant x}|f_{\lambda+1}(t)|$$

and thus, by the hypothesis of $f(t)$, X_r is bounded independently of u and x.

The lemma now follows from (6.6).

7. *Proof of Theorem* 1. *First version.*

For $\lambda = 0$, the theorem is an immediate consequence of Lemma 1.

Suppose now that $\lambda > 0$. It follows from the second hypothesis, by Lemma 2, that $\phi(t)$ is essentially bounded in $(1, \infty)$. Hence, in view of Lemma 7, since $\phi^{(\lambda-[\lambda])}(t)$ is absolutely continuous, there is an absolutely continuous function $\psi(t)$ such that

$$\psi(t) = \phi(t) \text{ p.p. in } (1, \infty). \quad (7.1)$$

In virtue of Lemma 1, we have

$$\int_1^{\infty}|\psi'(t)|\,dt < \infty, \quad (7.2)$$

from which (i) follows.

From (1.4) it is clear that there is no loss in generality in now supposing that

$$\psi(t) = o(1) \quad \text{as} \quad t \to \infty. \quad (7.3)$$

Let s be the integer such that $s < \lambda \leqslant s+1$, and write $\delta = \lambda - s$.

It follows from (7.1) and (7.2), by Lemma 8 (iii), that

$$\int_1^{\infty} t^{\delta-1}|\phi^{(\delta)}(t)|\,dt < \infty, \quad (7.4)$$

and from (7.1), (7.2) and (7.3), by Lemma 9, that, for $t \geqslant 1$,

$$\frac{1}{\Gamma(\delta)}\int_t^{\infty}(u-t)^{\delta-1}\phi^{(\delta)}(u)\,du = -\psi(t). \quad (7.5)$$

Now assume that

$$\int_1^{\infty} t^{\lambda-1}|\phi^{(\lambda)}(t)|\,dt < \infty \quad (7.6)$$

and

$$\int_1^{\infty} t^{\lambda}|\phi^{(\lambda+1)}(t)|\,dt = \infty. \quad (7.7)$$

It follows from (7.4), (7.5), and (7.6), by Lemma 10 (iii), that

$$\frac{1}{\Gamma(\lambda)}\int_t^{\infty}(u-t)^{\lambda-1}\phi^{(\lambda)}(u)\,du = (-1)^{s+1}\psi(t) \quad (t \geqslant 1). \quad (7.8)$$

As a consequence of (7.6) and (7.7) we have

$$\int_1^{\infty}\left|\frac{d}{dt}\{t^{\lambda}\phi^{(\lambda)}(t)\}\right| dt = \infty;$$

and hence it follows, by Lemma 1 with $\phi(t)$ replaced by $t^{\lambda}\phi^{(\lambda)}(t)$ and $f(t)$ by $t^{-\lambda}g(t)$, that there is a function $g(t)$, such that

$$g^{(s)}(t) \text{ is absolutely continuous,} \quad g(1) = g'(1) = \ldots = g^{(s)}(1) = 0, \quad (7.9)$$

$$\int_1^{\infty} t^{-\lambda}g(t)\,dt \text{ is convergent} \quad (7.10)$$

and

$$\int_1^{\infty} g(t)\phi^{(\lambda)}(t)\,dt = \infty. \quad (7.11)$$

We now define, for $t \geqslant 1$,

$$f(t) = I_{s+1-\lambda}\,g^{(s+1)}(t). \quad (7.12)$$

Then it is familiar that, in view of (7.9),

$$f_{\lambda}(t) = g(t). \quad (7.13)$$

It follows from (7.10) and (7.13), by Lemma 6, that

$$\int_1^\infty f(t)\,dt \text{ is summable } (C, \lambda). \tag{7.14}$$

Consequently, by the second hypothesis,

$$\int_1^\infty f(t)\phi(t)\,dt \text{ is bounded } (C). \tag{7.15}$$

On the other hand we have, in virtue of (7.13), (7.8) and (7.1),

$$\int_1^x g(u)\phi^{(\lambda)}(u)\,du = \frac{1}{\Gamma(\lambda)}\int_1^x \phi^{(\lambda)}(u)\,du \int_1^u (u-t)^{\lambda-1} f(t)\,dt$$

$$= \frac{1}{\Gamma(\lambda)}\int_1^x f(t)\,dt \int_t^x (u-t)^{\lambda-1}\phi^{(\lambda)}(u)\,du$$

$$= (-1)^{s+1}\int_1^x f(t)\phi(t)\,dt - \frac{1}{\Gamma(\lambda)}\int_1^x f(t)\,dt \int_x^\infty (u-t)^{\lambda-1}\phi^{(\lambda)}(u)\,du \quad (x > 1). \tag{7.16}$$

It follows from (7.6) and (7.14), by Lemma 11 with $\phi(t)$ replaced by $\phi^{(\lambda)}(t)$, that the final repeated integral in (7.16) is $o(1)$ $(C, s+1)$ as $x \to \infty$. Hence, by (7.11) and (7.16), in contradiction to (7.15),

$$\int_1^\infty f(t)\phi(t)\,dt \text{ is not bounded } (C).$$

Therefore the assumption is false, and thus, since $\phi(t)$ satisfies the hypotheses with λ replaced by $\delta+r$ $(r = 0, 1, \ldots, s)$,

$$\textit{if } \int_1^\infty t^{\delta+r-1}|\phi^{(\delta+r)}(t)|\,dt < \infty \textit{ then } \int_1^\infty t^{\delta+r}|\phi^{(\delta+r+1)}(t)|\,dt < \infty \quad (r = 0, 1, \ldots, s).$$

The result now follows in consequence of (7.4).

*Second version**. We note that $\phi(t)$ in this case satisfies the hypotheses of the first version. Result (ii) follows and, as before, there is an absolutely continuous function $\psi(t)$ and a number l, such that

$$\text{(i)}' \ \psi(t) \equiv \phi(t), \ \psi(t) - l = o(1) \text{ as } t \to \infty \text{ and } \text{(ii)}' \int_1^\infty \left|\frac{d}{dt}\{\psi(t) - l\}\right| dt < \infty.$$

It is familiar that $\int_1^\infty \frac{\cos(\log t)}{t}\,dt$ is bounded $(C, 0)$ but not summable (C).

* *Cf.* L. S. Bosanquet, *Journal London Math. Soc.*, 20 (1945), 47.

It follows from (i)′ and (ii)′, by the second version of Theorem A with $\lambda = 0$ and $f(t) = t^{-1}\cos(\log t)$, that

$$\int_1^\infty \frac{\cos(\log t)}{t}\{\phi(t) - l\}\,dt \text{ is summable } (C, 0).$$

Thus, in view of the second hypothesis of the theorem,

$$\int_1^\infty \frac{l\cos(\log t)}{t}\,dt \text{ is summable } (C),$$

which is only possible when $l = 0$.

This completes the proof of the theorem.

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